Research Article

On Approximately p -Wright Affine Functions in Ultrametric Spaces

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We prove some results for mappings taking values in ultrametric spaces and satisfying approximately a generalization of the equation of p -Wright affine functions. They are motivated by the notion of stability for functional equations.

1. Introduction and Preliminaries

Let X_1 and X_2 be linear spaces over fields $\mathbb{F}_1$ and $\mathbb{F}_2$, respectively, and let $p, \hat{p} \in \mathbb{F}_1$ be fixed. The functional equation

$$f(px + \hat{p}y) + f(\hat{p}x + py) = f(x) + f(y), \quad (1)$$

for function $f: X_1 \rightarrow X_2$, generalizes the equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y). \quad (2)$$

For $\mathbb{F}_1 = \mathbb{R}$ and $p \in (0, 1)$, solutions of (2) are called the p -Wright affine functions, which are both p -Wright convex and concave (see [1]). For information on the p -Wright convexity and the p -Wright concavity we refer, for example, to [1, 2].

Note also that for $p = 1/2$ (2) becomes the well-known Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}. \quad (3)$$

For $p = 1/3$ (2) takes the form $f(2x + y) + f(x + 2y) = f(3x) + f(3y)$, which has been studied in [3] (see also [4]) in connection with some investigations of the generalized (σ, τ) -Jordan derivations on Banach algebras. The cases of more arbitrary p have been studied in [1, 5] (cf. [2]).

In this paper we consider (1) in a bit generalized form, with p and $\hat{p}$ being some group endomorphisms. Motivated by the notion of the Hyers-Ulam stability (see, e.g., [6–9] for further details), we prove some results concerning the

mappings that take values in the ultrametric spaces and satisfy (1) approximately, that is, fulfil inequality (8) (with suitable assumptions on θ). Our outcomes correspond to those in [10], where some issues (analogous as in Theorems 2 and 6) have been considered for functions mapping a classical normed space (real or complex) into a classical Banach space, with p being a scalar (real or complex, resp.) and for θ of the following two forms:

$$\theta(x, y) = A(\|x\|^k + \|y\|^k), \quad \theta(x, y) = A\|x\|^k\|y\|^k, \quad (4)$$

with some real $A > 0$ and $k > 0$ (see also [4] for similar outcomes but only for $p = 1/3$). The main tool in our investigations is a fixed point theorem from [11] (for information on related results see [12–15]).

Recall that an ultrametric space is a metric space (Y, d) with the metric d satisfying the condition $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for every $x, y, z \in Y$; such a metric d is called an ultrametric. One of important examples of the ultrametric spaces is a non-Archimedean normed space.

Let us remind yet that a linear space X over a field $\mathbb{K}$, with a function $\|\cdot\|: X \rightarrow [0, \infty)$, is said to be a non-Archimedean normed space provided $\|x\| = 0$ if and only if $x = 0$ and $\|rx\| = |r|\|x\|$ and $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$, $r \in \mathbb{K}$, where $|\cdot|$ denotes a non-Archimedean valuation in $\mathbb{K}$, that is a function $|\cdot|: \mathbb{K} \rightarrow [0, \infty)$ such that $|r| = 0$ if and only if $r = 0$ and $|rs| = |r||s|$ and $|r + s| \leq \max\{|r|, |s|\}$ for $r, s \in \mathbb{K}$.

Clearly, if X is a non-Archimedean normed space, then the formula $d(x, y) := \|x - y\|$ defines an ultrametric d in X , that is invariant (i.e., $d(x + z, y + z) = d(x, y)$ for every $x, y, z \in X$).

Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. If the valuation is nontrivial (i.e., there is an $r_0 \in \mathbb{K}$ such that $0 \neq |r_0| \neq 1$), then we have $|1| = |-1| = 1$ and $|n \times 1| \leq 1$ for all $n \in \mathbb{N}$ (positive integers), where 1 is the neutral element of the semigroup $(\mathbb{K}, \cdot)$, $1 \times 1 = 1$ and $(n + 1) \times 1 = (n \times 1) + 1$ for $n \in \mathbb{N}$.

The first example of a non-Archimedean field was provided by Hensel in [16], where he gave a description of the p -adic numbers (for each fixed prime number p and any non-zero rational number x , there exists a unique integer n_x such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p ; then $|x|_p := p^{-n_x}$ defines a non-Archimedean valuation in $\mathbb{Q}$ and therefore also a non-Archimedean norm in $\mathbb{Q}$).

Let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$, with respect to the metric $d(x, y) = |x - y|_p$. Then $\mathbb{Q}_p$ (called the p -adic number field) can be identified with the set of all formal series $x = \sum_{k \geq n_x} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The operation of addition and multiplication between any two elements of $\mathbb{Q}_p$ is defined naturally. The non-Archimedean norm $\|\cdot\|_p$ in $\mathbb{Q}_p$ is defined by $\|\sum_{k \geq n_x} a_k p^k\|_p = p^{-n_x}$; $\mathbb{Q}_p$ endowed with it turns out to be a locally compact field (see [17]). Let us mention yet that the p -adic numbers have gained the interest of physicists because of their connections with some issues in quantum physics, p -adic strings and superstrings (see [18]).

The problem of stability of functional equations was motivated by a question of S.M. Ulam asked in 1940 and an answer to it published by Hyers [19]. Since then numerous papers on this subject have been published, and we refer to [6–9, 13, 15, 20, 21] for more details, some discussions, and further references.

Let us mention yet that the issue of stability of functional equations is related to the notions of shadowing (see [22, 23]), the theory of perturbation (see [24]), and optimization.

Moslehian and Rassias [25] (see [26, 27] for some related outcomes) have proved the first stability results for the Cauchy and quadratic functional equations in non-Archimedean normed spaces. Afterwards several stability results for other equations in such spaces have been published, for example, in [28–30] (for further references see [13, 15]).

2. Auxiliary Result

In this section X denotes a nonempty set and (Y, d) stands for a complete ultrametric space. The main tool in the proofs of the main theorems of this paper is a fixed point result that can be derived from [11, Theorem 2]. To present it we need to introduce some notions.

For any $\delta_1, \delta_2 \in \mathbb{R}_+^X$ (A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$) we write $\delta_1 \leq \delta_2$ provided $\delta_1(x) \leq \delta_2(x)$ for $x \in X$, and we say that an operator $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ is nondecreasing if it satisfies the condition $\Lambda\delta_1 \leq \Lambda\delta_2$ for all $\delta_1, \delta_2 \in \mathbb{R}_+^X$ with $\delta_1 \leq \delta_2$. Moreover, given a sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^X$, we write $\lim_{n \rightarrow \infty} g_n = 0$ provided $\lim_{n \rightarrow \infty} g_n(x) = 0$ for $x \in X$.

We use the following hypothesis concerning operators $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$: ($\mathcal{C}$) $\lim_{n \rightarrow \infty} \Lambda\delta_n = 0$ for every sequence $(\delta_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^X$ satisfying the condition $\lim_{n \rightarrow \infty} \delta_n = 0$.

Moreover, to simplify some notations, we define $\Delta : (Y^X)^2 \rightarrow \mathbb{R}_+^X$ by

$$\Delta(\xi, \mu)(x) := d(\xi(x), \mu(x)), \quad \xi, \mu \in Y^X, x \in X. \quad (5)$$

Now we are in a position to present the mentioned fixed point result.

Theorem 1. Let $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ be a nondecreasing operator satisfying hypothesis ($\mathcal{C}$). If $\mathcal{T} : Y^X \rightarrow Y^X$, $\varepsilon : X \rightarrow \mathbb{R}_+$, and $\varphi : X \rightarrow Y$ satisfy

$$\begin{aligned} d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) &\leq \Lambda(\Delta(\xi, \mu))(x), \\ \xi, \mu &\in Y^X, x \in X, \\ d((\mathcal{T}\varphi)(x), \varphi(x)) &\leq \varepsilon(x), \quad x \in X, \\ \lim_{n \rightarrow \infty} \Lambda^n \varepsilon &= 0, \end{aligned} \quad (6)$$

then the limit $\lim_{n \rightarrow \infty} (\mathcal{T}^n \varphi)(x) =: \psi(x)$ exists for every $x \in X$ and the function $\psi \in Y^X$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$d(\varphi(x), \psi(x)) \leq \sup_{n \in \mathbb{N}_0} (\Lambda^n \varepsilon)(x), \quad x \in X. \quad (7)$$

3. The Main Results

We say that $(Y, +, d)$ is an ultrametric group if $(Y, +)$ is a group and d is ultrametric in Y such that the group operation $+$ is continuous with respect to d . In what follows $(X, +)$ is a group (though we use the additive notation for the group operation in X , it does not mean that the group must be commutative), $(Y, +, d)$ is an ultrametric commutative group, and the ultrametric d is complete and invariant, unless explicitly stated otherwise. Given a mapping $r : X \rightarrow X$, for simplicity of notation, we write $rx := r(x)$ for each $x \in X$.

The following two theorems concern stability of functional equation (1).

Theorem 2. Let p and $\hat{p}$ be endomorphisms of X with $p \circ \hat{p} = \hat{p} \circ p$, $B \in (0, 1)$, and $g : X \rightarrow Y$ and $\theta : X^2 \rightarrow [0, \infty)$ satisfy the inequalities

$$\begin{aligned} d(g(px + \hat{p}y) + g(\hat{p}x + py), g(x) + g(y)) &\leq \theta(x, y), \\ x, y &\in X, \end{aligned} \quad (8)$$

$$\max\{\theta(px, py), \theta(\hat{p}x, \hat{p}y)\} \leq B\theta(x, y), \quad x, y \in X. \quad (9)$$

Then there exists a unique function $G : X \rightarrow Y$ such that

$$G(px + \hat{p}y) + G(\hat{p}x + py) = G(x) + G(y), \quad x, y \in X, \quad (10)$$

$$d(g(x), G(x)) \leq \min\{\theta(x, 0), \theta(0, x)\}, \quad x \in X. \quad (11)$$

Moreover, G is the unique solution of (10) for which there exists a constant $M \in (0, \infty)$ with

$$d(g(x), G(x)) \leq M \min\{\theta(x, 0), \theta(0, x)\}, \quad x \in X. \quad (12)$$

Proof. Taking in (8) first $y = 0$ and next $x = 0$ we obtain

$$\begin{aligned} d(g(px) + g(\hat{p}x), g(x) + g(0)) &\leq \theta(x, 0), \quad x \in X, \\ d(g(\hat{p}x) + g(px), g(x) + g(0)) &\leq \theta(0, x), \quad x \in X. \end{aligned} \quad (13)$$

Write $g_0(x) := g(x) - g(0)$ and $\mathcal{T}\xi(x) := \xi(px) + \xi(\hat{p}x)$ for $x \in X, \xi \in Y^X$. Then (13) implies the inequality

$$d(\mathcal{T}g_0(x), g_0(x)) \leq \varepsilon(x), \quad x \in X, \quad (14)$$

where $\varepsilon(x) := \min\{\theta(x, 0), \theta(0, x)\}$. Define an operator $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ by

$$\Lambda\eta(x) := \max\{\eta(px), \eta(\hat{p}x)\}, \quad \eta \in \mathbb{R}_+^X, x \in X. \quad (15)$$

Then Λ is non-decreasing, satisfies hypothesis (C) and

$$\begin{aligned} d(\mathcal{T}\xi(x), \mathcal{T}\mu(x)) &= d(\xi(px) + \xi(\hat{p}x), \mu(px) + \mu(\hat{p}x)) \\ &\leq \max\{d(\xi(px), \mu(px)), d(\xi(\hat{p}x), \mu(\hat{p}x))\} \\ &= \max\{\Delta(\xi, \mu)(px), \Delta(\xi, \mu)(\hat{p}x)\} \\ &= \Lambda\Delta(\xi, \mu)(x), \quad \xi, \mu \in Y^X, x \in X. \end{aligned} \quad (16)$$

Note that by (9)

$$\begin{aligned} \varepsilon(px) &= \min\{\theta(px, 0), \theta(0, px)\} \\ &\leq \min\{B\theta(x, 0), B\theta(0, x)\} \\ &= B \min\{\theta(x, 0), \theta(0, x)\} = B\varepsilon(x), \quad x \in X, \end{aligned} \quad (17)$$

and analogously $\varepsilon(\hat{p}x) \leq B\varepsilon(x)$ for $x \in X$. Consequently

$$\max\{\varepsilon(px), \varepsilon(\hat{p}x)\} \leq B\varepsilon(x), \quad x \in X. \quad (18)$$

Further, it is easy to show by induction that

$$\Lambda^n \varepsilon(x) \leq B^n \varepsilon(x), \quad x \in X, n \in \mathbb{N}. \quad (19)$$

Since $0 < B < 1$, this means that $\lim_{n \rightarrow \infty} (\Lambda^n \varepsilon)(x) = 0$ for $x \in X$. Moreover, in view of (18) we see that $\Lambda^{n+1} \varepsilon(x) \leq \Lambda^n \varepsilon(x)$ for $n \in \mathbb{N}_0$ and $x \in X$, whence

$$\begin{aligned} \sup_{n \in \mathbb{N}_0} (\Lambda^n \varepsilon)(x) &= (\Lambda^0 \varepsilon)(x) = \varepsilon(x), \\ \sup_{n \in \mathbb{N}_0} (\Lambda^{n+1} \varepsilon)(x) &= \Lambda \varepsilon(x). \end{aligned} \quad (20)$$

Consequently, by Theorem 1, there is a solution $G_0 : X \rightarrow Y$ of the equation

$$G_0(x) = G_0(px) + G_0(\hat{p}x) \quad (21)$$

such that $d(g_0(x), G_0(x)) \leq \varepsilon(x)$ for $x \in X$. Moreover,

$$G_0(x) := \lim_{n \rightarrow \infty} (\mathcal{T}^n g_0)(x), \quad x \in X. \quad (22)$$

Now, we show that, for every $n \in \mathbb{N}_0$,

$$\begin{aligned} d(\mathcal{T}^n g_0(px + \hat{p}y) + \mathcal{T}^n g_0(\hat{p}x + py), \\ \mathcal{T}^n g_0(x) + \mathcal{T}^n g_0(y)) \\ \leq B^n \theta(x, y), \quad x, y \in X. \end{aligned} \quad (23)$$

Clearly, for $n = 0$, (23) reduces to (8). Next, fix $m \in \mathbb{N}_0$ and assume that (23) holds for $n = m$. Then, by (9),

$$\begin{aligned} d(\mathcal{T}^{m+1} g_0(px + \hat{p}y) + \mathcal{T}^{m+1} g_0(\hat{p}x + py), \\ \mathcal{T}^{m+1} g_0(x) + \mathcal{T}^{m+1} g_0(y)) \\ = d(\mathcal{T}^m g_0(p(px + \hat{p}y)) + \mathcal{T}^m g_0(\hat{p}(px + \hat{p}y)) \\ + \mathcal{T}^m g_0(p(\hat{p}x + py)) + \mathcal{T}^m g_0(\hat{p}(\hat{p}x + py)), \\ \mathcal{T}^m g_0(px) + \mathcal{T}^m g_0(\hat{p}x) \\ + \mathcal{T}^m g_0(py) + \mathcal{T}^m g_0(\hat{p}y)) \\ \leq \max\{d(\mathcal{T}^m g_0(ppx + \hat{p}py) + \mathcal{T}^m g_0(\hat{p}px + ppy), \\ \mathcal{T}^m g_0(px) + \mathcal{T}^m g_0(py)), \\ d(\mathcal{T}^m g_0(p\hat{p}x + \hat{p}\hat{p}y) + \mathcal{T}^m g_0(\hat{p}\hat{p}x + p\hat{p}y), \\ \mathcal{T}^m g_0(\hat{p}x) + \mathcal{T}^m g_0(\hat{p}y))\} \\ \leq \max\{B^m \theta(px, py), B^m \theta(\hat{p}x, \hat{p}y)\} \\ \leq B^{m+1} \theta(x, y), \quad x, y \in X. \end{aligned} \quad (24)$$

Thus we obtain (23) for $n = m + 1$, which completes the induction. Letting $n \rightarrow \infty$ in (23), we obtain that

$$\begin{aligned} G_0(px + \hat{p}y) + G_0(\hat{p}x + py) &= G_0(x) + G_0(y), \\ x, y \in X. \end{aligned} \quad (25)$$

Write $G(x) := G_0(x) + g(0)$ for $x \in X$. Then (11) holds and

$$G(px + \hat{p}y) + G(\hat{p}x + py) = G(x) + G(y), \quad x, y \in X. \quad (26)$$

It remains to prove the uniqueness of G . So, let $M \in (0, \infty)$ and $G_1 : X \rightarrow Y$ a solution to (10) with $d(g(x), G_1(x)) \leq M\varepsilon(x)$ for $x \in X$. Then

$$\begin{aligned} G_1(px) + G_1(\hat{p}x) &= G_1(x) + G_1(0), \quad x \in X, \\ G(px) + G(\hat{p}x) &= G(x) + G(0), \quad x \in X, \end{aligned} \quad (27)$$

and, by (11),

$$d(G(x), G_1(x)) \leq \max\{1, M\} \varepsilon(x), \quad x \in X. \quad (28)$$

Further, by (9), $\theta(0, 0) = 0$, whence $\varepsilon(0) = 0$. This and (28) yield

$$G(0) = G_1(0). \quad (29)$$

We show that, for each $n \in \mathbb{N}_0$,

$$d(G(x), G_1(x)) \leq \max\{1, M\} \varepsilon(x) \sup_{j \geq n} \{B^j\}, \quad x \in X. \quad (30)$$

The case $n = 0$ is just (28). So fix $m \in \mathbb{N}_0$ and assume that (30) holds for $n = m$. Then, from (27), (29), and (18) we obtain

$$\begin{aligned} d(G(x), G_1(x)) &= d(G(px) + G(\hat{p}x), G_1(px) + G_1(\hat{p}x)) \\ &\leq \max\{d(G(px), G_1(px)), d(G(\hat{p}x), G_1(\hat{p}x))\} \\ &\leq \max\left\{\max\{1, M\} \varepsilon(px) \sup_{j \geq m} \{B^j\}, \right. \\ &\quad \left. \max\{1, M\} \varepsilon(\hat{p}x) \sup_{j \geq m} \{B^j\}\right\} \\ &= \max\{1, M\} \sup_{j \geq m} \{B^j\} \max\{\varepsilon(px), \varepsilon(\hat{p}x)\} \\ &\leq \max\{1, M\} B \varepsilon(x) \sup_{j \geq m} \{B^j\} \\ &= \max\{1, M\} \varepsilon(x) \sup_{j \geq m+1} \{B^j\}, \quad x \in X. \end{aligned} \quad (31)$$

Thus we have shown (30). Now, letting $n \rightarrow \infty$ in (30) we get $G_1 = G$. $\square$

Remark 3. Let X be a normed space (either classical or non-Archimedean) over a field $\mathbb{F}$. Clearly, if X is classical, then we assume that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$; if X is non-Archimedean, then $\mathbb{F}$ is a nontrivial non-Archimedean field. Given $r : X \rightarrow X$, write $\|r\| := \inf\{\xi \in (0, \infty) : \|rx - ry\| \leq \xi\|x - y\| \text{ for } x, y \in X\}$. It is easy to see that if $\|p\| < 1$, $\|\hat{p}\| < 1$, and θ has one of the following forms

- (i) $\theta(x, y) = A(\|x\|^k + \|y\|^l)^m$,
- (ii) $\theta(x, y) = A(\max\{\|x\|^k, \|y\|^l\})^m$,
- (iii) $\theta(x, y) = A\|x\|^k\|y\|^l$,

with some $A \in (0, \infty)$ and $k, l, m \in \mathbb{N}$, then the assumptions of the above theorem are fulfilled. Multiplying and/or adding those functions we can obtain numerous further examples.

Note that in the situation when θ is of form (iii), $\theta(x, 0) = \theta(0, x) = 0$ for $x \in X$, whence $g = G$ in the statement of Theorem 2. This means that (under the assumptions of Theorem 2) with θ given by (iii), every $g : X \rightarrow Y$ satisfying (8) must be actually a solution to (1).

In particular, from Theorem 2, we get immediately the following result on stability of (1).

Corollary 4. Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in \mathbb{F}$,

$$\max\{|p|, |1 - p|\} < 1, \quad (32)$$

and, $\theta : X^2 \rightarrow [0, \infty)$ be of one let the forms (i)–(iii). Assume that $g : X \rightarrow Y$ satisfies the inequality

$$\begin{aligned} d(g(px + (1 - p)y) + g((1 - p)x + py), g(x) + g(y)) \\ \leq \theta(x, y), \quad x, y \in X. \end{aligned} \quad (33)$$

Then the following two statements are valid.

- (α) If (i) or (ii) holds, there exists a unique solution $G : X \rightarrow Y$ of (1) such that $d(g(x), G(x)) \leq A(\min\{\|x\|^k, \|x\|^l\})^m$ for $x \in X$.
- (β) If (iii) holds, then g is a solution to (1).

Remark 5. Let X be a non-archimedean normed space over a field $\mathbb{F}$ that is non-Archimedean and $p \in \mathbb{F}$. Then results analogous to Corollary 4 cannot be derived from Theorem 2, because we have $1 = |p + 1 - p| \leq \max\{|p|, |1 - p|\}$, which means that, with $|p| < 1$, we get $1 \leq |1 - p| \leq \max\{|p|, 1\} = 1$.

However, if $|p| > 1$, then $1 < |p| = |1 - p - 1| \leq \max\{1, |1 - p|\}$, which means that $|1 - p| > 1$. Consequently, we obtain results analogous to Corollary 4 for $|p| > 1$ and the function θ is one of the following forms

- (a) $\theta(x, y) = A(\|x\|^k + \|y\|^l)^m$ for all $(x, y) \in X^2 \setminus \{(0, 0)\}$ and $\theta(0, 0) = 0$,
- (b) $\theta(x, y) = A(\max\{\|x\|^k, \|y\|^l\})^m$ for all $(x, y) \in X^2 \setminus \{(0, 0)\}$ and $\theta(0, 0) = 0$,
- (c) $\theta(x, y) = A\|x\|^k\|y\|^l$ for all $x, y \in X \setminus \{0\}$ and $\theta(x, 0) = \theta(0, x) = 0$ for all $x \in X$,

with some $A, k, l \in (0, \infty)$ and $m \in (-\infty, 0)$. Unfortunately, we must define θ at the point $(0, 0)$ in a bit artificial way in (a) and (b), similar (but even bigger) problem we have in (c). The subsequent modified version of Theorem 2 amends this situation to some extent for (a) and (b).

Theorem 6. Let $p, \hat{p} : X \rightarrow X$ be monomorphisms with $p \circ \hat{p} = \hat{p} \circ p$, $B \in (0, 1)$, and $g : X \rightarrow Y$, $\theta : X^2 \setminus \{(0, 0)\} \rightarrow [0, \infty)$ satisfy

$$\begin{aligned} d(g(px + \hat{p}y) + g(\hat{p}x + py), g(x) + g(y)) &\leq \theta(x, y), \\ \max\{\theta(px, py), \theta(\hat{p}x, \hat{p}y)\} &\leq B\theta(x, y) \end{aligned} \quad (34)$$

for $(x, y) \in X^2 \setminus \{(0, 0)\}$. Then there is a solution $G : X \rightarrow Y$ of (10) with

$$d(g(x), G(x)) \leq \min\{\theta(x, 0), \theta(0, x)\}, \quad x \in X \setminus \{0\}. \quad (35)$$

Proof. Arguing in the same way as in the proof of Theorem 2, with X being replaced by $X_0 := X \setminus \{0\}$, from Theorem 1, we deduce that there is a solution $G_1 : X_0 \rightarrow Y$ of the equation

$$G_1(x) = G_1(px) + G_1(\hat{p}x) \quad (36)$$

such that $d(g_0(x), G_1(x)) \leq \min\{\theta(x, 0), \theta(0, x)\}$ for $x \in X_0$. Moreover, $G_1(x) := \lim_{n \rightarrow \infty} (\mathcal{T}^n g_0)(x)$ for $x \in X_0$. Define $G_0 : X \rightarrow Y$ by $G_0(x) = G_1(x)$ for $x \in X_0$ and $G_0(0) = 0$.

In the same way as in the proof of Theorem 2, we obtain that

$$\begin{aligned} & d(\mathcal{T}^n g_0(px + \hat{p}y) + \mathcal{T}^n g_0(\hat{p}x + py), \\ & \mathcal{T}^n g_0(x) + \mathcal{T}^n g_0(y)) \\ & \leq B^n \theta(x, y), \quad x, y \in X_0, px + \hat{p}y, \hat{p}x + py \in X_0 \end{aligned} \quad (37)$$

for all $n \in \mathbb{N}_0$. We need yet to prove that, for every $n \in \mathbb{N}_0$,

$$\begin{aligned} & d(\mathcal{T}^n g_0(\hat{p}x + py), \mathcal{T}^n g_0(x) + \mathcal{T}^n g_0(y)) \leq B^n \theta(x, y), \\ & x, y \in X_0, px + \hat{p}y = 0, \hat{p}x + py \neq 0, \end{aligned} \quad (38)$$

$$\begin{aligned} & d(\mathcal{T}^n g_0(px + \hat{p}y), \mathcal{T}^n g_0(x) + \mathcal{T}^n g_0(y)) \leq B^n \theta(x, y), \\ & x, y \in X_0, px + \hat{p}y \neq 0, \hat{p}x + py = 0, \end{aligned} \quad (39)$$

$$\begin{aligned} & d(0, \mathcal{T}^n g_0(x) + \mathcal{T}^n g_0(y)) \leq B^n \theta(x, y), \\ & x, y \in X_0, px + \hat{p}y = \hat{p}x + py = 0. \end{aligned} \quad (40)$$

We show only (38); the proofs for (39) and (40) are analogous.

Clearly, for $n = 0$, (38) follows from (34). Next, if (37) holds for a fixed $n \in \mathbb{N}$, then, for every $x, y \in X_0$ with $px + \hat{p}y = 0$ and $\hat{p}x + py \neq 0$,

$$\begin{aligned} & d(\mathcal{T}^{m+1} g_0(\hat{p}x + py), \mathcal{T}^{m+1} g_0(x) + \mathcal{T}^{m+1} g_0(y)) \\ & = d(\mathcal{T}^m g_0(p(\hat{p}x + py)) + \mathcal{T}^m g_0(\hat{p}(\hat{p}x + py)), \\ & \mathcal{T}^m g_0(px) + \mathcal{T}^m g_0(\hat{p}x) \\ & + \mathcal{T}^m g_0(py) + \mathcal{T}^m g_0(\hat{p}y)) \\ & \leq \max\{d(\mathcal{T}^m g_0(\hat{p}px + ppy), \\ & \mathcal{T}^m g_0(px) + \mathcal{T}^m g_0(py)), \\ & d(\mathcal{T}^m g_0(\hat{p}\hat{p}x + p\hat{p}y), \\ & \mathcal{T}^m g_0(\hat{p}x) + \mathcal{T}^m g_0(\hat{p}y))\} \\ & \leq \max\{B^m \theta(px, py), B^m \theta(\hat{p}x, \hat{p}y)\} \leq B^{m+1} \theta(x, y). \end{aligned} \quad (41)$$

This completes the induction.

Letting $n \rightarrow \infty$ in (37)–(40), we obtain that

$$\begin{aligned} & G_0(px + \hat{p}y) + G_0(\hat{p}x + py) = G_0(x) + G_0(y), \\ & x, y \in X. \end{aligned} \quad (42)$$

Writing $G(x) := G_0(x) + g(0)$ for $x \in X$ we obtain that (35) holds and G is a solution to (10). $\square$

Below we present a theorem that is somewhat complementary to Theorem 2.

Theorem 7. Let p and $\hat{p}$ be endomorphisms of X with $p \circ \hat{p} = \hat{p} \circ p$, and p be bijective, $B \in (0, 1)$, and $g : X \rightarrow Y$ and $\theta : X^2 \rightarrow [0, \infty)$ satisfy the inequalities

$$\max\{\theta(x, y), \theta(\hat{p}x, \hat{p}y)\} \leq B\theta(px, py), \quad x, y \in X. \quad (43)$$

Assume that $g : X \rightarrow Y$ satisfies (8). Then there exists a unique solution $G : X \rightarrow Y$ of (10) such that

$$\begin{aligned} & d(g(x), G(x)) \leq \min\{\theta(p^{-1}x, 0), \theta(0, p^{-1}x)\}, \\ & x \in X. \end{aligned} \quad (44)$$

Moreover, G is the unique solution of (10) for which there exists a constant $M \in (0, \infty)$ with

$$\begin{aligned} & d(g(x), G(x)) \leq M \min\{\theta(p^{-1}x, 0), \theta(0, p^{-1}x)\}, \\ & x \in X. \end{aligned} \quad (45)$$

Proof. The proof is very similar to that of Theorem 2, but for the convenience of readers we present it here.

Taking in (8) first $y = 0$ and next $x = 0$ we obtain

$$\begin{aligned} & d(g(px) + g(\hat{p}x), g(x) + g(0)) \leq \min\{\theta(x, 0), \theta(0, x)\}, \\ & x \in X, \end{aligned} \quad (46)$$

whence replacing x by $p^{-1}x$ we derive the following inequality:

$$\begin{aligned} & d(g(x) + g(\hat{p}p^{-1}x), g(p^{-1}x) + g(0)) \\ & \leq \min\{\theta(p^{-1}x, 0), \theta(0, p^{-1}x)\}, \quad x \in X. \end{aligned} \quad (47)$$

Since d is invariant, this yields

$$d(g(x) - g(0), g(qx) - g(\hat{p}qx)) \leq \varepsilon(x), \quad x \in X, \quad (48)$$

with $q := p^{-1}$ and $\varepsilon(x) := \min\{\theta(qx, 0), \theta(0, qx)\}$. Writing $g_0(x) := g(x) - g(0)$ and $\mathcal{T}\xi(x) := \xi(qx) - \xi(\hat{p}qx)$ for $x \in X$, $\xi \in Y^X$, we finally get

$$d(g_0(x), \mathcal{T}g_0(x)) \leq \varepsilon(x), \quad x \in X. \quad (49)$$

Define an operator $\Lambda : \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ by $\Lambda\eta(x) := \max\{\eta(qx), \eta(\hat{p}qx)\}$ for $\eta \in \mathbb{R}_+^X$, $x \in X$. Clearly, Λ is non-decreasing and satisfies hypothesis (C), and, according to (43),

$$\begin{aligned} & d(\mathcal{T}\xi(x), \mathcal{T}\mu(x)) \\ & = d(\xi(qx) - \xi(\hat{p}qx), \mu(qx) - \mu(\hat{p}qx)) \\ & \leq \max\{d(\xi(qx), \mu(qx)), d(\xi(\hat{p}qx), \mu(\hat{p}qx))\} \\ & = \max\{\Delta(\xi, \mu)(qx), \Delta(\xi, \mu)(\hat{p}qx)\} \\ & = \Lambda\Delta(\xi, \mu)(x), \quad \xi, \mu \in Y^X, x \in X. \end{aligned} \quad (50)$$

Note that

$$\begin{aligned}\varepsilon(qx) &= \min\{\theta(qqx, 0), \theta(0, qqx)\} \\ &\leq \min\{B\theta(qx, 0), B\theta(0, qx)\} \\ &= B \min\{\theta(qx, 0), \theta(0, qx)\} \\ &= B\varepsilon(x), \quad x \in X,\end{aligned}\quad (51)$$

and analogously $\varepsilon(\hat{p}qx) \leq B\varepsilon(x)$ for $x \in X$. Consequently

$$\max\{\varepsilon(qx), \varepsilon(\hat{p}qx)\} \leq B\varepsilon(x), \quad x \in X. \quad (52)$$

It is easy to show by induction that $\Lambda^n \varepsilon(x) \leq B^n \varepsilon(x)$ for $x \in X$, $n \in \mathbb{N}$. Since $0 < B < 1$, this means that $\lim_{n \rightarrow \infty} (\Lambda^n \varepsilon)(x) = 0$ for $x \in X$. Further, by (52), for every $n \in \mathbb{N}_0$ and $x \in X$ we have $\Lambda^{n+1} \varepsilon(x) \leq \Lambda^n \varepsilon(x)$, whence

$$\begin{aligned}\sup_{n \in \mathbb{N}_0} (\Lambda^n \varepsilon)(x) &= (\Lambda^0 \varepsilon)(x) = \varepsilon(x), \\ \sup_{n \in \mathbb{N}_0} (\Lambda^{n+1} \varepsilon)(x) &= \Lambda \varepsilon(x).\end{aligned}\quad (53)$$

Consequently, by Theorem 1, there is a function $G_0 : X \rightarrow Y$ such that $d(g_0(x), G_0(x)) \leq \varepsilon(x)$, $G_0(x) = G_0(qx) - G_0(\hat{p}qx)$, and $G_0(x) = \lim_{n \rightarrow \infty} (\mathcal{T}^n g_0)(x)$ for $x \in X$. It is easily seen that

$$G_0(x) = G_0(px) + G_0(\hat{p}x), \quad x \in X. \quad (54)$$

Now, we show that, for every $n \in \mathbb{N}_0$,

$$\begin{aligned}d(\mathcal{T}^n g_0(px + \hat{p}y) + \mathcal{T}^n g_0(\hat{p}x + py), \\ \mathcal{T}^n g_0(x) + \mathcal{T}^n g_0(y)) \\ \leq B^n \theta(x, y), \quad x, y \in X.\end{aligned}\quad (55)$$

Clearly, for $n = 0$, (55) reduces to (8). Next, fix $m \in \mathbb{N}_0$ and assume that (55) holds for every $x, y \in X$ with $n = m$. Then, by (43),

$$\begin{aligned}d(\mathcal{T}^{m+1} g_0(px + \hat{p}y) + \mathcal{T}^{m+1} g_0(\hat{p}x + py), \\ \mathcal{T}^{m+1} g_0(x) + \mathcal{T}^{m+1} g_0(y)) \\ = d(\mathcal{T}^m g_0(q(px + \hat{p}y)) - \mathcal{T}^m g_0(\hat{p}q(px + \hat{p}y)) \\ + \mathcal{T}^m g_0(q(\hat{p}x + py)) - \mathcal{T}^m g_0(\hat{p}q(\hat{p}x + py)),\end{aligned}$$

$$\begin{aligned}\mathcal{T}^m g_0(qx) - \mathcal{T}^m g_0(\hat{p}qx) \\ + \mathcal{T}^m g_0(qy) - \mathcal{T}^m g_0(\hat{p}qy)) \\ \leq \max\{d(\mathcal{T}^m g_0(pqx + \hat{p}qy) + \mathcal{T}^m g_0(\hat{p}qx + pqy), \\ \mathcal{T}^m g_0(qx) + \mathcal{T}^m g_0(qy)), \\ d(\mathcal{T}^m g_0(p\hat{p}qx + \hat{p}\hat{p}qy) \\ + \mathcal{T}^m g_0(\hat{p}\hat{p}qx + p\hat{p}qy), \\ \mathcal{T}^m g_0(\hat{p}qx) + \mathcal{T}^m g_0(\hat{p}qy))\} \\ \leq \max\{B^m \theta(qx, qy), B^m \theta(\hat{p}qx, \hat{p}qy)\} \\ \leq B^{m+1} \theta(x, y), \quad x, y \in X.\end{aligned}\quad (56)$$

Thus we obtain (55) for $n = m + 1$, which completes the induction. Letting $n \rightarrow \infty$ in (55), we get $G_0(px + \hat{p}y) + G_0(\hat{p}x + py) = G_0(x) + G_0(y)$ for $x, y \in X$. Next, writing $G(x) := G_0(x) + g(0)$ for $x \in X$, we obtain that G is a solution to (1).

To complete the proof let us yet mention that we show the uniqueness of G in the same way as in the proof of Theorem 2. $\square$

Now, it is easily seen that Theorem 7 yields the subsequent corollary, which is an analogue of Corollary 4.

Corollary 8. *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in \mathbb{F}$,*

$$\max\{1, |1 - p|\} < |p|, \quad (57)$$

and let $\theta : X^2 \rightarrow [0, \infty)$ be one of the forms (i)–(iii). If $g : X \rightarrow Y$ satisfies inequality (33), then the following two statements are valid.

- (α) *If (i) or (ii) holds, there exists a unique solution $G : X \rightarrow Y$ of (1) such that $d(g(x), G(x)) \leq A(\min\{\|p^{-1}x\|^k, \|p^{-1}x\|^l\})^m$ for $x \in X$.*
- (β) *If (iii) holds, then g is a solution to (1).*

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