Research Article

Functions of Bounded $\kappa\phi$ -Variation in the Sense of Riesz-Korenblum

Mariela Castillo,¹ Sergio Rivas,² María Sanoja,¹ and Iván Zea¹

¹ Escuela de Matemática, Universidad Central de Venezuela, Los Chaguaramos, Caracas 1050, Venezuela

² Área de Matemática, Universidad Nacional Abierta, San Bernardino, Caracas 1010, Venezuela

Correspondence should be addressed to María Sanoja; sanoja_maria@hotmail.com

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We present the space of functions of bounded $\kappa\phi$ -variation in the sense of Riesz-Korenblum, denoted by $\kappa BV_{\phi}[a, b]$, which is a combination of the notions of bounded ϕ -variation in the sense of Riesz and bounded κ -variation in the sense of Korenblum. Moreover, we prove that the space generated by this class of functions is a Banach space with a given norm and we prove that the uniformly bounded composition operator satisfies Matkowski's weak condition.

1. Introduction

The concept of functions of bounded variation has been well known since Jordan [1] gave the complete characterization of functions of a bounded variation as the difference of two increasing functions in 1881. This class of functions immediately proved to be important in connection with the rectification of curves and with Dirichlet's theorem on the convergence of Fourier series. Functions of a bounded variation exhibit many interesting properties that make them a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics (see [2–4]).

Riesz [5] in 1910 generalized the notion of Jordan and introduced the concept of bounded p -variation ($1 < p < \infty$) and showed that, for $1 < p < \infty$, this class coincides with the class of functions absolutely continuous with the derivative in the space L_p . On the other hand, this notion of bounded p -variation was generalized by Medvedev [6] in 1953 who introduced the concept of bounded ϕ -variation in the sense of Riesz and also showed a Riesz's lemma for this class of functions.

Korenblum [7] in 1975 introduced the notion of bounded κ -variation. This concept differs from others due to the fact that it introduces a distortion function κ that measures intervals in the domain of the function and not in the range. In 1985, Cyphert and Kelingos [8] showed that a function u is of bounded κ -variation if it can be written as the difference

of two κ -decreasing functions. In 1986, S. K. Kim and J. Kim [9] and Park [10], in 2010, introduced the notion of functions of $\kappa\phi$ -bounded variation on compact interval $[a, b] \subset \mathbb{R}$ which is a combination of concepts of bounded κ -variation and bounded ϕ -variation in the sense of Schramm [11], and in 2011 Aziz et al. [12] showed that the space of bounded κ -variation satisfies Matkowski's weak condition.

Recently in [13] Castillo et al. introduce the notion of bounded κ -variation in the sense of Riesz-Korenblum, which is a combination of the notions of bounded p -variation in the sense of Riesz and bounded κ -variation in the sense of Korenblum.

The purpose of this paper is twofold. First, to introduce the concept of bounded $\kappa\phi$ -variation in the sense of Riesz-Korenblum, which is a combination of the notions of bounded ϕ -variation in the sense of Riesz and bounded κ -variation in the sense of Korenblum. We prove some properties of this class of functions and its relation with the functions of bounded κ -variation and bounded ϕ -variation in the sense of Riesz. Second we prove that the space generated by this class of functions is a Banach space with a given norm and that the uniformly bounded composition operator satisfies Matkowski's weak condition in this space. The Matkowski property has been studied by several authors (see [14–16]), and for Matkowski's weak property, see also [3, 17–21]. In [22–24] Matkowski, Merentes, and others authors have been studying a weaker condition on the composition operator

such as uniformly bounded and uniformly continuous composition operators.

2. Preliminaries

In this section we present some definitions and preliminary results related to the notion of functions of bounded $\kappa\varphi$ -variation in the sense of Riesz-Korenblum.

Definition 1. Let $u : [a, b] \rightarrow \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$, we define

$$V(u; [a, b]) := \sup_{\pi} \sum_{i=1}^n |u(t_i) - u(t_{i-1})|, \quad (1)$$

where the supremum is taken over all partitions π of the interval $[a, b]$. If $V(u; [a, b]) < \infty$, we say that u has bounded variation. We denote by $BV[a, b]$ the collection of all functions of bounded variation on $[a, b]$.

Now, we will give some well-known properties of the space of functions $BV[a, b]$.

- (1) If the function u is monotone, then $V(u; [a, b]) = |u(b) - u(a)|$.
- (2) If $u \in BV[a, b]$, then u is bounded on $[a, b]$.
- (3) A function u has bounded variation in an interval $[a, b]$ if and only if it can be decomposed as a difference of increasing functions.
- (4) Every function of bounded variation has left- and right-hand limits at each point of its domain.
- (5) If $u \in \text{Lip}[a, b]$, then $u \in BV[a, b]$ that is, $\text{Lip}[a, b] \subset BV[a, b]$.
- (6) $BV[a, b]$ is a Banach space endowed with the norm

$$\|u\|_{BV} = |u(a)| + V(u; [a, b]), \quad u \in BV[a, b]. \quad (2)$$

In 1937, Young (see [25]) introduced the definition of φ -function as follows.

Definition 2. A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be a φ -function if it satisfies the following properties.

- (a) φ is continuous on $[0, \infty)$.
- (b) $\varphi(t) = 0$ if and only if $t = 0$.
- (c) φ is strictly increasing.
- (d) $\lim_{t \rightarrow \infty} \varphi(t) = \infty$.

Definition 3 (conditions ∞_1 and Δ_2). Let φ be a convex φ -function, then

- (a) φ satisfies the condition ∞_1 if $\lim_{t \rightarrow \infty} (\varphi(t)/t) = \infty$,
- (b) φ satisfies the condition $\Delta_2(\infty)$ if there is $C > 0$, $x_0 > 0$ such that

$$\varphi(2t) \leq C\varphi(t), \quad t \geq x_0. \quad (3)$$

The notion of bounded variation due to Jordan (Definition 1) was generalized by Medvedev (see [6]) as follows.

Definition 4. Let φ be a φ -function and $u : [a, b] \rightarrow \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$, we define

$$V_{\varphi}^R(u) = V_{\varphi}^R(u; [a, b]) := \sup_{\pi} \sum_{i=1}^n \varphi\left(\frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|}\right) |t_i - t_{i-1}|, \quad (4)$$

where the supremum is taken over all partitions π of the interval $[a, b]$. If $V_{\varphi}^R(u; [a, b]) < \infty$, we say that u has bounded φ -variation in the sense of Riesz. We denote by $V_{\varphi}^R[a, b]$ the class of all functions of bounded φ -variation in the sense of Riesz on $[a, b]$.

Now we give some properties of the class of functions $V_{\varphi}^R[a, b]$.

- (1) If $u \in \text{Lip}[a, b]$, then $u \in V_{\varphi}^R[a, b]$; that is, $\text{Lip}[a, b] \subset V_{\varphi}^R[a, b]$.
- (2) If φ is convex then $V_{\varphi}^R[a, b] \subset BV[a, b]$ and if $\lim_{t \rightarrow \infty} (\varphi(t)/t) = \gamma < \infty$, then $V_{\varphi}^R[a, b] = BV[a, b]$.
- (3) If φ satisfies the condition ∞_1 and $V_{\varphi}^R(u; [a, b]) < \infty$, then u is bounded on $[a, b]$.
- (4) $V_{\varphi}^R[a, b]$ is a symmetric convex set.
- (5) If φ is convex, then $\kappa V_{\varphi}^R[a, b]$ is a vector space if and only if φ satisfies the condition $\Delta_2(\infty)$.
- (6) If φ is convex, then the set $RV_{\varphi}[a, b] = \{u : [a, b] \rightarrow \mathbb{R} : \exists \lambda > 0, V_{\varphi}^R(\lambda u) < \infty\}$ is a Banach space endowed with the norm

$$\|u\|_{\varphi} = |u(a)| + \inf \left\{ \varepsilon > 0 : V_{\varphi} \left(\frac{u}{\varepsilon} \right) \leq 1 \right\}. \quad (5)$$

- (7) Let φ be a convex φ -function such that it satisfies the condition ∞_1 . A function u has bounded φ -variation in an interval $[a, b]$ if and only if u is absolutely continuous in $[a, b]$ and $\int_a^b \varphi(|u'(t)|) dt < \infty$. Also,

$$V_{\varphi}^R(u; [a, b]) = \int_a^b \varphi(|u'(t)|) dt. \quad (6)$$

Other generalization of the notion of bounded variation was introduced by Korenblum. Korenblum employed a function $\kappa : [0, 1] \rightarrow [0, 1]$ called κ -function. This function κ can be viewed as a rescaling of lengths of subintervals of $[a, b]$ such that the length of $[a, b]$ is 1 if $\kappa(1) = 1$.

Definition 5. A function $\kappa : [0, 1] \rightarrow [0, 1]$ is said to be a κ -function if it satisfies the following properties:

- (a) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$,
- (b) κ is concave (down), increasing, and
- (c) $\lim_{t \rightarrow 0^+} (\kappa(t)/t) = \infty$.

The set of all κ -functions will be denoted by $\mathcal{K}$. Note that, every κ -function κ is subadditive; that is,

$$\kappa(t_1 + t_2) \leq \kappa(t_1) + \kappa(t_2), \quad t_1, t_2 \in [0, 1]. \quad (7)$$

Then, for all partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$, we have

$$1 = \kappa(1) = \kappa\left(\sum_{i=1}^n \frac{t_i - t_{i-1}}{b - a}\right) \leq \sum_{i=1}^n \kappa\left(\frac{t_i - t_{i-1}}{b - a}\right). \quad (8)$$

Korenblum (see [7]) introduces the definition of bounded κ -variation as follows.

Definition 6. A real function u on $[a, b]$ is said to be of bounded κ -variation, if

$$\begin{aligned} \kappa V(u) &= \kappa V(u; [a, b]) \\ &:= \sup_{\pi} \frac{\sum_{i=1}^n |u(t_i) - u(t_{i-1})|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} < \infty, \end{aligned} \quad (9)$$

where the supremum is taken over all partitions π of the interval $[a, b]$. We denote by $\kappa BV[a, b]$ the collection of all functions of bounded κ -variation on $[a, b]$.

Next, some properties of the space $\kappa BV[a, b]$ are exposed (see [8]).

- (1) If the function u is monotone, then $\kappa V(u; [a, b]) = |u(b) - u(a)|$.
- (2) If $u \in \kappa BV[a, b]$, then u is bounded on $[a, b]$.
- (3) If $u \in BV[a, b]$, then $u \in \kappa BV[a, b]$; that is, $BV[a, b] \subset \kappa BV[a, b]$.
- (4) A function u has bounded κ -variation in an interval $[a, b]$ if and only if it can be decomposed as a difference of κ -decreasing functions.
- (5) Every function of bounded κ -variation has left- and right-hand limits at each point of its domain.
- (6) $\kappa BV[a, b]$ is a Banach space endowed with the norm

$$\|u\|_{\kappa} = |u(a)| + \kappa V(u; [a, b]), \quad u \in \kappa BV[a, b]. \quad (10)$$

3. Main Results

In this section we present the principal results of this paper. Next, we introduce the definition of function of bounded $\kappa\varphi$ -variation in the sense of Riesz-Korenblum for the function $u : [a, b] \rightarrow \mathbb{R}$.

Definition 7. Let φ be a φ -function, $\kappa \in \mathcal{K}$, and $u : [a, b] \rightarrow \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$, we define

$$\begin{aligned} \kappa V_{\varphi}^R(u) &= \kappa V_{\varphi}^R(u; [a, b]) \\ &:= \sup_{\pi} \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))}, \end{aligned} \quad (11)$$

where the supremum is taken over all partitions π of the interval $[a, b]$. If $\kappa V_{\varphi}^R(u; [a, b]) < \infty$, we say that u has bounded $\kappa\varphi$ -variation in the sense of Riesz-Korenblum. We will denote by $\kappa V_{\varphi}^R[a, b]$ the class of all functions of bounded $\kappa\varphi$ -variation in the sense of Riesz-Korenblum on $[a, b]$.

Remark 8. Note that the class $\kappa V_{\varphi}^R[a, b]$ is not empty since for an affine function $u : [a, b] \rightarrow \mathbb{R}$ is defined by $u(t) := dt + e$, $t \in [a, b]$, where d, e are fixed real numbers. For a given partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$ we have

$$\begin{aligned} &\frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ &= \frac{\varphi(|d|) (b - a)}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))}. \end{aligned} \quad (12)$$

Taking the supremum over all partitions π of the interval $[a, b]$, the greater value of the right side of the above expression is obtain for the partition $\pi : a = t_0 < t_1 = b$ and in this case we get

$$\frac{\varphi(|d|) (b - a)}{\kappa((t_1 - t_0) / (b - a))} = \frac{\varphi(|d|) (b - a)}{\kappa(1)} = \varphi(|d|) (b - a). \quad (13)$$

Therefore,

$$\kappa V_{\varphi}^R(u; [a, b]) = \varphi(|d|) (b - a). \quad (14)$$

In the following proposition, we prove two important properties of the space $\kappa RV_{\varphi}[a, b]$.

Proposition 9. Let φ be a convex φ -function, then

$$(a) \quad \kappa V_{\varphi}^R[a, b] \subset \kappa BV[a, b].$$

$$(b) \quad \text{If } \lim_{t \rightarrow \infty} (\varphi(t)/t) = \gamma < \infty, \text{ then } \kappa V_{\varphi}^R[a, b] = \kappa BV[a, b].$$

Proof. (a) Let $u \in \kappa V_{\varphi}^R[a, b]$, $\pi : a = t_0 < t_1 < \dots < t_n = b$, be a partition of the interval $[a, b]$ and

$$\sigma := \left\{ i = 1, \dots, n : \frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|} \leq 1 \right\}. \quad (15)$$

Since φ is a convex φ -function and $\varphi(0) = 0$, we have

$$\varphi(1) = \varphi\left(\frac{t}{t}\right) \leq \frac{1}{t}\varphi(t), \quad t \geq 1, \quad (16)$$

so $t \leq \varphi(t)/\varphi(1)$.

Hence, for $i \notin \sigma$ we get

$$\frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|} \leq \frac{\varphi(|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|)}{\varphi(1)} \quad (17)$$

multiplied by $|t_i - t_{i-1}|$ and applying the sum on both sides of the above inequality, we have

$$\begin{aligned} & \sum_{i \notin \sigma}^n \frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|} |t_i - t_{i-1}| \\ & \leq \frac{1}{\varphi(1)} \sum_{i \notin \sigma}^n \varphi\left(\frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|}\right) |t_i - t_{i-1}|, \end{aligned} \quad (18)$$

then

$$\begin{aligned} & \frac{\sum_{i=1}^n |u(t_i) - u(t_{i-1})|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & = \frac{\sum_{i=1}^n (|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & = \frac{\sum_{i \in \sigma} (|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \quad + \frac{\sum_{i \notin \sigma} (|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \leq \frac{\sum_{i \in \sigma} |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \quad + \frac{1}{\varphi(1)} \cdot \frac{\sum_{i \notin \sigma} \varphi(|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \leq \frac{(b-a)}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} + \frac{1}{\varphi(1)} \kappa V_\varphi^R(u) \\ & \leq (b-a) + \frac{1}{\varphi(1)} \kappa V_\varphi^R(u). \end{aligned} \quad (19)$$

Then

$$\frac{\sum_{i=1}^n |u(t_i) - u(t_{i-1})|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \leq (b-a) + \frac{1}{\varphi(1)} \kappa V_\varphi^R(u). \quad (20)$$

Considering the supremum over all partitions π of the interval $[a, b]$ in the above expression, we get

$$\kappa V(u) \leq b-a + \frac{1}{\varphi(1)} \kappa V_\varphi^R(u), \quad (21)$$

therefore $\kappa V_\varphi^R[a, b] \subset \kappa BV[a, b]$.

Now, we will show part (b). If

$$0 < \lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = \gamma < \infty, \quad (22)$$

then there exist $x_0 > 0$ and $c > 0$, such that

$$\varphi(t) \leq ct, \quad t \geq x_0. \quad (23)$$

Let us consider the partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$, $u \in \kappa BV[a, b]$, and

$$\alpha_{x_0} := \left\{ i = 1, \dots, n : \frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|} > x_0 \right\}, \quad (24)$$

then

$$\begin{aligned} & \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & = \frac{\sum_{i \in \alpha_{x_0}} \varphi(|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \quad + \frac{\sum_{i \notin \alpha_{x_0}} \varphi(|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \leq \frac{\sum_{i \in \alpha_{x_0}} c |u(t_i) - u(t_{i-1})|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \quad + \frac{\sum_{i \notin \alpha_{x_0}} \varphi(x_0) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))}, \end{aligned} \quad (25)$$

thus,

$$\begin{aligned} & \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})|/|t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \leq \frac{c \sum_{i \in \alpha_{x_0}} |u(t_i) - u(t_{i-1})|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \quad + \frac{\varphi(x_0) \sum_{i \notin \alpha_{x_0}} |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \leq c \cdot \kappa V(u) + \varphi(x_0) \frac{(b-a)}{\sum_{i=1}^n \kappa((t_i - t_{i-1})/(b-a))} \\ & \leq c \cdot \kappa V(u) + \varphi(x_0) (b-a). \end{aligned} \quad (26)$$

Then by considering the supremum over all partitions π of the interval $[a, b]$ of the left side, we get

$$\kappa V_\varphi^R(u) \leq c \cdot \kappa V(u) + \varphi(x_0) (b-a), \quad (27)$$

that is,

$$\kappa BV[a, b] \subset \kappa V_\varphi^R[a, b]. \quad (28)$$

Therefore, from part (a) and (28) we have $\kappa V_\varphi^R[a, b] = \kappa BV[a, b]$. $\square$

Proposition 10. Let $\text{Lip}[a, b]$ be the Banach space of all Lipschitz functions $u : [a, b] \rightarrow \mathbb{R}$. Then $\text{Lip}[a, b] \subset \kappa V_\varphi^R[a, b]$.

Proof. Let $\pi : a = t_0 < t_1 < \dots < t_n = b$ be a partition of the interval $[a, b]$ and $u \in \text{Lip}[a, b]$, then there exists $c \in \mathbb{R}$ such that for any $x, y \in [a, b]$ we have $|u(x) - u(y)| \leq c|x - y|$. Hence,

$$\begin{aligned} & \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ & \leq \frac{\sum_{i=1}^n \varphi(c |t_i - t_{i-1}| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ & = \frac{\sum_{i=1}^n \varphi(c) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ & = \frac{\varphi(c) (b - a)}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ & \leq \varphi(c) (b - a), \end{aligned} \quad (29)$$

considering the supremum over all partitions π of the interval $[a, b]$, we get

$$\kappa V_\varphi^R(u) \leq \varphi(c) (b - a) < \infty. \quad (30)$$

Therefore, $\text{Lip}[a, b] \subset \kappa V_\varphi^R[a, b]$. $\square$

The class of functions of a bounded $\kappa\varphi$ -variation has many interesting properties as the following proposition shows.

Proposition 11. Let φ be a φ -function, $\kappa \in \mathcal{K}$ and $u : [a, b] \rightarrow \mathbb{R}$ be a function, then

(a) The function $\kappa V_\varphi^R(\cdot) : \kappa V_\varphi^R[a, b] \rightarrow \mathbb{R}$ is an even function, that is, $\kappa V_\varphi^R(u) = \kappa V_\varphi^R(-u)$.

(b) $\kappa V_\varphi^R(u) = 0$ if and only if u is constant.

(c) If $\kappa V_\varphi^R(u) < \infty$ then u is bounded on $[a, b]$.

(d) φ is convex if and only if the function $\kappa V_\varphi^R(\cdot) : \kappa V_\varphi^R[a, b] \rightarrow [0, \infty)$ defined by

$$\kappa V_\varphi^R(u) := \kappa V_\varphi^R(u; [a, b]), \quad u \in \kappa V_\varphi^R[a, b] \quad (31)$$

is convex.

Proof. (a) From Definition 4, we have

$$\begin{aligned} & \kappa V_\varphi^R(u) \\ & = \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ & = \sup_\pi \frac{\sum_{i=1}^n \varphi(|-u(t_i) - (-u(t_{i-1}))| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ & = \kappa V_\varphi^R(-u). \end{aligned} \quad (32)$$

(b) If u is constant, then $\kappa V_\varphi^R(u) = 0$. Now if $\kappa V_\varphi^R(u) = 0$, we get that

$$0 = \kappa V_\varphi^R(u) \geq \frac{\varphi(|u(t) - u(a)| / |t - a|) |t - a|}{\kappa((t - a) / (b - a)) + \kappa((b - t) / (b - a))}, \quad (33)$$

for some $t \in (a, b]$. From Definitions 2 and 5, we have

$$0 \geq \frac{\varphi(|u(t) - u(a)| / |t - a|) |t - a|}{\kappa((t - a) / (b - a)) + \kappa((b - t) / (b - a))} \geq 0, \quad (34)$$

hence

$$\varphi\left(\frac{|u(t) - u(a)|}{|t - a|}\right) |t - a| = 0, \quad t \in (a, b]. \quad (35)$$

Since $\varphi(0) = 0$ if and only if $t = 0$, we get

$$\frac{|u(t) - u(a)|}{|t - a|} = 0, \quad (36)$$

therefore, $u(t) = u(a)$ $t \in [a, b]$. So, u is constant.

(c) Suppose that $u \in \kappa V_\varphi^R[a, b]$ is unbounded on $[a, b]$. Then, there exists a sequence $\{t_n\}_{n \geq 1}$, $t_n \in [a, b]$, for all $n \geq 1$, such that $\lim_{n \rightarrow \infty} |u(t_n)| = \infty$. Let $\{t_{n_k}\}_{n_k \geq 1}$ be a subsequence of $\{t_n\}_{n \geq 1}$ such that $\{t_{n_k}\}_{n_k \geq 1}$ converges to point $x \in [a, b]$. Then, $\{u(t_{n_k})\}_{n_k \geq 1}$ is a subsequence of $\{u(t_n)\}_{n \geq 1}$. So, $\lim_{k \rightarrow \infty} |u(t_{n_k})| = \infty$.

Case 1. Suppose that $x \neq a$. Since

$$\begin{aligned} & \frac{\varphi(|u(t_{n_k}) - u(a)| / |t_{n_k} - a|) |t_{n_k} - a|}{\kappa((t_{n_k} - a) / (b - a)) + \kappa((b - t_{n_k}) / (b - a))} \\ & \leq \kappa V_\varphi^R(u) < \infty, \end{aligned} \quad (37)$$

for all $n_k \geq 1$ and since φ is continuous, we have

$$\begin{aligned} & \frac{\varphi(\lim_{k \rightarrow \infty} (|u(t_{n_k}) - u(a)| / |x - a|)) |x - a|}{\kappa((x - a) / (b - a)) + \kappa((b - x) / (b - a))} \\ & = \lim_{k \rightarrow \infty} \frac{\varphi(|u(t_{n_k}) - u(a)| / |t_{n_k} - a|) |t_{n_k} - a|}{\kappa((t_{n_k} - a) / (b - a)) + \kappa((b - t_{n_k}) / (b - a))} \\ & \leq \kappa V_\varphi^R(u). \end{aligned} \quad (38)$$

On the other hand, $|u(t_{n_k}) - u(a)|$ tends to infinity as $k \rightarrow \infty$. Then, since $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} \frac{\varphi(|u(t_{n_k}) - u(a)| / |x - a|) |x - a|}{\kappa((x - a) / (b - a)) + \kappa((b - x) / (b - a))} = \infty, \quad (39)$$

then $\kappa V_\varphi^R(u) = \infty$, which is a contradiction.

Case 2. Suppose that $x = a$. Then, since

$$\begin{aligned} & \frac{\varphi(|u(b) - u(t_{n_k})| / |b - t_{n_k}|) |b - t_{n_k}|}{\kappa((t_{n_k} - a) / (b - a)) + \kappa((b - t_{n_k}) / (b - a))} \\ & \leq \kappa V_\varphi^R(u) < \infty, \end{aligned} \quad (40)$$

for all $n_k \geq 1$, and since φ is continuous, we have

$$\begin{aligned} & \frac{\varphi(\lim_{k \rightarrow \infty} |u(b) - u(t_{n_k})| / |b - x|) |b - x|}{\kappa((x - a) / (b - a)) + \kappa((b - x) / (b - a))} \\ &= \lim_{k \rightarrow \infty} \frac{\varphi(|u(b) - u(t_{n_k})| / |b - t_{n_k}|) |b - t_{n_k}|}{\kappa((t_{n_k} - a) / (b - a)) + \kappa((b - t_{n_k}) / (b - a))} \\ &\leq \kappa V_\varphi^R(u). \end{aligned} \quad (41)$$

Since $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and $\lim_{k \rightarrow \infty} |u(b) - u(t_{n_k})| = \infty$, we get

$$\lim_{k \rightarrow \infty} \frac{\varphi(|u(b) - u(t_{n_k})| / |b - x|) |b - x|}{\kappa((x - a) / (b - a)) + \kappa((b - x) / (b - a))} = \infty, \quad (42)$$

so $\kappa V_\varphi^R = \infty$ which is a contradiction. In both cases a contradiction is reached. Hence u is bounded.

(d) Let $u, v : [a, b] \rightarrow \mathbb{R}$ and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$. Suppose that φ is convex, then

$$\begin{aligned} & \alpha \kappa V_\varphi^R(u) + \beta \kappa V_\varphi^R(v) \\ &= \alpha \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ &+ \beta \sup_\pi \frac{\sum_{i=1}^n \varphi(|v(t_i) - v(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\ &\geq \sup_\pi \left(\sum_{i=1}^n \left[\alpha \varphi \left(\frac{|u(t_i) - u(t_{i-1})|}{|t_i - t_{i-1}|} \right) |t_i - t_{i-1}| \right. \right. \\ &\quad \left. \left. + \beta \varphi \left(\frac{|v(t_i) - v(t_{i-1})|}{|t_i - t_{i-1}|} \right) |t_i - t_{i-1}| \right] \right) \\ &\quad \times \left(\sum_{i=1}^n \kappa \left(\frac{t_i - t_{i-1}}{b - a} \right) \right)^{-1}, \end{aligned} \quad (43)$$

where the supremum is taken over all partitions π of the interval $[a, b]$.

Since φ is convex, we have

$$\begin{aligned} & \alpha \kappa V_\varphi^R(u) + \beta \kappa V_\varphi^R(v) \\ &\geq \sup_\pi \left(\sum_{i=1}^n \varphi \left(\frac{|\alpha(u(t_i) - u(t_{i-1})) + \beta(v(t_i) - v(t_{i-1}))|}{|t_i - t_{i-1}|} \right) |t_i - t_{i-1}| \right) \\ &\quad \times \left(\sum_{i=1}^n \kappa \left(\frac{t_i - t_{i-1}}{b - a} \right) \right)^{-1} \end{aligned}$$

$$\begin{aligned} &= \sup_\pi \left(\sum_{i=1}^n \varphi \left(\frac{|(\alpha u + \beta v)(t_i) - (\alpha u + \beta v)(t_{i-1})|}{|t_i - t_{i-1}|} \right) |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa \left(\frac{t_i - t_{i-1}}{b - a} \right) \right)^{-1} \\ &= \kappa V_\varphi^R(\alpha u + \beta v). \end{aligned} \quad (44)$$

So $\kappa V_\varphi^R(\alpha u + \beta v) \leq \alpha \kappa V_\varphi^R(u) + \beta \kappa V_\varphi^R(v)$ for all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$, and therefore $\kappa V_\varphi^R(\cdot)$ is convex.

In order to prove the order direction, let us consider the following functions. $u : [a, b] \rightarrow \mathbb{R}$ defined by $u(t) := xt$ and $v : [a, b] \rightarrow \mathbb{R}$ defined by $v(t) := yt$, $t \in [a, b]$. Therefore, we have

$$\begin{aligned} \kappa V_\varphi^R(u) &= \varphi(x)(b - a), & \kappa V_\varphi^R(v) &= \varphi(y)(b - a), \\ \kappa V_\varphi^R(\alpha u + \beta v) &= \varphi(\alpha x + \beta y)(b - a). \end{aligned} \quad (45)$$

Since $\kappa V_\varphi^R(\cdot)$ is convex, we get

$$\begin{aligned} \varphi(\alpha x + \beta y)(b - a) &= \kappa V_\varphi^R(\alpha u + \beta v) \\ &\leq \alpha \kappa V_\varphi^R(u) + \beta \kappa V_\varphi^R(v) \\ &= \alpha \varphi(x)(b - a) + \beta \varphi(y)(b - a), \end{aligned} \quad (46)$$

which implies that

$$\varphi(\alpha x + \beta y) \leq \alpha \varphi(x) + \beta \varphi(y). \quad (47)$$

Therefore, φ is convex, and the proof of the theorem is completed. $\square$

Remark 12. The part (c) of Proposition 11 is a consequence of the part (a) of Proposition 9 if the φ -function is convex.

Proposition 13. Let φ be a φ -function, $\kappa \in \mathcal{K}$, and $u, v : [a, b] \rightarrow \mathbb{R}$ be functions, then

$$\begin{aligned} \kappa V_\varphi^R(\alpha u + \beta v) &\leq \kappa V_\varphi^R(u) + \kappa V_\varphi^R(v) \\ \alpha, \beta &\in [0, 1], \quad \alpha + \beta = 1. \end{aligned} \quad (48)$$

Proof. Let $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$ and $x, y \in [0, \infty)$. Since φ is nondecreasing and nonnegative and $\alpha x + \beta y$ is one of the segment joining point x with y , then we have

$$\varphi(\alpha x + \beta y) \leq \max\{\varphi(x), \varphi(y)\} \leq \varphi(x) + \varphi(y). \quad (49)$$

From the inequality above, we deduce that

$$\begin{aligned}
& \kappa V_{\varphi}^R(u) + \kappa V_{\varphi}^R(v) \\
&= \sup_{\pi} \frac{\sum_{i=1}^n \varphi(|u(t_i) - u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\
&+ \sup_{\pi} \frac{\sum_{i=1}^n \varphi(|v(t_i) - v(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \\
&\geq \sup_{\pi} \left(\sum_{i=1}^n \varphi \left(\frac{|\alpha(u(t_i) - u(t_{i-1})) + \beta(v(t_i) - v(t_{i-1}))|}{|t_i - t_{i-1}|} \right) \right. \\
&\quad \left. \times |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa \left(\frac{t_i - t_{i-1}}{b - a} \right) \right)^{-1},
\end{aligned} \tag{50}$$

thus

$$\begin{aligned}
& \kappa V_{\varphi}^R(u) + \kappa V_{\varphi}^R(v) \\
&= \sup_{\pi} \left(\sum_{i=1}^n \varphi \left(\frac{|(\alpha u + \beta v)(t_i) - (\alpha u + \beta v)(t_{i-1})|}{|t_i - t_{i-1}|} \right) \right. \\
&\quad \left. \times |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa \left(\frac{t_i - t_{i-1}}{b - a} \right) \right)^{-1} \\
&= \kappa V_{\varphi}^R(\alpha u + \beta v).
\end{aligned} \tag{51}$$

Therefore, $\kappa V_{\varphi}^R(\alpha u + \beta v) \leq \kappa V_{\varphi}^R(u) + \kappa V_{\varphi}^R(v)$ for all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$. $\square$

Remark 14. From Propositions 11 and 13, we have $\kappa V_{\varphi}^R[a, b]$ to be convex and symmetric.

The following lemma allows us to give a characterization of the space $\kappa RV_{\varphi}[a, b]$.

Lemma 15. Let X be a vector space and $A \subset X$ a nonempty convex and symmetric set. Then one has the following.

(a) $0 \in A$.

(b) The vector space associated by A is equal to:

$$\langle A \rangle = \{x \in X : \exists \lambda > 0, \lambda x \in A\} = \bigcup_{\lambda > 0} \lambda A. \tag{52}$$

Proof. (a) For all $x \in A$, we have that $-x \in A$ $0 = (1/2)x + (1/2)(-x) \in A$.

(b) By definition, $A \subset \bigcup_{\lambda > 0} \lambda A \subset \langle A \rangle$. In order to show the other inclusion, we have to prove that $\bigcup_{\lambda > 0} \lambda A$ is a vector space. Indeed, if $\alpha \in \mathbb{R}$, we get

$$\begin{aligned}
& \alpha(\lambda x) \in \bigcup_{\lambda > 0} \lambda A, \quad \alpha > 0, \\
& \alpha(\lambda x) = 0 \in \bigcup_{\lambda > 0} \lambda A, \quad \alpha = 0, \\
& \alpha(\lambda x) = (-\alpha\lambda)(-x) \in \bigcup_{\lambda > 0} \lambda A, \quad \alpha < 0.
\end{aligned} \tag{53}$$

Also, if $x, y \in A$ and $\alpha > 0, \beta > 0$, then

$$\alpha x + \beta y = (\alpha + \beta) \left(\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \right) \in (\alpha + \beta) A. \tag{54}$$

Therefore, $\bigcup_{\lambda > 0} \lambda A$ is a vector space.

Let us now prove that $\bigcup_{\lambda > 0} \lambda A = \{x \in X : \exists \lambda > 0, \lambda x \in A\}$. Indeed, if $z = \lambda x$, $x \in A$, $\lambda > 0$, then $1/z = x \in A$. Conversely, if $\lambda x \in A$, for some $\lambda > 0$, then $x = (1/\lambda)(\lambda x) \in (1/\lambda)A$. $\square$

As a consequence of Lemma 15 and since κV_{φ}^R is a convex and symmetric set, we have the following corollary.

Corollary 16. Let φ be a φ -function and $\kappa \in \mathcal{K}$. Then, the vector space generated by the class $\kappa V_{\varphi}^R[a, b]$ is equal to the following

$$\kappa RV_{\varphi}[a, b] = \{u : [a, b] \rightarrow \mathbb{R} : \exists \lambda > 0, \kappa V_{\varphi}^R(\lambda u) < \infty\}. \tag{55}$$

Theorem 17. Let φ be a φ -function and $\kappa \in \mathcal{K}$, then

$$RV_{\varphi}[a, b] \subset \kappa RV_{\varphi}[a, b] \subset \kappa BV[a, b]. \tag{56}$$

Proof. First we prove that $RV_{\varphi}[a, b] \subset \kappa RV_{\varphi}[a, b]$. Let $u \in RV_{\varphi}[a, b]$, then there exists $\lambda > 0$ such that $V_{\varphi}^R(\lambda u) < \infty$ and $\pi : a = t_0 < t_1 < \dots < t_n = b$ is a partition of the interval $[a, b]$, then from inequality (8) we have

$$\begin{aligned}
& \sum_{i=1}^n \varphi \left(\frac{|\lambda u(t_i) - \lambda u(t_{i-1})|}{|t_i - t_{i-1}|} \right) |t_i - t_{i-1}| \\
& \leq V_{\varphi}^R(\lambda u) \\
& = V_{\varphi}^R(\lambda u) \kappa \left(\sum_{i=1}^n \frac{t_i - t_{i-1}}{b - a} \right) \\
& \leq V_{\varphi}^R(\lambda u) \sum_{i=1}^n \kappa \left(\frac{t_i - t_{i-1}}{b - a} \right).
\end{aligned} \tag{57}$$

Thus,

$$\frac{\sum_{i=1}^n \varphi(|\lambda u(t_i) - \lambda u(t_{i-1})| / |t_i - t_{i-1}|) |t_i - t_{i-1}|}{\sum_{i=1}^n \kappa((t_i - t_{i-1}) / (b - a))} \leq V_{\varphi}^R(\lambda u). \tag{58}$$

Then, considering the supremum of the left side, we get

$$\kappa V_{\varphi}^R(\lambda u) \leq V_{\varphi}^R(\lambda u), \tag{59}$$

therefore, $u \in \kappa RV_{\varphi}[a, b]$ and $RV_{\varphi}[a, b] \subset \kappa RV_{\varphi}[a, b]$. On the other hand, by part (a) of Proposition 9 we get that $\kappa RV_{\varphi}[a, b] \subset \kappa BV[a, b]$. $\square$

Theorem 18. Let φ be a convex φ -function, then

$$\Lambda = \{u \in \kappa RV_{\varphi}[a, b] : \kappa V_{\varphi}^R(u) \leq 1\} \tag{60}$$

is a symmetric convex absorbent subset of $\kappa RV_{\varphi}[a, b]$.

Proof. First we show the convexity. Let $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$ and $u, v \in \Lambda$. Then, by Proposition 11 we get

$$\kappa V_\varphi^R(\alpha u + \beta v) \leq \alpha \kappa V_\varphi^R(u) + \beta \kappa V_\varphi^R(v) \leq \alpha \cdot 1 + \beta \cdot 1 = 1, \quad (61)$$

thus $\alpha u + \beta v \in \Lambda$.

Now let $u \in \Lambda$ and $0 < |\lambda| \leq 1$. If $0 < \lambda \leq 1$ by Proposition 11, we have

$$\kappa V_\varphi^R(\lambda u) \leq \lambda \kappa V_\varphi^R(u) \leq \lambda \cdot 1 \leq 1. \quad (62)$$

For the case $-1 \leq \lambda < 0$, by the symmetric and convexity of the functional $\kappa V_\varphi^R(\cdot)$ given in Proposition 11 we get that

$$\kappa V_\varphi^R(\lambda u) = \kappa V_\varphi^R(-\lambda(-u)) \leq -\lambda \kappa V_\varphi^R(-u) = |\lambda| \kappa V_\varphi^R(u). \quad (63)$$

Hence, we have shown that Λ is balance. Now we will show that Λ is absorbent. Let $u \in \kappa RV_\varphi[a, b]$ then there exist $\alpha > 0$ such that $\kappa V_\varphi^R(\alpha u) < \infty$. If $\kappa V_\varphi^R(\alpha u) \leq 1$ then $\alpha u \in \Lambda$. On the other hand, if $\kappa V_\varphi^R(\alpha u) > 1$ we have

$$\kappa V_\varphi^R\left(\frac{\alpha u}{\kappa V_\varphi^R(\alpha u)}\right) \leq \frac{1}{\kappa V_\varphi^R(\alpha u)} \kappa V_\varphi^R(\alpha u) = 1, \quad (64)$$

in this case, $(\alpha/\kappa V_\varphi^R(\alpha u))u \in \Lambda$. Hence, Λ is absorbent. $\square$

Remark 19. As a consequence of Theorem 18, the Minkowski functional associated to the set Λ defines a seminorm on $\kappa RV_\varphi[a, b]$ and is defined by

$$\mu_\Lambda(u) = \inf \left\{ \lambda > 0 : \kappa V_\varphi^R\left(\frac{u}{\lambda}\right) \leq 1 \right\}. \quad (65)$$

Theorem 20. Let φ be a convex φ -function. Then, $(\kappa RV_\varphi[a, b], \|\cdot\|_{\kappa\varphi})$, where the functional $\|\cdot\|_{\kappa\varphi} : \kappa RV_\varphi[a, b] \rightarrow \mathbb{R}$, defined by

$$\|u\|_{\kappa\varphi} = |u(a)| + \mu_\Lambda(u), \quad u \in \kappa RV_\varphi[a, b], \quad (66)$$

is a normed space.

Proof. Let $u, v \in \kappa RV_\varphi[a, b]$, $\alpha \in \mathbb{R}$. Then, we have the following.

(a) $\|u\|_{\kappa\varphi} \geq 0$ since $|u(a)| \geq 0$ and $\mu_\Lambda(u) \geq 0$.

(b)

$$\begin{aligned} \|\alpha u\|_{\kappa\varphi} &= |\alpha u(a)| + |\alpha| \mu_\Lambda(u) \\ &= |\alpha| |u(a)| + |\alpha| \mu_\Lambda(u) \\ &= |\alpha| (|u(a)| + \mu_\Lambda(u)) \\ &= |\alpha| \|u\|_{\kappa\varphi}. \end{aligned} \quad (67)$$

Therefore, $\|\alpha u\|_{\kappa\varphi} = |\alpha| \|u\|_{\kappa\varphi}$.

(c)

$$\begin{aligned} \|u + v\|_{\kappa\varphi} &= |(u + v)(a)| + \mu_\Lambda(u + v) \\ &= |u(a) + v(a)| + \mu_\Lambda(u + v) \\ &\leq |u(a)| + |v(a)| + \mu_\Lambda(u) + \mu_\Lambda(v) \\ &= \|u\|_{\kappa\varphi} + \|v\|_{\kappa\varphi}. \end{aligned} \quad (68)$$

Thus, $\|u + v\|_{\kappa\varphi} \leq \|u\|_{\kappa\varphi} + \|v\|_{\kappa\varphi}$.

(d) Let us now prove that $\|u\|_{\kappa\varphi} = 0$ if and only if $u = 0$. Indeed, suppose that $\|u\|_{\kappa\varphi} = 0$, that is,

$$|u(a)| + \mu_\Lambda(u) = 0, \quad (69)$$

then $|u(a)| = 0$ and $0 = \mu_\Lambda(u) = \inf\{\lambda > 0 : \kappa V_\varphi^R(u/\lambda) \leq 1\}$. This implies that for each positive integer n , there exists λ_n such that $1/\lambda_n > \lambda_n > 0$ and $\kappa V_\varphi^R(u/\lambda_n) \leq 1$. As the function $\kappa V_\varphi^R(\cdot)$ is convex, we have $\kappa V_\varphi^R(u) \leq \lambda_n$. Taking the limit as $n \rightarrow \infty$, we get $\kappa V_\varphi^R(u) = 0$. Moreover, by part (b) of Proposition 11 we have that u is constant; that is, $u(t) = u(a)$, $t \in [a, b]$, therefore $u = 0$.

Now, suppose that $u = 0$. Then $|u(a)| = 0$ and $\mu_\Lambda(u) = 0$. Hence $\|u\|_{\kappa\varphi} = 0$. Therefore $(\kappa RV_\varphi[a, b], \|\cdot\|_{\kappa\varphi})$ is a normed space. $\square$

Lemma 21. Let $u \in \kappa RV_\varphi^R[a, b]$. If $\lambda > 0$, then $\mu_\Lambda(u) \leq \lambda$ if and only if $\kappa V_\varphi^R(u/\lambda; [a, b]) \leq 1$.

Proof

Case 1. Let $u \in \kappa RV_\varphi^R[a, b]$ and $\mu_\Lambda(u) < \lambda$. Then, by the infimum property, there is $0 < k < \lambda$ such that $\kappa V_\varphi^R(u/k) \leq 1$. Hence, by the convexity of $\kappa V_\varphi^R(\cdot)$,

$$\kappa V_\varphi^R\left(\frac{u}{\lambda}\right) = \kappa V_\varphi^R\left(\frac{u}{k} \frac{k}{\lambda}\right) \leq \frac{k}{\lambda} \kappa V_\varphi^R\left(\frac{u}{k}\right) \leq \frac{k}{\lambda} \leq 1. \quad (70)$$

Case 2. Let $u \in \kappa RV_\varphi^R[a, b]$ and $\mu_\Lambda(u) = \lambda$. Then, by the infimum property, there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ such that

$$\lambda_n > \lambda, \quad \kappa V_\varphi^R\left(\frac{u}{\lambda_n}\right) \leq 1, \quad n \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} \lambda_n = \lambda. \quad (71)$$

Since u/λ_n pointwise converges to u/λ on $[a, b]$ as $n \rightarrow \infty$, by the lower semicontinuity of $\kappa V_\varphi^R(\cdot)$, we obtain that

$$\kappa V_\varphi^R\left(\frac{u}{\lambda}\right) \leq \liminf_{n \rightarrow \infty} \kappa V_\varphi^R\left(\frac{u}{\lambda_n}\right) \leq 1. \quad (72)$$

As, by the definition of the infimum, the converse is obvious, this completes the proof. $\square$

In the next theorem, we prove that $(\kappa RV_\varphi[a, b], \|\cdot\|_{\kappa\varphi})$ is a Banach space.

Theorem 22. Let φ be a convex φ -function such that φ satisfies the condition ∞_1 , then the space $(\kappa RV_\varphi[a, b], \|\cdot\|_{\kappa\varphi})$ is a Banach space.

Proof. Let $\{u_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\kappa RV_\varphi[a, b], \|\cdot\|_{\kappa\varphi})$, then given that $0 < \varepsilon < 1$, there is $N \in \mathbb{N}$, such that for $n, m \geq N$ we have

$$\|u_n - u_m\|_{\kappa\varphi} < \varepsilon, \quad n, m \geq N, \quad (73)$$

that is,

$$|(u_n - u_m)(a)| + \mu_\Lambda(u_n - u_m) < \varepsilon, \quad n, m \geq N. \quad (74)$$

Then,

$$|u_n(a) - u_m(a)| < \varepsilon, \quad \mu_\Lambda(u_n - u_m) < \varepsilon \quad n, m \geq N, \quad (75)$$

by Lemma 21 and the last inequality, we have

$$\kappa V_\varphi^R\left(\frac{u_n - u_m}{\varepsilon}; [a, b]\right) < 1. \quad (76)$$

Then, for a partition $\pi : a \leq x < y \leq b$ we get that

$$\begin{aligned} & \varphi\left(\frac{|(u_n - u_m)(y) - (u_n - u_m)(x)|}{\varepsilon|y - x|}\right)|y - x| \\ & \times \left(\kappa\left(\frac{x - a}{b - a}\right) + \kappa\left(\frac{y - x}{b - a}\right) + \kappa\left(\frac{b - y}{b - a}\right)\right)^{-1} \\ & \leq \kappa V_\varphi^R\left(\frac{u_n - u_m}{\varepsilon}; [a, b]\right) < 1. \end{aligned} \quad (77)$$

Hence, we have

$$\begin{aligned} & \varphi\left(\frac{|(u_n - u_m)(y) - (u_n - u_m)(x)|}{\varepsilon|y - x|}\right)|y - x| \\ & < \kappa\left(\frac{x - a}{b - a}\right) + \kappa\left(\frac{y - x}{b - a}\right) + \kappa\left(\frac{b - y}{b - a}\right) \\ & \leq 3 \cdot \kappa(1) = 3, \end{aligned} \quad (78)$$

therefore

$$|(u_n - u_m)(y) - (u_n - u_m)(x)| < \varphi^{-1}\left(\frac{3}{|y - x|}\right)|y - x|\varepsilon. \quad (79)$$

Since φ satisfies the condition ∞_1 and φ^{-1} is continuous, we obtain that

$$\varphi^{-1}\left(\frac{3}{|y - x|}\right)|y - x|, \quad x, y \in [a, b], \quad x \neq y, \quad (80)$$

is bounded. Let $M > 0$ be an upper bound of (80), then

$$|(u_n - u_m)(y) - (u_n - u_m)(x)| < M\varepsilon. \quad (81)$$

As a consequence, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is a uniformly Cauchy sequence, on the interval $[a, b]$, and by the completeness of $\mathbb{R}$ $\lim_{n \rightarrow \infty} u_n(t)$ exist for all $t \in [a, b]$.

We defined on $[a, b]$ the function $u(t) := \lim_{n \rightarrow \infty} u_n(t)$. We claim that $u \in \kappa RV_\varphi[a, b]$. In fact, let $\pi : a = t_1 < t_2 < \dots < t_n = b$ be a partition of the interval $[a, b]$. Then, for $n \geq N$ and $\varepsilon > 0$ one has

$$\begin{aligned} & \kappa V_\varphi^R\left(\frac{u_n - u}{\varepsilon}\right) \\ & = \sup_\pi \left(\sum_{i=1}^n \varphi\left(\frac{|(u_n - u)(t_i) - (u_n - u)(t_{i-1})|}{\varepsilon|t_i - t_{i-1}|}\right) \right. \\ & \quad \left. \times |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa\left(\frac{t_i - t_{i-1}}{b - a}\right) \right)^{-1} \\ & = \sup_\pi \left(\sum_{i=1}^n \varphi\left(|(u_n - \lim_{m \rightarrow \infty} u_m)(t_i) - (u_n - \lim_{m \rightarrow \infty} u_m)(t_{i-1})|(\varepsilon|t_i - t_{i-1}|)^{-1}\right) \right. \\ & \quad \left. \times |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa\left(\frac{t_i - t_{i-1}}{b - a}\right) \right)^{-1} \\ & = \sup_\pi \lim_{m \rightarrow \infty} \left(\sum_{i=1}^n \varphi\left(\frac{|(u_n - u_m)(t_i) - (u_n - u_m)(t_{i-1})|}{\varepsilon|t_i - t_{i-1}|}\right) \right. \\ & \quad \left. \times |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa\left(\frac{t_i - t_{i-1}}{b - a}\right) \right)^{-1}. \end{aligned} \quad (82)$$

Since for $n, m \geq N$,

$$\begin{aligned} 1 & \geq \kappa V_\varphi^R\left(\frac{u_n - u_m}{\varepsilon}\right) \\ & \geq \left(\sum_{i=1}^n \varphi\left(\frac{|(u_n - u_m)(t_i) - (u_n - u_m)(t_{i-1})|}{\varepsilon|t_i - t_{i-1}|}\right) |t_i - t_{i-1}| \right) \\ & \quad \times \left(\sum_{i=1}^n \kappa\left(\frac{t_i - t_{i-1}}{b - a}\right) \right)^{-1}, \end{aligned} \quad (83)$$

then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \left(\sum_{i=1}^n \varphi\left(\frac{|(u_n - u_m)(t_i) - (u_n - u_m)(t_{i-1})|}{\varepsilon|t_i - t_{i-1}|}\right) \right. \\ & \quad \left. \times |t_i - t_{i-1}| \right) \left(\sum_{i=1}^n \kappa\left(\frac{t_i - t_{i-1}}{b - a}\right) \right)^{-1} \\ & \leq 1, \quad n \geq N. \end{aligned} \quad (84)$$

Thus,

$$\kappa V_\varphi^R\left(\frac{u_n - u}{\varepsilon}\right) \leq 1. \quad (85)$$

Therefore, $(u_n - u)/\varepsilon \in \kappa RV_\varphi[a, b]$ for all $n \geq N$. As $u_n \in \kappa RV_\varphi[a, b]$ for all $n \geq N$, and $\kappa RV_\varphi[a, b]$ is a vector space, then $u \in \kappa RV_\varphi[a, b]$.

Finally, let us prove that $\{u_n\}_{n \geq 1}$ converge in norm to u . Let $\varepsilon > 0$ be arbitrary, then

$$\begin{aligned} \|u_n - u\|_{\kappa\varphi} &= \left\| u_n - \lim_{m \rightarrow \infty} u_m \right\|_{\kappa\varphi} \\ &= \lim_{m \rightarrow \infty} \|u_n - u_m\|_{\kappa\varphi} < \varepsilon, \quad \forall n \geq N. \end{aligned} \quad (86)$$

Therefore, the sequence $\{u_n\}_{n \geq 1}$ converges to u in the norm $\|\cdot\|_{\kappa\varphi}$ and thus $\kappa RV_\varphi[a, b]$ is a Banach space. $\square$

3.1. Uniformly Bounded Composition Operator. In this section, we present the other main result of this paper; namely, we show that any uniformly bounded composition operator that maps the space $\kappa RV_\varphi[a, b]$ into itself necessarily satisfies the so-called Matkowski's weak condition.

Definition 23. For a given function $h : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, the composition operator $H : \mathbb{R}^{[a, b]} \rightarrow \mathbb{R}^{[a, b]}$ generated by h is defined by

$$Hu(t) = h(t, u(t)), \quad u \in \mathbb{R}^{[a, b]}, \quad t \in [a, b]. \quad (87)$$

Here $\mathbb{R}^{[a, b]}$ denotes the family of all functions $u : [a, b] \rightarrow \mathbb{R}$.

Remark 24. Since $\kappa RV_\varphi[a, b] \subset \kappa BV[a, b]$, then every function of bounded $\kappa\varphi$ -variation in the sense of Riesz-Korenblum has left- and right-hand limits at each point of its domain (see [8]).

Now, we will give the definition of left regularization of a function.

Definition 25. Let $u \in \kappa RV_\varphi[a, b]$, one defined its left regularization $u^- : (a, b] \rightarrow \mathbb{R}$ of mapping u by the following:

$$u^-(t) := \begin{cases} \lim_{s \rightarrow t^-} u(s), & t \in (a, b], \\ u(a), & t = a. \end{cases} \quad (88)$$

We will denote by $\kappa RV_\varphi^-[a, b]$ the subset in $\kappa RV_\varphi[a, b]$ which consists of those functions that are left continuous on $(a, b]$.

Lemma 26. If $u \in \kappa RV_\varphi[a, b]$, then $u^- \in \kappa RV_\varphi^-[a, b]$.

Thus, if a function u has bounded $\kappa\varphi$ -variation in the sense of Riesz-Korenblum, then its left regularization is a left continuous function.

Definition 27 (see [24, Definition 1]). Let $\mathbb{X}$ and $\mathbb{Y}$ be two metric (or normed) spaces. One says that a mapping $H : \mathbb{X} \rightarrow \mathbb{Y}$ is uniformly bounded if, for any $t > 0$, there exists a nonnegative real number $\gamma(t)$ such that for any nonempty set $B \subset \mathbb{X}$ we have

$$\text{diam } B \leq t \implies \text{diam } H(B) \leq \gamma(t). \quad (89)$$

Theorem 28. Let φ be a φ -function, $\kappa \in \mathcal{K}$, and $h : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function continuous with respect to the second variable. Suppose that the composition operator H generated by h maps $\kappa RV_\varphi[a, b]$ into itself and satisfies the following inequality:

$$\|H(u) - H(v)\|_{\kappa\varphi} \leq \gamma(\|u - v\|_{\kappa\varphi}), \quad u, v \in \kappa RV_\varphi[a, b] \quad (90)$$

for some function $\gamma : [0, \infty) \rightarrow [0, \infty)$. Then, there exist functions $\alpha, \beta \in \kappa RV_\varphi^-[a, b]$ such that

$$h^-(t, x) = \alpha(t)x + \beta(t), \quad t \in [a, b], \quad x \in \mathbb{R}, \quad (91)$$

where $h^-(\cdot, x) : (a, b] \rightarrow \mathbb{R}$ is the left regularization of $h(\cdot, x)$ for all $x \in \mathbb{R}$.

Proof. By hypothesis, for $x \in \mathbb{R}$ fixed the constant function $u(t) = x$, $t \in [a, b]$ belongs to $\kappa RV_\varphi[a, b]$. Since H maps $\kappa RV_\varphi[a, b]$ into itself, we have that the function $(Hu)(t) = h(t, u(t)) = h(t, x) \in \kappa RV_\varphi[a, b]$. By Lemma 26, the left regularization $h^-(\cdot, x) \in \kappa RV_\varphi^-[a, b]$ for every $x \in \mathbb{R}$.

From inequality (90) and the definition of the norm $\|\cdot\|_{\kappa\varphi}$ we obtain for $u_1, u_2 \in \kappa RV_\varphi[a, b]$ that

$$\begin{aligned} \mu_\Lambda(H(u_1) - H(u_2)) &\leq \|H(u_1) - H(u_2)\|_{\kappa\varphi} \\ &\leq \gamma(\|u_1 - u_2\|_{\kappa\varphi}). \end{aligned} \quad (92)$$

From the inequality (92) and Lemma 21, if $\gamma(\|u_1 - u_2\|_{\kappa\varphi}) > 0$, then

$$\kappa V_\varphi^R \left(\frac{H(u_1) - H(u_2)}{\gamma(\|u_1 - u_2\|_{\kappa\varphi})} \right) \leq 1. \quad (93)$$

On the other hand, if $a \leq r < s \leq b$, then from the definitions of the operator H , the functional κV_φ^R , and inequality (93), we have

$$\begin{aligned} &\varphi \left(\frac{|H(u_1)(s) - H(u_2)(s) - H(u_1)(r) + H(u_2)(r)|}{\gamma(\|u_1 - u_2\|_{\kappa\varphi})|s - r|} \right) \\ &\times |s - r| \left(\kappa \left(\frac{r - a}{b - a} \right) + \kappa \left(\frac{s - r}{b - a} \right) + \kappa \left(\frac{b - s}{b - a} \right) \right)^{-1} \\ &\leq \kappa V_\varphi^R \left(\frac{H(u_1) - H(u_2)}{\gamma(\|u_1 - u_2\|_{\kappa\varphi})} \right) \leq 1, \end{aligned} \quad (94)$$

whence

$$\begin{aligned} &\varphi \left(\frac{|h(s, u_1(s)) - h(s, u_2(s)) - h(r, u_1(r)) + h(r, u_2(r))|}{\gamma(\|u_1 - u_2\|_{\kappa\varphi})|s - r|} \right) \\ &\times |s - r| \left(\kappa \left(\frac{r - a}{b - a} \right) + \kappa \left(\frac{s - r}{b - a} \right) + \kappa \left(\frac{b - s}{b - a} \right) \right)^{-1} \leq 1. \end{aligned} \quad (95)$$

Let $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ and consider the functions $u_1, u_2 : [a, b] \rightarrow \mathbb{R}$ defined by

$$u_k(t) := \frac{x_1 - x_2}{2(s-r)}(t-r) + \frac{x_k + x_2}{2}, \quad t \in [a, b], \quad k = 1, 2 \quad (96)$$

then $u_k \in \kappa RV_\varphi[a, b]$, $k = 1, 2$. From (96), we have

$$u_1 - u_2 = \frac{x_1 - x_2}{2}, \quad (97)$$

therefore,

$$\|u_1 - u_2\|_{\kappa\varphi} = \left| \frac{x_1 - x_2}{2} \right|. \quad (98)$$

Moreover,

$$u_1(s) = x_1, \quad u_1(r) = \frac{x_1 + x_2}{2} = u_2(s), \quad u_2(r) = x_2. \quad (99)$$

From (95) we get

$$\begin{aligned} & \varphi \left(\left| h(s, x_1) - h\left(s, \frac{x_1 + x_2}{2}\right) - h\left(r, \frac{x_1 + x_2}{2}\right) + h(r, x_2) \right| \right) \\ & \times (\gamma(|(x_1 - x_2)/2|)|s-r|)^{-1} \leq \frac{3}{|s-r|}, \end{aligned} \quad (100)$$

then

$$\begin{aligned} & \left| h(s, x_1) - h\left(s, \frac{x_1 + x_2}{2}\right) - h\left(r, \frac{x_1 + x_2}{2}\right) + h(r, x_2) \right| \\ & \leq \varphi^{-1} \left(\frac{3}{|s-r|} \right) \gamma \left(\left| \frac{x_1 - x_2}{2} \right| \right) |s-r|. \end{aligned} \quad (101)$$

Since φ satisfies the condition ∞_1 , we have

$$\lim_{t \rightarrow 0} \varphi^{-1} \left(\frac{1}{t} \right) t = 0, \quad (102)$$

then from the definition of h^- and letting r tend to s in (101), we obtain

$$\left| h^-(s, x_1) - 2h^-\left(s, \frac{x_1 + x_2}{2}\right) + h^-(s, x_2) \right| \leq 0, \quad (103)$$

therefore

$$h^-\left(s, \frac{x_1 + x_2}{2}\right) = \frac{h^-(s, x_1) + h^-(s, x_2)}{2}, \quad (104)$$

$$s \in (a, b], \quad x_1, x_2 \in \mathbb{R}$$

which proves that for every fixed $s \in (a, b]$ the function $h^-(s, \cdot)$ satisfies the Jensen functional equation in $\mathbb{R}$ (see [26, page 315]). The continuity of h with respect to the second variable implies that for every $t \in [a, b]$ there exist $\alpha(t), \beta(t) \in \mathbb{R}$ such that

$$h^-(t, x) = \alpha(t)x + \beta(t), \quad t \in [a, b], \quad x \in \mathbb{R}. \quad (105)$$

Since $h^-(\cdot, x) \in \kappa RV_\varphi^-[a, b]$ for all $x \in \mathbb{R}$ in particular for $x = 0$,

$$\beta(t) := h^-(t, 0) \quad t \in [a, b], \quad (106)$$

therefore $\beta \in \kappa RV_\varphi^-[a, b]$.

On the other hand, considering $x = 1$ we obtain that

$$\alpha(t) = h^-(t, x) - \beta(t), \quad t \in [a, b], \quad (107)$$

as $\kappa RV_\varphi[a, b]$ is a vector space which implies that $\alpha \in \kappa RV_\varphi^-[a, b]$. $\square$

Theorem 29. Let φ be a function and $\kappa \in \mathcal{K}$. Suppose that a function $h : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with respect to the second variable. If the composition operator H generated by h maps the space $\kappa RV_\varphi[a, b]$ into itself and is uniformly bounded, then there exist the functions $\alpha, \beta \in \kappa RV_\varphi^-[a, b]$, such that

$$h^-(t, x) = \alpha(t)x + \beta(t), \quad t \in [a, b], \quad x \in \mathbb{R}. \quad (108)$$

Proof. Take any $t \geq 0$ and $u, v \in \kappa RV_\varphi[a, b]$ such that

$$\|u - v\|_{\kappa\varphi} \leq t. \quad (109)$$

Since $\text{diam}\{u, v\} \leq t$, by the uniform boundedness of H , we have

$$\text{diam } H(\{u, v\}) \leq \gamma(t), \quad (110)$$

that is,

$$\|H(u) - H(v)\|_{\kappa\varphi} = \text{diam } H(\{u, v\}) \leq \gamma(\|u - v\|_{\kappa\varphi}), \quad (111)$$

therefore, by the Theorem 28 we have

$$h^-(t, x) = \alpha(t)x + \beta(t), \quad t \in [a, b], \quad x \in \mathbb{R}. \quad (112)$$

$\square$

Remark 30. Observe that similar results hold for the right regularization h^+ of h defined by

$$h^+(t, x) := \lim_{s \rightarrow t^+} h(s, x) \quad t \in [a, b]. \quad (113)$$

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