Research Article

$\tilde{F}(p, q, s)$ Holomorphic Spaces in the Unit Ball of $\mathbb{C}^n$

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We introduce the $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\phi(p, q, s)$ vector spaces of holomorphic functions defined in the unit ball of $\mathbb{C}^n$, generalizing previous work like Ouyang et al. (1998), Stroethoff (1989), and Choa et al. (1992). Likewise, we characterize those spaces in terms of harmonic majorants as a generalization of Arellano et al. (2000).

1. Introduction

1.1. Preliminaries in One Complex Variable. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$, let $\varphi_a : \mathbb{C} \setminus \{1/\bar{a}\} \rightarrow \mathbb{C}$ be the Möbius transformation defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}. \quad (1)$$

For $z, a \in \mathbb{D}$, we denote Green's function of $\mathbb{D}$ with logarithmic singularity at a by

$$g(z, a) = \log \frac{1}{|\varphi_a(z)|} = \log \frac{|1 - \bar{a}z|}{|a - z|}. \quad (2)$$

The Bloch space $\mathcal{B}$ is defined as the set of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$, such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < +\infty. \quad (3)$$

For $1 \leq p < \infty$, Aulaskari and Lappan [1] introduced in 1994 the $\mathcal{Q}_p$ spaces as the family of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ satisfying

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z, a) dA(z) < +\infty. \quad (4)$$

For $0 < p < 1$, this definition was extended in [2].

Theorem 1 (see [2]). *Let $p \geq 0$ and $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. Then, $f \in \mathcal{Q}_p$ if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dA(z) < +\infty. \quad (5)$$

With the aim of generalizing and enclosing several weighted function spaces, Zhao in [3] introduced the $F(p, q, s)$ spaces in the next way.

Let $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$, and $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. We will say that f belongs to $F(p, q, s)$ if it satisfies the integral condition

$$\{f\}_{p,q,s} := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < +\infty. \quad (6)$$

So far, Theorem 1 is generalized for functions of $F(p, q, s)$ by the following.

Theorem 2 (see [3]). *Let $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$, and $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. Then, $f \in F(p, q, s)$ if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) < +\infty. \quad (7)$$

Zhao has shown that for certain intervals of p , q , and s , $|f(0)| + \{f\}_{p,q,s}$ makes $F(p, q, s)$ as a Banach space.

In this paper, we present the $\tilde{F}(p, q, s)$ spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, that generalize the $F(p, q, s)$ spaces introduced by Zhao in [3] for analytic functions in the unit disk. At the same time, this work generalizes, mainly, several results of Ouyang et al for $\mathcal{Q}_p$ -holomorphic functions appearing in [4]. However, the techniques, the methods, and the structure of our work results are completely different to the quoted reference. At the beginning of Section 2, we introduce the spaces $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$, that is, when we use in the integral representation, the invariant Green function, or when we use the biholomorphism φ . We remark that the generalization to several complex variables of $F(p, q, s)$ requires, for q and s , different intervals to those used in the one-dimensional case. In Theorems 12 and 13, we draw attention to the continuity of the integral expressions defining these spaces to conclude easily the inclusions of the little classes $\tilde{F}_{G,0}(p, q, s)$ and $\tilde{F}_{\varphi,0}(p, q, s)$ in $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$, respectively.

In Section 2.2, we present what is the most natural Bloch space associated to $\tilde{F}_G(p, q, s)$ and several characterizations that we give in Theorem 22. It is important to compare this statement with the results of [4, Proposition 3.6] and [5, Theorem 2.4]. In Section 3, we present in Corollary 30 the equivalence between $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$, generalizing Proposition 3.4 of [4] and Theorem 1 of [3]. Finally in Section 4, we obtain the characterizations of $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$ in terms of harmonic majorants. Giving in this way, the holomorphic version of the results of Arellano et al. in [6].

1.2. Preliminaries in Several Variables. Consider the vector space $\mathbb{C}^n = \{z = (z_1, z_2, \dots, z_n) : z_j \in \mathbb{C}, j = 1, 2, \dots, n\}$ with the Hermitian product defined as

$$\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}, \quad (8)$$

and the Euclidean norm on $\mathbb{C}^n$ defined as

$$|z| = \langle z, z \rangle^{1/2} = \left(\sum_{j=1}^n |z_j|^2 \right)^{1/2}. \quad (9)$$

The open ball with center $a \in \mathbb{C}^n$ and radius $r > 0$ denoted by $B_r(a)$ is defined as

$$B_r(a) = \{z \in \mathbb{C}^n : |z - a| < r\}, \quad (10)$$

and its closure is

$$\overline{B_r(a)} = \{z \in \mathbb{C}^n : |z - a| \leq r\}. \quad (11)$$

We will denote $B_1(0) = B$, $B_r(0) = B_r$, and $A(r) = B \setminus B_r$.

The function f , of class $\mathcal{C}^2(\Omega)$, is harmonic in Ω if

$$\Delta f(x) = 0, \quad (12)$$

for all $x \in \Omega$. Moreover, f results harmonic in Ω if and only if

$$\sum_{j=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0. \quad (13)$$

Let $a \in B \setminus \{0\}$. The Möbius function $\varphi_a(z)$ is defined as

$$\varphi_a(z) = \frac{a - P_a(z) - (1 - |a|^2)^{1/2} Q_a(z)}{1 - \langle z, a \rangle}, \quad (14)$$

with $P_a(z) = \langle z, a \rangle a / |a|^2$ and $Q_a(z) = z - P_a(z)$, for $z \in B$, and define $\varphi_0(z) = -z$ (see [7]).

This set of functions is a part of the family of automorphisms on B denoted by $\text{Aut}(B)$.

Lemma 3 (see [7, 8]). *The function φ_a satisfies the following properties:*

- (i) $\varphi_a(\varphi_a(z)) = z$;
- (ii) $1 - |\varphi_a(z)|^2 = (1 - |a|^2)(1 - |z|^2) / |1 - \langle z, a \rangle|^2$;
- (iii) the real Jacobian for φ_a in $z \in B$ is given by

$$J_{\mathbb{R}} \varphi_a(z) = \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{n+1} = \left(\frac{1 - |\varphi_a(z)|^2}{1 - |z|^2} \right)^{n+1}. \quad (15)$$

Lemma 4. *The real Jacobian of φ_a satisfies*

$$\left(\frac{1 - |a|}{1 + |a|} \right)^{n+1} \leq |J_{\mathbb{R}} \varphi_a(z)| \leq \left(\frac{1 + |a|}{1 - |a|} \right)^{n+1}. \quad (16)$$

Proof. It follows directly from $1 - |a| \leq |1 - \langle z, a \rangle| \leq 1 + |a|$. $\square$

Define, for $a \in B$ and $0 < R < 1$,

$$D(a, R) = \varphi_a(B(R)) = \{z \in B : |\varphi_a(z)| < R\}. \quad (17)$$

1.3. Invariant Gradient and Measure. The collection of all holomorphic functions in B is denoted by $\mathcal{O}$. Let $\nabla f = (\partial f / \partial z_1, \dots, \partial f / \partial z_n)$ denote the complex gradient of f and $Rf = \sum_{j=1}^n z_j (\partial f / \partial z_j)$ denote the radial derivative of f . Let $d\lambda(z) := d\nu(z) / (1 - |z|^2)^{n+1}$, where $d\nu$ is the normalized

volume measure in $\mathbb{C}^n$. Then, $d\lambda$ is a Möbius invariant; that is

$$\int_B f(z) d\lambda(z) = \int_B f \circ \psi(z) d\lambda(z), \quad (18)$$

for each λ -integrable function f and $\psi \in \text{Aut}(B)$.

Let $\tilde{\Delta}f(z) = \Delta(f \circ \varphi_z)(0)$ denote the invariant Laplacian of f [8], and let $\tilde{\nabla}f(z) = \nabla(f \circ \varphi_z)(0)$ denote the invariant gradient of f [7]. It is well known [7] that, for $f \in \mathcal{O}$,

$$|\tilde{\nabla}f(z)|^2 = (1 - |z|^2) (|\nabla f(z)|^2 - |Rf(z)|^2), \quad (19)$$

$$|\tilde{\nabla}(f \circ \varphi)(z)|^2 = |(\tilde{\nabla}f) \circ \varphi(z)|^2, \quad (20)$$

where $\varphi \in \text{Aut}(B)$. In [7], the invariant Green's function is defined as $G(z, a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt. \quad (21)$$

As in [7], the n -dimensional invariant Bloch space $\mathcal{B}$ is the set of holomorphic functions $f : B \rightarrow \mathbb{C}$, such that their Bloch invariant norm $|f|_{\mathcal{B}}$ satisfies

$$|f|_{\mathcal{B}} = \sup_{z \in B} |\tilde{\nabla}f(z)| < +\infty; \quad (22)$$

this is equivalent [4, Lemma 3.2] to, say,

$$\sup_{z \in B} (1 - |z|^2) |\nabla f(z)| < +\infty. \quad (23)$$

Proposition 5 (see [7]). *Let $n \geq 2$ be an integer; then there are positive constants C_1 and C_2 , such that, for all $z \in B \setminus \{0\}$,*

$$C_1(1 - |z|^2)^n |z|^{-2(n-1)} \leq g(z) \leq C_2(1 - |z|^2)^n |z|^{-2(n-1)}. \quad (24)$$

Proposition 6 (see [9]). *For $f \in \mathcal{O}$, one has*

$$\begin{aligned} (1 - |z|^2)^2 |\nabla f(z)|^2 &\leq |\tilde{\nabla}f(z)|^2 \\ &\leq (1 - |z|^2) |\nabla f(z)|^2 \leq |\nabla f(z)|^2. \end{aligned} \quad (25)$$

Theorem 7 (see [7, Theorem 1.12]). *Suppose that c is real and $t > -1$. Then, the integral*

$$J_{c,t}(z) = \int_B \frac{(1 - |w|^2)^t dv(w)}{|1 - \langle z, \zeta \rangle|^{n+1+t+c}}, \quad z \in B \quad (26)$$

has the following asymptotic properties.

(i) *If $c < 0$, $J_{c,t}$ is bounded in B .*

(ii) *If $c = 0$, then*

$$J_{c,t}(z) \sim \ln \frac{1}{1 - |z|^2}, \quad (27)$$

as $|z| \rightarrow 1^-$.

(iii) *If $c > 0$, then*

$$J_{c,t}(z) \sim \frac{1}{(1 - |z|^2)^c}, \quad (28)$$

as $|z| \rightarrow 1^-$.

Let $0 < p < \infty$, $-(p/2) < q$. We say that p , q , and s satisfy condition (A) if

$$\begin{aligned} 1 - \frac{p+2q}{2n} < s < \frac{n}{n-1} &\quad \text{if } 0 < \frac{p+2q}{2n} < 1, \\ 0 < s < \frac{n}{n-1} &\quad \text{if } 1 \leq \frac{p+2q}{2n} < \infty; \end{aligned} \quad (29)$$

and satisfy condition (B) if

$$\begin{aligned} 1 - \frac{p+2q}{2n} < s < \infty &\quad \text{if } 0 < \frac{p+2q}{2n} < 1, \\ 0 < s < \infty &\quad \text{if } 1 \leq \frac{p+2q}{2n} < \infty. \end{aligned} \quad (30)$$

Proposition 8. *Suppose that p , q , and s satisfy condition (A). Then,*

$$\begin{aligned} \sup_{|a| < 1} \int_B (1 - |z|^2)^{(p/2)+q} \frac{(|1 - |\varphi_a(z)|^2|)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) &< \infty, \\ \lim_{|a| \rightarrow 1^-} \int_B (1 - |z|^2)^{(p/2)+q} \frac{(|1 - |\varphi_a(z)|^2|)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) &= 0. \end{aligned} \quad (31)$$

Proof. We have the following equalities:

$$\begin{aligned} &\int_B (1 - |z|^2)^{(p/2)+q} \frac{(|1 - |\varphi_a(z)|^2|)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\ &= \int_B (1 - |\varphi_a(w)|^2)^{(p/2)+q} \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} dv(w) \\ &= \int_B \frac{(1 - |a|^2)^{(p/2)+q} (1 - |w|^2)^{(p/2)+q+ns-n-1}}{|1 - \langle w, a \rangle|^{p+2q} |w|^{2(n-1)s}} dv(w) \\ &= (1 - |a|^2)^{(p/2)+q} \int_B \frac{(1 - |w|^2)^{(p/2)+q+ns-n-1}}{|1 - \langle w, a \rangle|^{p+2q} |w|^{2(n-1)s}} dv(w). \end{aligned} \quad (32)$$

The singularity at 0 is integrable if $s < n/(n-1)$. Now, if we estimate

$$\int_B \frac{(1 - |w|^2)^{(p/2)+q+ns-n-1}}{|1 - \langle w, a \rangle|^{p+2q}} dv(w), \quad (33)$$

as $(p/2) + q + ns - n - 1 > -1$ if $1 - ((p+2q)/2n) < s$ and $0 < (p/2) + q$, $0 < s < \infty$, then the result follows from Theorem 7 for any case of $c = (p/2) + q - ns$. $\square$

In a similar way, we have the following.

Proposition 9. Suppose that p, q , and s satisfy condition (B). Then,

$$\begin{aligned} \sup_{|a| < 1} \int_B (1 - |z|^2)^{(p/2)+q} (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z) &< \infty, \\ \lim_{|a| \rightarrow 1^-} \int_B (1 - |z|^2)^{(p/2)+q} (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z) &= 0. \end{aligned} \quad (34)$$

2. Function Spaces $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$

2.1. Main Definitions. In this section, we define the spaces $\tilde{F}_G(p, q, s)$, $\tilde{F}_\varphi(p, q, s)$ and their little associated spaces. We study some properties and relations between them.

Suppose that p, q , and s satisfy condition (A), and let $f \in \mathcal{O}$. Define

$$\tilde{I}_{p,q,s} f(a) = \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z). \quad (35)$$

Suppose that p, q , and s satisfy condition (B), and let $f \in \mathcal{O}$. Define

$$\tilde{J}_{p,q,s} f(a) = \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z). \quad (36)$$

It is clear that

$$\tilde{J}_{p,q,s} f(a) \leq \tilde{I}_{p,q,s} f(a), \quad \text{for each } a \in B. \quad (37)$$

Associated to these functions, we define the following vector spaces of holomorphic functions.

- (i) Suppose that p, q , and s satisfy condition (A). $f \in \mathcal{O}$ belongs to $\tilde{F}_G(p, q, s)$ if it satisfies

$$\sup_{a \in B} \tilde{I}_{p,q,s} f(a) < +\infty \quad (38)$$

and belongs to the little space $\tilde{F}_{G,0}(p, q, s)$ if it satisfies

$$\lim_{|a| \rightarrow 1^-} \tilde{I}_{p,q,s} f(a) = 0. \quad (39)$$

- (ii) Suppose that p, q , and s satisfy condition (B). $f \in \mathcal{O}$ belongs to $\tilde{F}_\varphi(p, q, s)$ if it satisfies

$$\sup_{a \in B} \tilde{J}_{p,q,s} f(a) < +\infty \quad (40)$$

and belongs to the little space $\tilde{F}_{\varphi,0}(p, q, s)$ if it satisfies

$$\lim_{|a| \rightarrow 1^-} \tilde{J}_{p,q,s} f(a) = 0. \quad (41)$$

- (iii) Let $n < (p + 2q)/2$. $f \in \mathcal{O}$ belongs to the Dirichlet space $\tilde{D}_{p,q}$ if

$$\int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q d\lambda(z) < \infty. \quad (42)$$

As $\tilde{\nabla}(f + h)(z) = \tilde{\nabla} f(z) + \tilde{\nabla} h(z)$, then for $1 \leq p < \infty$, the sets $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$ are normed vector function spaces, with norms given by

$$\begin{aligned} \|f\|_G &:= |f(0)| + \sup_{a \in B} (\tilde{I}_{p,q,s} f(a))^{1/p}, \\ \|f\|_\varphi &:= |f(0)| + \sup_{a \in B} (\tilde{J}_{p,q,s} f(a))^{1/p}. \end{aligned} \quad (43)$$

This is a direct consequence of the usual $L^p(B, d\mu)$ theory when we consider the measures $d\mu(z) = (1 - |z|^2)^q G^s(z, a) d\lambda(z)$, $d\mu(z) = (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z)$, respectively. For $0 < p < 1$ and nonnegative x, y , we have $(x + y)^p \leq x^p + y^p$; as a consequence, the sets $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$ result in metric vector spaces, with the metric given by

$$\begin{aligned} d_G(f, h) &= |f(0) - h(0)| + \sup_{a \in B} \tilde{I}_{p,q,s}(f - h)(a), \\ d_\varphi(f, h) &= |f(0) - h(0)| + \sup_{a \in B} \tilde{J}_{p,q,s}(f - h)(a), \end{aligned} \quad (44)$$

respectively (see [10]). We will prove their completeness in Theorem 26.

Example 10. Let $\tilde{\mathcal{O}}$ denotes the class of holomorphic functions on a neighborhood of $\bar{B}$. Let $f \in \tilde{\mathcal{O}}$. Suppose that p, q , and s satisfy condition (A). By (19) we have, for example, the estimation

$$\begin{aligned} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ \leq \max_{z \in \bar{B}} |\nabla f(z)|^p \int_B (1 - |z|^2)^{(p/2)+q} G^s(z, a) d\lambda(z). \end{aligned} \quad (45)$$

So, in the definitions, the parameters p, q , and s were chosen to guarantee that the vector spaces $F_G(p, q, s)$ and $F_{G,0}(p, q, s)$ include at least this wide class of holomorphic functions, as Propositions 5, 8, and 9 show. Compare this example with Proposition 3.7 and Theorem 3.8 from [4].

Let $f : B \rightarrow \mathbb{C}$, with power series development given by

$$f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}, \quad (46)$$

such that $\sum_{n=0}^{\infty} n \sum_{|\alpha|=n} |a_{\alpha}| < \infty$. Then, f belongs to each one of the previous spaces. By (37), we get the same result for $F_\varphi(p, q, s)$ and $F_{\varphi,0}(p, q, s)$.

The spaces $\tilde{F}_G(p, q, s)$ are Möbius invariant in the following sense.

Theorem 11. Suppose that p, q , and s satisfy condition (A). If $f \in \tilde{F}_G(p, q, s)$ and φ is a Möbius transformation, then $f \circ \varphi \in \tilde{F}_G(p, q, s)$.

Proof. Let $a, b \in B$, $w = \varphi_a(z)$, and $c = \varphi_a(b)$. By the change of variables formula and the Möbius invariance of $G(z, a)$, $\tilde{\nabla}$, and $d\lambda$, we have

$$\begin{aligned}
 & \int_B |\tilde{\nabla}(f \circ \varphi_a)(w)|^p (1 - |w|^2)^q G^s(w, b) d\lambda(w) \\
 &= \int_B |(\tilde{\nabla}f) \circ \varphi_a(w)|^p (1 - |w|^2)^q G^s(w, b) d\lambda(w) \\
 &= \int_B |(\tilde{\nabla}f) \circ \varphi_a(\varphi_a(z))|^p (1 - |\varphi_a(z)|^2)^q \\
 &\quad \times G^s(\varphi_a(z), \varphi_a(c)) d\lambda(z) \\
 &= \int_B |\tilde{\nabla}f(z)|^p (1 - |\varphi_a(z)|^2)^q G^s(z, c) d\lambda(z) \\
 &\leq \left(\frac{1 + |a|}{1 - |a|} \right)^{ql} \int_B |\tilde{\nabla}f(z)|^p (1 - |z|^2)^q G^s(z, c) d\lambda(z), \tag{47}
 \end{aligned}$$

where we have used in the last line the estimations of Lemmas 3 and 4. $\square$

Observe that if $q = 0$ and $f \in \tilde{F}_G(p, 0, s)$, then

$$\begin{aligned}
 & \int_B |\tilde{\nabla}(f \circ \varphi_a)(w)|^p G^s(w, b) d\lambda(w) \\
 &= \int_B |\tilde{\nabla}f(z)|^p G^s(z, c) d\lambda(z), \tag{48}
 \end{aligned}$$

and these spaces are Möbius invariant in the classical way.

Theorem 12. Suppose that p, q , and s satisfy condition (A). Suppose that

$$\tilde{I}_{p,q,s}f(a) < \infty, \tag{49}$$

for all $a \in B$. Then, the function $\tilde{I}_{p,q,s}f$ is continuous on B .

Proof. Let $a \in B$ be fixed and R , such that $|a| < R < 1$. Consider a neighborhood $B_\delta(a)$ with $\delta = (R - |a|)/2$. Let $\{a_m\}_{m \in \mathbb{N}}$ be a sequence contained in $B_\delta(a)$ and convergent to a . Consider now $\tilde{I}_{p,q,s}f(a_m)$ and define $d\mu(z) = |\tilde{\nabla}f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$. Then, $G^s(z, a_m)(1 - |z|^2)^{-ns}$ is bounded in $(B \setminus B_R) \times B_\delta(a)$. Thus, by bounded convergence theorem,

$$\int_{B \setminus B_R} \frac{G^s(z, a_m)}{(1 - |z|^2)^{ns}} d\mu(z) \tag{50}$$

converges to

$$\int_{B \setminus B_R} \frac{G^s(z, a)}{(1 - |z|^2)^{ns}} d\mu(z). \tag{51}$$

Now, integrating in B_R and using change of variables,

$$\begin{aligned}
 & \int_{B_R} |\tilde{\nabla}f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\
 &= \int_{D(a,R)} |\tilde{\nabla}f(\varphi_a(w))|^p (1 - |\varphi_a(w)|^2)^q \\
 &\quad \times g^s(w) d\lambda(w) \tag{52} \\
 &= \int_{D(a,R)} |\tilde{\nabla}f(\varphi_a(w))|^p \frac{(1 - |a|^2)^q}{|1 - \langle w, a \rangle|^{2q}} \\
 &\quad \times (1 - |w|^2)^q g^s(w) d\lambda(w).
 \end{aligned}$$

Consider

$$|\tilde{\nabla}f(\varphi_{a_m}(w))|^p \frac{(1 - |a_m|^2)^q}{|1 - \langle w, a_m \rangle|^{2q}}, \tag{53}$$

and define the measure $d\mu(w)$ by

$$d\mu(w) = (1 - |w|^2)^q g^s(w) d\lambda(w). \tag{54}$$

Now, if we take $|w| < 1$, $|a| < R$, then $1 - R \leq |1 - \langle w, a_m \rangle|$. Besides, on $D(a, R)$, $|\tilde{\nabla}f(\varphi_{a_m}(w))|$ results are bounded by a constant $M > 0$, because $\varphi_{a_m}(w) \in B_R$. Now,

$$\begin{aligned}
 & \int_{D(a_m,R)} |\tilde{\nabla}f(\varphi_{a_m}(w))|^p \frac{(1 - |a_m|^2)^q}{|1 - \langle w, a_m \rangle|^{2q}} d\mu(w) \\
 &= \int_B |\tilde{\nabla}f(\varphi_{a_m}(w))|^p \frac{(1 - |a_m|^2)^q}{|1 - \langle w, a_m \rangle|^{2q}} \\
 &\quad \times \chi_{\varphi_{a_m}(B_R)}(w) d\mu(w), \tag{55}
 \end{aligned}$$

where $\chi_{\varphi_{a_m}(B_R)}$ denotes the characteristic function of the set $D(a_m, R)$. As

$$|\tilde{\nabla}f(\varphi_{a_m}(w))|^p \frac{(1 - |a_m|^2)^q}{|1 - \langle w, a_m \rangle|^{2q}} \chi_{\varphi_{a_m}(B_R)}(w) \leq \frac{M}{|1 - R|^{2q}}, \tag{56}$$

then by Lebesgue's dominated convergence theorem, we have that (55) converges to

$$\int_B |\tilde{\nabla}f(\varphi_a(w))|^p \frac{(1 - |a|^2)^q}{|1 - \langle w, a \rangle|^{2q}} \chi_{\varphi_a(B_R)}(w) d\mu(w). \tag{57}$$

$\square$

Theorem 13. Suppose that p, q , and s satisfy condition (B). Suppose that

$$\tilde{J}_{p,q,s}f(a) < \infty, \tag{58}$$

for all $a \in B$. Then, the function $\tilde{J}_{p,q,s}f$ is continuous on B .

Proof. Let $a \in B$ and $\delta > 0$, such that $\overline{B_\delta(a)} \subset B$. If $K := \overline{B} \times \overline{B_\delta(a)}$, the function $l : K \rightarrow \mathbb{R}$ defined by

$$(\eta, \xi) \mapsto l(\eta, \xi) = \frac{(1 - |\xi|^2)^s}{|1 - \langle \eta, \xi \rangle|^{2s}} \quad (59)$$

results to be uniformly continuous on K . As $f \in \tilde{F}_\varphi(p, q, s)$, then

$$\int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) = \tilde{J}_{p,q,s} f(0) < \infty. \quad (60)$$

If f is nonconstant, given that $\varepsilon > 0$, there exists $\rho > 0$, such that $|\eta' - \eta| < \rho$ and $|\xi' - \xi| < \rho$; then

$$|l(\eta', \xi') - l(\eta, \xi)| < \frac{\varepsilon}{\int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)}. \quad (61)$$

Thus, if $|a - b| < \rho$, then

$$\begin{aligned} & |\tilde{J}_{p,q,s} f(a) - \tilde{J}_{p,q,s} f(b)| \\ & \leq \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} |l(\eta, a) - l(\eta, b)| d\lambda(z) < \varepsilon. \end{aligned} \quad (62)$$

□

Remark 14. Last two theorems are the versions in $\mathbb{C}^n$ of Theorem 3.1 of [6].

Some relations between the little spaces $\tilde{F}_{G_0}(p, q, s)$, $\tilde{F}_{\varphi_0}(p, q, s)$ and the spaces $\tilde{F}_G(p, q, s)$, $\tilde{F}_\varphi(p, q, s)$ are given in the following results.

Proposition 15. Suppose that p, q , and s satisfy condition (A). If $f \in \tilde{F}_{G,0}(p, q, s)$, then the function

$${}_G h(a) = \begin{cases} \tilde{J}_{p,q,s} f(a), & \text{if } |a| < 1, \\ 0, & \text{if } |a| = 1, \end{cases} \quad (63)$$

is continuous on $\overline{B}$.

Proof. By the Theorem 12, it suffices to prove that if $|a_0| = 1$, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|a - a_0| < \delta$, then $|{}_G h(a) - {}_G h(a_0)| < \varepsilon$; that is, $|{}_G h(a)| < \varepsilon$. By hypothesis, $\lim_{|a| \rightarrow 1^-} \tilde{J}_{p,q,s} f(a) = 0$. So, given $\varepsilon > 0$, there exists $\delta > 0$, such that if $||a| - 1| < \delta$, then $|\tilde{J}_{p,q,s} f(a)| < \varepsilon$, or $1 - \delta < |a| < 1$ implies $|{}_G h(a)| < \varepsilon$. Moreover, as $\overline{B}$ is compact, ${}_G h$ results uniformly continuous on $\overline{B}$. □

Proposition 16. Suppose that p, q , and s satisfy condition (A). Then, $\tilde{F}_{G,0}(p, q, s)$ is contained in $\tilde{F}_G(p, q, s)$.

Proof. In fact if $f \in \tilde{F}_{G_0}(p, q, s)$, as the function ${}_G h$ is uniformly continuous on $\overline{B}$, then is bounded and so

$$\max_{a \in B} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) < \infty. \quad (64)$$

□

With the same arguments, we get $\tilde{F}_{\varphi,0}(p, q, s) \subset \tilde{F}_\varphi(p, q, s)$.

2.2. Characterization as Bloch Spaces. Now, we are going to present several characterizations of Bloch spaces similar to the characterizations obtained by Stroethoff [11, Theorems 4 and 7] and Zhao [3, Theorem 1].

For $-(1/2) \leq \alpha$, the α -Bloch spaces, $\mathcal{B}_\alpha$, are the families of holomorphic functions $f : B \rightarrow \mathbb{C}$ satisfying

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z \in B} (1 - |z|^2)^\alpha |\tilde{\nabla} f(z)| < +\infty. \quad (65)$$

It is easy to verify that $\|\cdot\|_{\mathcal{B}_\alpha}$ is a norm on the vector space $\mathcal{B}_\alpha$ and contains at least the holomorphic functions on domains Ω , with $\Omega \supset \overline{B}$. The respective little α -Bloch spaces $\mathcal{B}_{\alpha,0}$ satisfy

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |\tilde{\nabla} f(z)| = 0. \quad (66)$$

It is clear that $\mathcal{B}_{\alpha,0} \subset \mathcal{B}_\alpha$.

Proposition 17. (i) Suppose that p, q , and s satisfy condition (A). Then,

$$\tilde{F}_G(p, q, s) \subset \mathcal{B}_{q/p}, \quad (67)$$

and there exists $C > 0$, such that

$$\|f\|_{\mathcal{B}_{q/p}} \leq C \|f\|_G. \quad (68)$$

(ii) If $n < (p/2) + q$. Then,

$$\tilde{D}_{p,q} \subset \mathcal{B}_{q/p,0}. \quad (69)$$

Proof. (i) Let $f \in \tilde{F}_G(p, q, s)$; then by Propositions 5 and 6 and (20)

$$\begin{aligned} \infty & > \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \geq \int_{D(a,R)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q \frac{C_1^s (1 - |\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\ & \geq \int_{D(a,R)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q \frac{C_1^s (1 - R^2)^{ns}}{R^{2(n-1)s}} d\lambda(z) \\ & \geq \frac{C_1^s (1 - R^2)^{ns}}{R^{2(n-1)s}} \int_{D(a,R)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q d\lambda(z) \\ & \geq \frac{C_1^s (1 - R^2)^{ns}}{R^{2(n-1)s}} \int_{B_R} |\tilde{\nabla} (f \circ \varphi_a)(w)|^p \\ & \quad \times (1 - |\varphi_a(w)|^2)^q d\lambda(w) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{C_1^s (1-R^2)^{ns}}{R^{2(n-1)s}} \int_{B_R} \frac{|\tilde{\nabla}(f \circ \varphi_a)(w)|^p}{(1-|w|^2)^p} \\
&\quad \times \frac{(1-|a|^2)^q (1-|w|^2)^{q-n-1-p}}{|1-\langle w, a \rangle|^{2q}} d\nu(w) \\
&\geq \frac{C_1^s (1-R^2)^{ns}}{R^{2(n-1)s}} C_2(p, q, n) \\
&\quad \times \int_{B_R} \frac{|\tilde{\nabla}(f \circ \varphi_a)(w)|^p}{(1-|w|^2)^p} (1-|a|^2)^q d\nu(w) \\
&= C(p, q, s, n, R) (1-|a|^2)^q \int_{B_R} |\nabla(f \circ \varphi_a)(w)|^p d\nu(w) \\
&\geq C(p, q, s, n, R) (1-|a|^2)^q |\nabla(f \circ \varphi_a)(0)|^p \\
&= C(p, q, s, n, R) (1-|a|^2)^q |\tilde{\nabla}f(a)|^p \\
&= C(p, q, s, n, R) \left((1-|a|^2)^{q/p} |\tilde{\nabla}f(a)| \right)^p, \tag{70}
\end{aligned}$$

where we have used that $|\nabla(f \circ \varphi_a)|^p$ is a subharmonic function; (i) follows from this estimation.

(ii) Let $f \in \tilde{D}_{p,q}$ and $0 < R < 1$ be fixed. By the absolute continuity of the integral and by Lemma 3, given $\varepsilon > 0$, there exists $0 < r < 1$, such that if we choose $|a|$ sufficiently close to 1 then

$$\begin{aligned}
\varepsilon &> \int_{A(r)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q d\lambda(z) \\
&\geq \int_{D(a,R)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q d\lambda(z). \tag{71}
\end{aligned}$$

With the same kind of arguments used in (i), we obtain

$$\varepsilon \geq C(p, q, n, R) \left((1-|a|^2)^{q/p} |\tilde{\nabla}f(a)| \right)^p, \tag{72}$$

for all $a \in B$, sufficiently close to 1, and we conclude the proof. $\square$

The following result says that Dirichlet spaces are the limit case of $F_{G,0}(p, q, s)$ and $F_{\varphi,0}(p, q, s)$, when $s \rightarrow 0^+$.

Theorem 18. (i) Suppose that p, q , and s satisfy condition (A). Then,

$$\tilde{D}_{p,q} \subset \bigcap_{0 < s < 3} \tilde{F}_{g,0}(p, q, s). \tag{73}$$

(ii) Suppose that p, q , and s satisfy condition (B). Then,

$$\tilde{D}_{p,q} \subset \bigcap_{0 < s < \infty} \tilde{F}_{\varphi,0}(p, q, s). \tag{74}$$

Proof. We prove the first inclusion. The proof of the second one is similar. Let $f \in \tilde{D}_{p,q}$ and $\varepsilon > 0$. By Proposition 17, there exists $0 < \rho < 1$, such that

$$\int_{A(\rho)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q d\lambda(z) < \varepsilon, \tag{75}$$

$$(1-|z|^2)^q |\tilde{\nabla}f(z)|^p < \varepsilon, \quad \forall 0 < \rho < |z| < 1. \tag{76}$$

Let $0 < \rho < R < R' < 1$, such that if $R' < |a| < 1$; then $D(a, R) \subset A(R)$. Thus,

$$\begin{aligned}
I_{p,q,s}f(a) &\leq C_2^s \int_{B_R} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\
&\quad + C_2^s \int_{A(R) \setminus D(a,R)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q \\
&\quad \times \frac{(1-|\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\
&\quad + C_2^s \int_{D(a,R)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q \\
&\quad \times \frac{(1-|\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z). \tag{77}
\end{aligned}$$

By Proposition 8, the first integral goes to zero as $|a| \rightarrow 1^-$. For the second one, consider (75) as following:

$$\begin{aligned}
&\int_{A(R) \setminus D(a,R)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\
&\leq \frac{1}{R^{2(n-1)s}} \int_{A(\rho)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q d\lambda(z) < \frac{\varepsilon}{R^{2(n-1)s}}. \tag{78}
\end{aligned}$$

Moreover, for the third, using the change of variable $\varphi_a(w) = z$,

$$\begin{aligned}
&\int_{D(a,R)} |\tilde{\nabla}f(z)|^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\
&< \int_{D(a,R)} \varepsilon \frac{(1-|\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\
&= \varepsilon \int_{B_R} \frac{(1-|w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w). \tag{79}
\end{aligned}$$

and this concludes the proof. $\square$

Proposition 19. Suppose that p, q , and s satisfy condition (A), with $1 < s < n/(n-1)$. Then,

$$\mathcal{B}_{q/p} \subset \tilde{F}_G(p, q, s). \tag{80}$$

Proof. Let $f \in \mathcal{B}_{q/p}$. Then,

$$\sup_{z \in B} (1 - |z|^2)^{q/p} |\tilde{\nabla} f(z)| \leq \|f\|_{\mathcal{B}_{q/p}} < \infty. \quad (81)$$

Using change of variables and by Proposition 5 we have

$$\begin{aligned} & \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \leq \int_B \frac{\|f\|_{\mathcal{B}_{q/p}}^p}{(1 - |z|^2)^q} (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & = \|f\|_{\mathcal{B}_{q/p}}^p \int_B G^s(z, a) d\lambda(z) \\ & \leq \|f\|_{\mathcal{B}_{q/p}}^p \int_B \frac{C_2^s (1 - |\varphi_a(z)|^2)^{ns}}{|\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\ & = \|f\|_{\mathcal{B}_{q/p}}^p C_2^s \int_B \frac{(1 - |w|^2)^{ns-n-1}}{|w|^{2(n-1)s}} d\nu(w), \end{aligned} \quad (82)$$

and this integral results convergent because

$$\int_0^1 (1-r)^{ns-n-1} r^{2(n-1)s-1} dr < \infty, \quad (83)$$

when $1 < s < n/(n-1)$. Finally, $\mathcal{B}_{q/p} \subseteq \tilde{F}_G(p, q, s)$ if $1 < s < n/(n-1)$. $\square$

Similar results to Propositions 17 and 19 can be obtained for $\tilde{F}_\varphi(p, q, s)$.

Proposition 20. Suppose that p, q , and s satisfy condition (B), with $1 < s < \infty$. Then,

$$\tilde{F}_\varphi(p, q, s) \subset \mathcal{B}_{q/p}, \quad (84)$$

and there exists $C > 0$, such that

$$\|f\|_{\mathcal{B}_{q/p}} \leq C \|f\|_\varphi. \quad (85)$$

Proof. Let $f \in \tilde{F}_\varphi(p, q, s)$; then

$$\begin{aligned} \infty & > \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z) \\ & = \int_B |\tilde{\nabla} (f \circ \varphi_a)(w)|^p (1 - |w|^2)^q \\ & \quad \times (1 - |w|^2)^{ns-n-1} d\nu(w) \\ & \geq \int_{B_{1/2}} \frac{|\tilde{\nabla} (f \circ \varphi_a)(w)|^p (1 - |a|^2)^q (1 - |w|^2)^q}{(1 - |w|^2)^p |1 - \langle w, a \rangle|^{2q}} \\ & \quad \times (1 - |w|^2)^{ns-n-1+p} d\nu(w) \end{aligned}$$

$$\begin{aligned} & \geq C(p, q, s, n) \int_{B_{1/2}} \frac{|\tilde{\nabla} (f \circ \varphi_a)(w)|^p}{(1 - |w|^2)^p} (1 - |a|^2)^q d\nu(w) \\ & \geq C(p, q, s, n) (1 - |a|^2)^q \int_{B_{1/2}} |\nabla (f \circ \varphi_a)(w)|^p d\nu(w), \end{aligned} \quad (86)$$

and we finish the proof as in the Proposition 17. $\square$

Proposition 21. Suppose that p, q , and s satisfy condition (A), with $1 < s < \infty$. Then,

$$\mathcal{B}_{q/p} \subset \tilde{F}_\varphi(p, q, s). \quad (87)$$

Proof. Let $f \in \mathcal{B}_{q/p}$. By hypothesis,

$$\sup_{z \in B} (1 - |z|^2)^{q/p} |\tilde{\nabla} f(z)| \leq \|f\|_{\mathcal{B}_\alpha} < \infty. \quad (88)$$

Thus,

$$\begin{aligned} & \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z) \\ & \leq \int_B \frac{\|f\|_{\mathcal{B}_\alpha}^p}{(1 - |z|^2)^q} (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z) \\ & = \|f\|_{\mathcal{B}_\alpha}^p \int_B \frac{(1 - |a|^2)^{ns} (1 - |z|^2)^{ns-n-1}}{|1 - \langle z, a \rangle|^{2ns}} d\nu(z) \\ & = \|f\|_{\mathcal{B}_\alpha}^p (1 - |a|^2)^{ns} \int_B \frac{(1 - |z|^2)^{ns-n-1}}{|1 - \langle z, a \rangle|^{2ns}} d\nu(z), \end{aligned} \quad (89)$$

and by Theorem 7, we obtain that if $s > 1$,

$$\int_B \frac{(1 - |z|^2)^{ns-n-1}}{|1 - \langle z, a \rangle|^{2ns}} d\nu(z) \approx \frac{1}{(1 - |a|^2)^{ns}}, \quad (90)$$

and the result follows. $\square$

We can synthesize the last propositions in the following.

Theorem 22. Suppose that p, q , and s satisfy condition (B). The following statements are equivalent.

- (i) $f \in \mathcal{B}_{q/p}$;
- (ii) $f \in \tilde{F}_G(p, q, s)$ for each $s \in (1, n/(n-1))$;
- (iii) $f \in \tilde{F}_G(p, q, s)$ for some $s \in (1, n/(n-1))$;
- (iv) $f \in \tilde{F}_\varphi(p, q, s)$ for each $s \in (1, \infty)$;
- (v) $f \in \tilde{F}_\varphi(p, q, s)$ for some $s \in (1, \infty)$.

Jointly with these properties, there are similar results for the little spaces $\tilde{F}_{G,0}(p, q, s)$, $\tilde{F}_{\varphi,0}(p, q, s)$, and $\mathcal{B}_{q/p,0}$.

For these cases, the inclusions $\tilde{F}_{G,0}(p, q, s)$, $\tilde{F}_{\varphi,0}(p, q, s) \subseteq \mathcal{B}_{q/p,0}$ follow immediately from the previous estimations. For the other inclusions we have the following.

Proposition 23. Suppose that p, q , and s satisfy condition (A) and $1 < s < n/(n-1)$. Then,

$$\mathcal{B}_{q/p,0} \subset \tilde{F}_{G,0}(p, q, s). \quad (91)$$

Proof. Since $1 < s < n/(n-1)$, then $C = \int_B G^s(z, a) d\lambda(z) < \infty$. Suppose that $f \in \mathcal{B}_{q/p,0}$; then

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{q/p} |\tilde{\nabla} f(z)| = 0. \quad (92)$$

Given $\varepsilon > 0$, there exists $0 < R < 1$, such that

$$|\tilde{\nabla} f(z)|^p < \frac{\varepsilon}{C(1 - |z|^2)^q} \quad (93)$$

if $R < |z| < 1$. We split the integral

$$\int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z), \quad (94)$$

as

$$\begin{aligned} & \int_{B_R} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & + \int_{B \setminus B_R} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z). \end{aligned} \quad (95)$$

By hypothesis, the second integral satisfies

$$\begin{aligned} & \int_{B \setminus B_R} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \leq \frac{\varepsilon}{C} \int_{B \setminus B_R} G^s(z, a) d\lambda(z) < \varepsilon. \end{aligned} \quad (96)$$

Now,

$$\begin{aligned} & \int_{B_R} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \leq C_2^s \int_{B_R} \frac{\|f\|_{\mathcal{B}_\alpha}^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^{ns}}{(1 - |z|^2)^q |\varphi_a(z)|^{2(n-1)s}} d\lambda(z) \\ & \leq C_2^s \|f\|_{\mathcal{B}_\alpha}^p (1 - |a|^2)^{ns} \\ & \quad \times \int_{B_R} \frac{(1 - |z|^2)^{ns-n-1}}{|1 - \langle z, a \rangle|^{2(n-1)s} |\varphi_a(z)|^{2(n-1)s}} d\nu(z), \end{aligned} \quad (97)$$

and the function

$$(a, z) \mapsto \frac{(1 - |z|^2)^{ns-n-1}}{|1 - \langle z, a \rangle|^{2(n-1)s} |\varphi_a(z)|^{2(n-1)s}} \quad (98)$$

attains its maximum on the compact set $\{z \in B : |z| \leq R\} \times \{a \in B : (R+1)/2 \leq |a| \leq 1\}$. Therefore, if $|a| \rightarrow 1^-$, the last estimation tends to be zero. $\square$

The proof of

$$\mathcal{B}_{q/p,0} \subset \tilde{F}_{\varphi,0}(p, q, s) \quad (99)$$

is similar.

Lemma 24. Let $-(1/2) \leq \alpha < \infty$ and $f \in \mathcal{B}_\alpha$. Then, f is a uniformly continuous function from the metric space B equipped with the Bergman metric ρ to the metric space $\mathbb{C}$ equipped with the Euclidean metric.

Proof. Let $\gamma : [0, 1] \rightarrow B$ be a piecewise smooth curve. Thus, the length of γ in the Bergman metric ρ of B is

$$l(\gamma) = \int_0^1 \sqrt{\langle Q(\gamma(t)) \gamma'(t), \gamma'(t) \rangle} dt, \quad (100)$$

where $Q(z)$ is the Bergman matrix of B (see [7]). Let $f : B \rightarrow \mathbb{C}$ be a holomorphic function. Then,

$$|\tilde{\nabla} f(z)| = \sup \left\{ \frac{|\langle \nabla f(z), \bar{w} \rangle|}{\sqrt{\langle Q(z) w, w \rangle}} : w \in \mathbb{C}^n - \{0\} \right\}. \quad (101)$$

Now, if $\gamma : [0, 1] \rightarrow B$ is a geodesic in the Bergman metric, with $\gamma(0) = z_0$, $\gamma(1) = z_1, \dots$, and $f \in \mathcal{B}_\alpha$, we have

$$\begin{aligned} |f(z_1) - f(z_0)| & \leq \int_0^1 |(f \circ \gamma)'(t)| dt \\ & = \int_0^1 |\langle \nabla f(\gamma(t)), \gamma'(t) \rangle| dt \\ & \leq \int_0^1 |\tilde{\nabla} f(\gamma(t))| \sqrt{\langle Q(\gamma(t)) \gamma'(t), \gamma'(t) \rangle} dt \\ & \leq \int_0^1 |\tilde{\nabla} f(\gamma(t))| \frac{(1 - |\gamma(t)|^2)^{q/p}}{(1 - |\gamma(t)|^2)^{q/p}} \\ & \quad \times \sqrt{\langle Q(\gamma(t)) \gamma'(t), \gamma'(t) \rangle} dt \\ & \leq \frac{\|f\|_{\mathcal{B}_\alpha}}{(1 - \sup_{t \in [0,1]} |\gamma(t)|^2)^{q/p}} \\ & \quad \times \int_0^1 \sqrt{\langle Q(\gamma(t)) \gamma'(t), \gamma'(t) \rangle} dt \\ & = \frac{\|f\|_{\mathcal{B}_\alpha}}{(1 - \sup_{t \in [0,1]} |\gamma(t)|^2)^{q/p}} \rho(z_0, z_1), \end{aligned} \quad (102)$$

and the proof is complete. $\square$

Theorem 25. Let $-(1/2) \leq \alpha < \infty$. The spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha,0}$ are complete metric spaces.

Proof. Let $\{f_k\} \subset \mathcal{B}_\alpha$ be a Cauchy sequence. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\|f_l - f_k\|_{\mathcal{B}_\alpha} < \varepsilon$, for all $k > l > N$. In particular, $\{f_k(0)\}$ is convergent. Let $0 < r < 1$. We

apply the estimation of Lemma 24 to $f_l - f_k$, $z_0 = 0$, and $z_1 = z \in B_r$. Thus,

$$|f_l(z) - f_k(z) - f_l(0) + f_k(0)| \leq \frac{\|f_l - f_k\|_{\mathcal{B}_\alpha}}{(1 - r^2)^{q/p}} \log \frac{1 + r}{1 - r}. \quad (103)$$

Then, $\{f_k\}$ converges uniformly on compact subsets of B to some holomorphic function, say f . Therefore, $\partial f_k / \partial z_j$ converges uniformly on compact subsets of B to $\partial f / \partial z_j$, for $j = 1, \dots, n$. So, we can get $|\tilde{\nabla}(f_l - f_k)|^p$ converges to $|\tilde{\nabla}(f - f_k)|^p$ uniformly on compact sets of B when $l \rightarrow \infty$. For $k, l > N$, we have

$$|f_k(0) - f_l(0)| + (1 - |z|^2)^\alpha |\tilde{\nabla}(f_k - f_l)(z)| \leq \|f_k - f_l\|_{\mathcal{B}_\alpha} < \varepsilon, \quad (104)$$

and taking limit when $k \rightarrow \infty$, we get

$$|f(0) - f_l(0)| + (1 - |z|^2)^\alpha |\tilde{\nabla}(f - f_l)(z)| \leq \varepsilon, \quad (105)$$

that is, $f - f_l \in \mathcal{B}_\alpha$ and hence $f = (f - f_l) + f_l \in \mathcal{B}_\alpha$.

Let $\{f_k\} \subset \mathcal{B}_{\alpha,0}$ be a Cauchy sequence. Then, f_k converges to $f \in \mathcal{B}_\alpha$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\|f - f_k\|_{\mathcal{B}_\alpha} < \varepsilon/2$, for all $k > N$. Moreover, there exists $0 < r < 1$, such that

$$(1 - |z|^2)^\alpha |\tilde{\nabla}(f_{N+1})(z)| < \frac{\varepsilon}{2}, \quad \forall r < |z| < 1. \quad (106)$$

Thus,

$$\begin{aligned} & (1 - |z|^2)^\alpha |\tilde{\nabla}(f)(z)| \\ & \leq (1 - |z|^2)^\alpha |\tilde{\nabla}(f - f_{N+1})(z)| \\ & \quad + (1 - |z|^2)^\alpha |\tilde{\nabla}f_{N+1}(z)| < \varepsilon, \end{aligned} \quad (107)$$

for all $r < |z| < 1$; therefore, $\mathcal{B}_{\alpha,0}$ is a complete space. $\square$

Theorem 26. Suppose that p, q , and s satisfy condition (A). The spaces $\tilde{F}_G(p, q, s)$ and $\tilde{F}_{G,0}(p, q, s)$ are complete metric spaces.

Proof. We give the proof only in the case $1 \leq p < \infty$. Let $\{f_k\} \subset \tilde{F}_G(p, q, s)$ be a Cauchy sequence. Since $\{f_k\}$ is too a Cauchy sequence in $\mathcal{B}_{q/p}$, then converges to some $f \in \mathcal{B}_{q/p}$.

We will prove that $f \in \tilde{F}_G(p, q, s)$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$\tilde{I}_{p,q,s}(f_l - f_k)(a) \leq \|f_l - f_k\|_G^p < \varepsilon^p, \quad \forall l > k > N. \quad (108)$$

By Fatou's Lemma, as $l > N$,

$$\begin{aligned} & \tilde{I}_{p,q,s}(f_l - f)(a) \\ & = \int_B |\tilde{\nabla}(f_l - f)(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & = \int_B \lim_{k \rightarrow \infty} |\tilde{\nabla}(f_l - f_k)(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \leq \liminf_{k \rightarrow \infty} \int_B |\tilde{\nabla}(f_l - f_k)(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \leq \lim_{k \rightarrow \infty} \|f_l - f_k\|_G^p \leq \varepsilon^p. \end{aligned} \quad (109)$$

Since $\{f_k(0)\}$ converges to $\{f(0)\}$, if we take supremum on B and p roots in both sides of the last inequality, we obtain

$$\lim_{k \rightarrow \infty} \|f_k - f\|_G = 0. \quad (110)$$

Since $f = (f - f_k) + f_k$, we get that $f \in \tilde{F}_G(p, q, s)$.

If $\{f_k\}$ is a Cauchy sequence in $\tilde{F}_{G,0}(p, q, s)$, then $\{f_k\}$ converges to $f \in \tilde{F}_G(p, q, s)$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that if $k > N$,

$$\begin{aligned} \tilde{I}_{p,q,s}f(a) & \leq 2^p \tilde{I}_{p,q,s}(f - f_k)(a) \\ & \quad + 2^p \tilde{I}_{p,q,s}f_k(a) \leq 2^p \varepsilon^p + 2^p \tilde{I}_{p,q,s}f_k(a). \end{aligned} \quad (111)$$

Thus,

$$\lim_{|a| \rightarrow 1^-} \tilde{I}_{p,q,s}f(a) \leq \lim_{|a| \rightarrow 1^-} 2^p \varepsilon^p + \lim_{|a| \rightarrow 1^-} 2^p \tilde{I}_{p,q,s}f_k(a) = 2^p \varepsilon^p; \quad (112)$$

therefore, $f \in \tilde{F}_{G,0}(p, q, s)$.

In the case $0 < p < 1$, the proof is similar. $\square$

As a consequence of the previous results and Proposition 17, we have the following.

Corollary 27. Suppose that $1 \leq p < \infty$, q , and s satisfy condition (A). The spaces $\tilde{F}_G(p, q, s)$, $\tilde{F}_{G,0}(p, q, s)$, $\mathcal{B}_{q/p}$, and $\mathcal{B}_{q/p,0}$ are Banach spaces, and the inclusion operators $I : \tilde{F}_G(p, q, s) \rightarrow \mathcal{B}_{q/p}$ and $I : \tilde{F}_{G,0}(p, q, s) \rightarrow \mathcal{B}_{q/p,0}$ are bounded.

Similar results are valid for the spaces $\tilde{F}_\varphi(p, q, s)$ and $\tilde{F}_{\varphi,0}(p, q, s)$.

3. The Equality $\tilde{F}_G(p, q, s) = \tilde{F}_\varphi(p, q, s)$

In this section, we prove the equality $\tilde{F}_G(p, q, s) = \tilde{F}_\varphi(p, q, s)$.

Theorem 28. Let $0 < p < \infty$, $q \in \mathbb{R}$, $0 < s < n/(n-1)$, $1/2 < R < 1$, and $f \in \mathcal{O}$. Then,

$$\begin{aligned} & \int_{B_{1/2}} |\nabla f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ & \leq t \int_{B_R} |\nabla f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z), \end{aligned} \quad (113)$$

where $t = t(q, s, n, R)$.

Proof. Let $1/2 < R < 1$. Define

$$0 < \frac{1}{\tau(q, s, n, R)} = \int_{1/2}^R (1 - r^2)^{q+ns-n-1} r^{2n-1} dr. \quad (114)$$

For $u \in S^{2n-1}$, as $|\nabla f|^p$ is subharmonic on B , we have

$$\begin{aligned} & \frac{1}{\tau(q, s, n, R)} \int_{S^{2n-1}} \left| \nabla f \left(\frac{u}{2} \right) \right|^p d\sigma \\ &= \int_{1/2}^R (1 - r^2)^{q+ns-n-1} r^{2n-1} dr \\ & \quad \cdot \int_{S^{2n-1}} \left| \nabla f \left(\frac{u}{2} \right) \right|^p d\sigma \\ &\leq \int_{1/2}^R (1 - r^2)^{q+ns-n-1} r^{2n-1} dr \\ & \quad \cdot \int_{S^{2n-1}} |\nabla f(ru)|^p d\sigma \\ &= \int_{B_R \setminus B_{1/2}} |\nabla f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) \\ &\leq \int_{B_R} |\nabla f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z). \end{aligned} \quad (115)$$

Thus,

$$\begin{aligned} & \int_{S^{2n-1}} \left| \nabla f \left(\frac{u}{2} \right) \right|^p d\sigma \\ &\leq \tau(q, s, n, R) \int_{B_R} |\nabla f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z). \end{aligned} \quad (116)$$

By Proposition 5,

$$\begin{aligned} & \int_0^{1/2} (1 - r^2)^{q-n-1} r^{2n-1} g^s(ru) dr \\ &\leq C_2^s \int_0^{1/2} (1 - r^2)^{q-n-1} r^{2n-1} \frac{(1 - r^2)^{ns}}{r^{2(n-1)s}} dr \\ &= C_2^s \int_0^{1/2} (1 - r^2)^{q+ns-n-1} r^{2(n-ns+s)-1} dr. \end{aligned} \quad (117)$$

This integral results convergent because $0 < s < n/(n-1)$. Denote the last integral by $\tau'(q, s, n)$. Then,

$$\begin{aligned} & \int_{B_{1/2}} |\nabla f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ &\leq C_2^s \int_0^{1/2} \int_{S^{2n-1}} \left| \nabla f \left(\frac{u}{2} \right) \right|^p \\ & \quad \times (1 - r^2)^{q+ns-n-1} r^{2(n-ns+s)-1} d\sigma dr \\ &\leq C_2^s \tau'(q, s, n) \tau(q, s, n, R) \\ & \quad \times \int_{B_R} |\nabla f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z), \end{aligned} \quad (118)$$

and the result follows. $\square$

Theorem 29. Suppose that p, q , and s satisfy condition (A). If $f \in \tilde{F}_G(p, q, s)$, then

$$\begin{aligned} & \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ &\leq t \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z), \end{aligned} \quad (119)$$

where $t = t(p, q, s, n, R)$ for some $1/2 < R < 1$.

Proof. From Proposition 6 and (113),

$$\begin{aligned} & \int_{B_{1/2}} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ &\leq \int_{B_{1/2}} |\nabla f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ &\leq C_2^s \tau(q, s, n) \tau(q, s, n, R) \\ & \quad \times \int_{B_R} |\nabla f(z)|^p \frac{(1 - |z|^2)^p}{(1 - |z|^2)^p} (1 - |z|^2)^{q+ns} d\lambda(z) \\ &\leq \frac{C_2^s \tau(q, s, n) \tau(q, s, n, R)}{(1 - R^2)^p} \\ & \quad \times \int_{B_R} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z). \end{aligned} \quad (120)$$

Now, if $1/2 \leq |z| < 1$,

$$\begin{aligned} & \int_{B \setminus B_{1/2}} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ &\leq C_2^s \int_{B \setminus B_{1/2}} |\tilde{\nabla} f(z)|^p \frac{(1 - |z|^2)^{q+ns}}{|z|^{2(n-1)s}} d\lambda(z) \\ &\leq C_2^s 2^{2(n-1)s} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z). \end{aligned} \quad (121)$$

Finally,

$$\begin{aligned} & \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ &\leq t \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z), \end{aligned} \quad (122)$$

with $t := t(p, q, s, n, R) := C_2^s (\tau'(q, s, n) \tau(q, s, n, R) + 2^{2(n-1)s}) / (1 - R^2)^p$. $\square$

By the Möbius invariance of the space $\tilde{F}_G(p, q, s)$, if we apply the last result to $f \circ \phi_a$, we obtain the following.

Corollary 30. Suppose that p, q , and s satisfy condition (A) and $f \in \tilde{F}_G(p, q, s)$. Then, for $1/2 < R < 1$,

$$\begin{aligned} & \int_B |\tilde{\nabla}(f \circ \varphi_a)(z)|^p (1 - |z|^2)^q g^s(z) d\lambda(z) \\ & \leq t(p, q, s, n, R) \int_B |\tilde{\nabla}(f \circ \varphi_a)(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z). \end{aligned} \quad (123)$$

Theorem 31. Suppose that p, q , and s satisfy condition (A) and $f \in \tilde{F}_G(p, q, s)$. Then,

$$\sup_{a \in B} \int_B |\tilde{\nabla} f(w)|^p (1 - |w|^2)^q G^s(w, a) d\lambda(w) < \infty, \quad (124)$$

if and only if

$$\sup_{a \in B} \int_B |\tilde{\nabla} f(w)|^p (1 - |w|^2)^q (1 - |\varphi_a(w)|^2)^{ns} d\lambda(w) < \infty. \quad (125)$$

Proof. We will abbreviate $t = t(p, q, s, n, R)$, then from the last corollary and (20)

$$\begin{aligned} 0 & \leq \int_B |\tilde{\nabla}(f \circ \varphi_a)(z)|^p (1 - |z|^2)^q \\ & \quad \times [t(1 - |z|^2)^{ns} - g^s(z)] d\lambda(z) \\ & = \int_B |(\tilde{\nabla} f)(\varphi_a(z))|^p (1 - |z|^2)^q \\ & \quad \times [t(1 - |z|^2)^{ns} - g^s(z)] d\lambda(z); \end{aligned} \quad (126)$$

using the change of variables $z = \varphi_a(w)$, we obtain

$$\begin{aligned} 0 & \leq \int_B |\tilde{\nabla} f(w)|^p (1 - |\varphi_a(w)|^2)^q \\ & \quad \times [t(1 - |\varphi_a(w)|^2)^{ns} - G^s(w, a)] d\lambda(w). \end{aligned} \quad (127)$$

Then,

$$\begin{aligned} 0 & \leq \int_B |\tilde{\nabla} f(w)|^p \frac{(1 - |a|^2)^q (1 - |w|^2)^q}{|1 - \langle w, a \rangle|^{2q}} \\ & \quad \times [t(1 - |\varphi_a(w)|^2)^{ns} - G^s(w, a)] d\lambda(w) \\ & \leq \frac{(1 + |a|)^q}{(1 - |a|)^q} \int_B |\tilde{\nabla} f(w)|^p (1 - |w|^2)^q \\ & \quad \times [t(1 - |\varphi_a(w)|^2)^{ns} - G^s(w, a)] d\lambda(w), \end{aligned} \quad (128)$$

and from here we obtain the result. $\square$

Corollary 32. Suppose that p, q , and s satisfy condition (A). Then,

$$\tilde{F}_G(p, q, s) = \tilde{F}_\varphi(p, q, s), \quad \tilde{F}_{G,0}(p, q, s) = \tilde{F}_{\varphi,0}(p, q, s). \quad (129)$$

4. Harmonic Majorants in

$\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$

For $z \in B$ and $u \in S^{2n-1}$, the function P defined by

$$P(z, u) = \frac{1 - |z|^2}{|z - u|^{2n}} \quad (130)$$

is the real Poisson kernel for the ball $B \subset \mathbb{R}^{2n}$. If $h(u)$ is a real function, integrable on S^{2n-1} , we define the Poisson integral formula of h as

$$\begin{aligned} H(z) &= \frac{1}{\sigma(S^{2n-1})} \int_{S^{2n-1}} h(u) P(z, u) d\sigma(u) \\ &= \frac{1}{\sigma(S^{2n-1})} \int_{S^{2n-1}} h(u) \frac{1 - |z|^2}{|z - u|^{2n}} d\sigma(u), \end{aligned} \quad (131)$$

where $\sigma(S^{2n-1})$ is the area measure of S^{2n-1} , for $n \geq 2$. This formula gives the classic solution to the Dirichlet problem for h . Moreover, if $0 \leq H < \infty$, then H is harmonic on B and has nontangential limit $h(u)$ a. e. on S^{2n-1} (see [12]).

Theorem 33 ((Harnack) see [13]). Consider B . Let $\{H_m\}$ be a monotone sequence of real harmonic functions on B . Suppose that $\{H_m(x_0)\}$ is bounded for some $x_0 \in B$. Then, the sequence $\{H_m\}$ converges uniformly on compact sets to a harmonic function on B .

Proposition 34. Let $h : B \rightarrow \mathbb{R}^+ \cup \{0\}$ be a continuous function on B . For $0 < R < 1$, let the function $h_R : S^{2n-1} \rightarrow \mathbb{R}$ be defined by

$$h_R(u) = \max_{0 \leq r \leq R} h(ru). \quad (132)$$

Then, h_R results in a uniformly continuous function on S^{2n-1} . Besides, if $0 < R_1 < R_2 < 1$, then

$$h_{R_1}(u) \leq h_{R_2}(u), \quad (133)$$

for every $u \in S^{2n-1}$.

Proof. The function $h|_{\overline{B}_r}$ is uniformly continuous; that is, given that $\varepsilon > 0$, there exists $\delta > 0$, such that $|h(z) - h(z')| < \varepsilon$, for all $z, z' \in \overline{B}_r$, with $|z - z'| < \delta$. Let $u, v \in S^{2n-1}$. There exist $r_u, r_v \in [0, R]$, such that $h_R(u) = h(r_u u)$ and $h_R(v) = h(r_v v)$. Thus, if $|u - v| < \delta$, then $|h(r_u u) - h(r_u v)| < \varepsilon$ and $|h(r_v v) - h(r_v u)| < \varepsilon$. In particular, $h_R(u) < \varepsilon + h(r_u v) \leq \varepsilon + h_R(v)$ and $h_R(v) < \varepsilon + h(r_v u) \leq \varepsilon + h_R(u)$. Therefore, if $|u - v| < \delta$, then $|h_R(u) - h_R(v)| < \varepsilon$.

It follows directly that if $0 < R_1 < R_2$, then

$$\max_{0 \leq r \leq R_1} h(ru) \leq \max_{0 \leq r \leq R_2} h(ru), \quad (134)$$

so the result follows. $\square$

Observe that Poisson's integral formula defines a continuous harmonic function H_R on B , with boundary values given by h_R .

Corollary 35. If h is as in the previous proposition and for every $u \in S^{2n-1}$, $\sup_{0 < R < 1} h_R(u) < \infty$, define $h_1 : S^{2n-1} \rightarrow \mathbb{R}$ as

$$h_1(u) = \sup_{0 < R < 1} h_R(u). \quad (135)$$

Then, one has besides that

$$h_1(u) = \lim_{R \rightarrow 1^-} h_R(u). \quad (136)$$

Theorem 36. If $h : B \rightarrow [0, \infty)$ is a continuous function and $h_1 : S^{2n-1} \rightarrow \mathbb{R}$ is the function defined in the previous corollary, let $H_1 : B \rightarrow \mathbb{R}$ be defined by

$$H_1(z) = \sup_{0 \leq R < 1} H_R(z), \quad (137)$$

for every $z \in B$. Then, H_1 results in a harmonic function, or $H_1(z) \equiv \infty$ on B . In the first case, H_1 can be obtained through Poisson's integral formula applied to h_1 . Besides,

$$\lim_{R \rightarrow 1^-} H_R(u) = h_1(u). \quad (138)$$

If $h_1(u) \geq 0$, then the nontangential limits

$$\lim_{z \rightarrow u} H_1(z) = h_1(u) \quad (139)$$

exist almost everywhere on S^{2n-1} .

Proof. Let $\{R_m\}$ be an increasing sequence of positive numbers in $(0, 1)$, such that $\lim_{m \rightarrow \infty} R_m = 1$. For every $m \in \mathbb{N}$, we define the harmonic continuous function

$$H_{R_m}(z) = \frac{1}{\sigma(S^{2n-1})} \int_{S^{2n-1}} h_{R_m}(u) \frac{1 - |z|^2}{|z - u|^{2n}} d\sigma(u). \quad (140)$$

Then, if $R_1 \leq R_2$, it follows that

$$H_{R_1}(z) \leq H_{R_2}(z). \quad (141)$$

Then, the sequence $\{H_{R_m}\}$ is increasing, so

$$\lim_{m \rightarrow \infty} H_{R_m}(z) = H_1(z), \quad (142)$$

for every $z \in B$.

By Harnack's theorem, $H_1 \equiv \infty$ or H_1 is a harmonic function on B , and the limit does not depend on the sequence. As

$$H_R(u) = h_R(u) \leq h_1(u), \quad (143)$$

it follows that $H_R(u) \rightarrow h_1(u)$ if $R \rightarrow 1^-$ and $\lim_{z \rightarrow u} H_1(u)$ there exists almost everywhere.

For every $z \in B$ fixed, let

$$\frac{1}{\sigma(S^{2n-1})} h_{R_m}(u) \frac{1 - |z|^2}{|z - u|^{2n}}, \quad (144)$$

this is a sequence of positive Lebesgue's integrable functions, such that for $R_{m_1} < R_{m_2}$,

$$\begin{aligned} 0 &\leq \frac{1}{\sigma(S^{2n-1})} h_{R_{m_1}}(u) \frac{1 - |z|^2}{|z - u|^{2n}} \\ &\leq \frac{1}{\sigma(S^{2n-1})} h_{R_{m_2}}(u) \frac{1 - |z|^2}{|z - u|^{2n}}. \end{aligned} \quad (145)$$

Then, by Lebesgue's monotone convergence theorem,

$$\begin{aligned} 0 &\leq \frac{1}{\sigma(S^{2n-1})} \int_{S^{2n-1}} h_{R_{m_1}}(u) \frac{1 - |z|^2}{|z - u|^{2n}} d\sigma(u) \\ &\rightarrow \frac{1}{\sigma(S^{2n-1})} \int_{S^{2n-1}} h_1(u) \frac{1 - |z|^2}{|z - u|^{2n}} d\sigma(u), \end{aligned} \quad (146)$$

as $m_1 \rightarrow \infty$. $\square$

Let $\Omega \subset \mathbb{C}^n$ be an open set and let $G \neq -\infty$. We say that a harmonic function $H : \Omega \rightarrow \mathbb{R}$ is the least harmonic majorant of G if

- (a) $G(z) \leq H(z)$, for every $z \in \Omega$;
- (b) I is any harmonic function on Ω , such that $G(z) \leq I(z)$, for every $z \in \Omega$, then $H(z) \leq I(z)$.

The least harmonic majorant of a continuous function, bounded and radially increasing is not necessarily the Poisson extension of its boundary values; see, for instance, Example 2.2 of [14].

In a similar way as [6, 14], consider a function $f \in \tilde{F}_G(p, q, s)$. For $0 < R < 1$, let us introduce the family of functions ${}_G h_R : S^{2n-1} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} {}_G h_R(u) &:= \max_{0 \leq r \leq R} {}_G h(ru) \\ &= \max_{a=ru, 0 \leq r \leq R} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z). \end{aligned} \quad (147)$$

By Theorem 12 and Proposition 34, this function is well defined for every $0 < R < 1$ and results in a continuous function on S^{2n-1} . Moreover, as $f \in \tilde{F}_G(p, q, s)$, for $0 < R_1 < R_2 < 1$,

$${}_G h_{R_1}(u) \leq {}_G h_{R_2}(u) \leq {}_G h_1(u) \leq \sup_{u \in S^{2n-1}} {}_G h_1(u) = \{f\}_{p,q,s}. \quad (148)$$

Let ${}_G H_R$ be the harmonic function associated with ${}_G h_R$, that is, the solution of Dirichlet's problem, with boundary values given by ${}_G h_R$. It follows from Proposition 34 that

$$\sup_{z \in B} {}_G H_R(z) \leq \sup_{z \in B} {}_G H_1(z) = \{f\}_{p,q,s}. \quad (149)$$

In the same way as we did for ${}_G h_R$, we introduce the family of functions ${}_\varphi h_R : S^{2n-1} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} {}_\varphi h_R(u) &:= \max_{0 \leq r \leq R} {}_\varphi h(ru) \\ &= \max_{a=ru, 0 \leq r \leq R} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q \\ &\quad \times (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z). \end{aligned} \quad (150)$$

In this case we obtain Theorem 13 and Proposition 34.

Consider

$$\sup_{z \in B} {}_\varphi H_R(z) \leq \sup_{z \in B} {}_\varphi H_1(z) = \sup_{u \in S^{2n-1}} {}_\varphi h_1(u) = [f]_{p,q,s}. \quad (151)$$

So, we can summarize the previous results in the next theorems.

Theorem 37. Suppose that p , q , and s satisfy condition (A). Then, f belongs to the class $\tilde{F}_G(p, q, s)$ if and only if the function ${}_G h_1 : S^{2n-1} \rightarrow \mathbb{R}$ defined by

$${}_G h_1(u) = \sup_{a=ru, 0 \leq r < 1} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \quad (152)$$

is bounded. The function ${}_G H_1$ is then the Dirichlet solution, with boundary values given by ${}_G h_1$. Further, ${}_G H_1$ is the least harmonic majorant of the family of harmonic functions $\{{}_G H_R\}$, $0 < R < 1$.

Analogous result is obtained for $\{{}_\varphi H_R\}$ and ${}_\varphi H_1$.

Now, we can formulate an analogous statement to the Theorem 3.1 in [14] to characterize functions of $\tilde{F}_G(p, q, s)$ and $\tilde{F}_\varphi(p, q, s)$ through their harmonic majorant, as in Theorem 3.9 of [6].

Theorem 38. Suppose that p , q , and s satisfy condition (A) and $f \in \tilde{F}_G(p, q, s)$. Then, the following conditions are equivalent:

- (i) $f \in \tilde{F}_\varphi(p, q, s)$;
- (ii) ${}_G h_1$ is bounded and ${}_G H_1$ is the classical Dirichlet solution with boundary values given by ${}_G h_1$;
- (iii) ${}_\varphi h_1$ is bounded and ${}_\varphi H_1$ is the classical Dirichlet solution with boundary values given by ${}_\varphi h_1$.

With this development, we have obtained a new version of Theorem 1.3 in [6], in terms of harmonic majorants ${}_G H_1$ and ${}_\varphi H_1$ and their corresponding boundary values ${}_G h_1$ and ${}_\varphi h_1$, for the invariant measure.

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