Research Article

Toeplitz Operators on Weighted Bergman Spaces

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We characterize the boundedness and compactness of a Toeplitz-type operator on weighted Bergman spaces satisfying the Bekollé-Bonami condition in terms of the Berezin transform.

1. Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane, and let dA denote the normalized Lebesgue measure on $\mathbb{D}$. For a given nonnegative integrable function u on $\mathbb{D}$, we define the *weighted Bergman space* $A^2(u)$ as the space of all analytic functions on $\mathbb{D}$ that belong to the weighted space $L^2(u)$. That is, $f \in A^2(u)$ if f is analytic on $\mathbb{D}$ and satisfies

$$\|f\|_{A^2(u)}^2 := \int_{\mathbb{D}} |f(z)|^2 u(z) dA(z). \quad (1)$$

Weighted Bergman spaces have been studied by several authors in different contexts (see, e.g., [1–5]). Most of the researches about Toeplitz-type operators in this spaces have been done by considering radial weights. In this paper we will consider weights satisfying the the so-called Békollé-Bonami condition.

Definition 1. A function u in $L^1(\mathbb{D})$ is said to satisfy the Bekolle-Bonami condition B_2 if there exists a constant $C > 0$ such that

$$\frac{1}{(A(S(I)))^2} \int_{S(I)} u(z) dA(z) \int_{S(I)} \frac{1}{u(z)} dA(z) \leq C \quad (2)$$

for every interval $I \subset \partial\mathbb{D}$. Here, $S(I)$ denotes the *Carleson square*:

$$S(I) = \left\{ re^{it} : e^{it} \in I, 1 - \frac{|I|}{2\pi} \leq r < 1 \right\}. \quad (3)$$

Condition B_2 was introduced by Bekolle and Bonami in [6]. They showed that B_2 is necessary and sufficient

for the Bergman projection to be bounded on $L^2(u dA)$. Condition B_2 was used by Luecking in [7, 8] to study Carleson measures in weighted Bergman spaces. Then in [9], Nakazi and Yamada generalized Luecking's results by introducing a more restrictive condition $(A_2)_\partial$ and found, under this condition, a characterization for Carleson measures on $A^2(u)$ spaces. They also gave several examples of weights satisfying this condition. In [10], Constantin generalized Luecking's result and studied Toeplitz-type operators on spaces $A^2(u)$. The characterization of Carleson measures is as follows.

Theorem 2 (see [7, 10]). *Suppose that $u > 0$ a.e. in $\mathbb{D}$ and satisfies the B_2 condition; let $d\mu = u dA$ and let ν be a positive Borel measure on $\mathbb{D}$. Then, the following are equivalent.*

- (1) *There exist a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f|^2 d\nu \leq C \int_{\mathbb{D}} |f|^2 d\mu, \quad (4)$$

for every polynomial f , that is, ν is a μ -Carleson measure.

- (2) *There exists $r > 0$ and $\gamma > 0$ satisfying*

$$\nu(D_r(a)) \leq \gamma \mu(D_r(a)), \quad (5)$$

for all $a \in \mathbb{D}$.

Theorem 3 (see [8]). *Suppose that $u \in L^1(dA)$ satisfies condition B_2 ; then, the dual of $A^2(u)$ can be identified with $A^2(u^{-1})$. The pairing is given by*

$$\langle f, g \rangle = \int_{\mathbb{D}} f \bar{g} dA. \quad (6)$$

Remark 4. Notice that if u satisfies the B_2 condition, then it is not hard to show (see, e.g., [8]) that there is a constant $C > 0$ such that for any function f analytic in $\mathbb{D}$ and any $a \in \mathbb{D}$; the following holds:

$$|f(a)|^2 \leq C \left(\int_{D_r(a)} u dA \right)^{-1} \|f\|_{A^2(u)}^2. \quad (7)$$

Consequently, evaluation functionals are bounded on $A^2(u)$, and so for every complex number $a \in \mathbb{D}$, there exists a function $K_u(\cdot, a) \in A^2(u)$ such that

$$\langle f, K_u(\cdot, a) \rangle_{A^2(u)} = f(a), \quad (8)$$

for every function $f \in A^2(u)$; that is, the functions $K_u(\cdot, a)$ are the reproducing kernels for $A^2(u)$. Notice also that the set $\{K_u(\cdot, w) : w \in \mathbb{D}\}$ is dense in $A^2(u)$.

Moreover, using Theorem 3, we have that there exists a bounded, bijective, linear operator $F_1 : A^2(u^{-1}) \rightarrow (A^2(u))^*$ such that $F_1(f)(g) = \langle g, \bar{f} \rangle_{A^2}$ for every $f \in A^2(u^{-1})$ and $g \in A^2(u)$. On the other hand, since $A^2(u)$ is a Hilbert space, there exists a bounded, bijective, linear operator $F_2 : (A^2(u))^* \rightarrow A^2(u)$ such that $\langle g, \overline{F_2(\gamma)} \rangle_{A^2(u)} = \gamma(g)$ for every $\gamma \in (A^2(u))^*$ and $g \in A^2(u)$.

Let $\gamma_a \in (A^2(u))^*$ denote the evaluation functional at a ; then, $F_2(\gamma_a) = K_u(\cdot, a)$. On the other hand, if we let $B_a(z)$, with $a, z \in \mathbb{D}$, denote the Bergman kernel

$$B_a(z) := \frac{1}{(1 - \bar{a}z)^2}, \quad (9)$$

then $B_a \in A^2(u^{-1})$, and the pairing is given by the A^2 product; we have that $F_1(B_a) = \gamma_a$.

Define $T : A^2(u) \rightarrow A^2(u^{-1})$ by $T = F_1^{-1}F_2^{-1}$. Then T is a bounded operator and $T(K_u(\cdot, a)) = F_1^{-1}(\gamma_a) = B_a$. Also, the operator $T^{-1} : A^2(u^{-1}) \rightarrow A^2(u)$ is well defined, and consequently we have that

$$\|B_a\|_{A^2(u^{-1})} \sim \|K_u(\cdot, a)\|_{A^2(u)}. \quad (10)$$

Definition 5. Let ν be a positive Borel measure on $\mathbb{D}$. Define for each polynomial p the Toeplitz operator T_ν

$$T_\nu p(z) := \int_{\mathbb{D}} K_u(z, w) p(w) d\nu(w). \quad (11)$$

We will study the conditions under which Toeplitz operators can be extended to $A^2(u)$, and we will characterize the boundedness and compactness of Toeplitz operators acting on the $A^2(u)$ spaces in terms of the so-called Berezin transform of the measure ν .

Definition 6. Let ν be a positive Borel measure on $\mathbb{D}$; the Berezin transform $\tilde{\nu}$ of ν is defined as

$$\tilde{\nu}(a) := \frac{1}{\|B_a\|_{A^2(u)}} \int_{\mathbb{D}} |B_a(z)|^2 d\nu(z). \quad (12)$$

From now on, we will assume that ν is a probability measure.

The following result is well known (see [1–5]).

Theorem 7. Suppose that μ is a finite positive Borel measure on $\mathbb{D}$. Then, the following are equivalent (here, $dA_\alpha(z)$ represents the standard weighted Lebesgue measure $(1 - |z|^2)^\alpha dA(z)$, $\alpha > -1$).

- (a) T_μ is bounded on $A^2(dA_\alpha)$.
- (b) $\tilde{\mu}$ is a bounded function on $\mathbb{D}$.
- (c) μ is a Carleson measure for $A^2(dA_\alpha)$,

where

$$\tilde{\mu}(z) = \int_{\mathbb{D}} |b_z(w)|^2 d\mu(w), \quad z \in \mathbb{D} \quad (13)$$

is the Berezin transform of the Toeplitz operator, here

$$b_z(w) := \frac{1 - |z|^2}{(1 - \bar{z}w)^2}. \quad (14)$$

The proof of this result mainly relies on the behavior of the reproducing kernels $b_z(w)$ when z is close to w and on the characterization of Carleson measures for the Bergman space. In the case of $A^2(u)$ spaces, such a characterization is known for Carleson measures (see, e.g., [10]), but we do not have an explicit formula for the reproducing kernels; consequently, we will need to use a different technique. This will be an atomic decomposition developed in terms of the reproducing kernels of $A^2(u)$.

2. Atomic Decomposition for $A^2(u)$

In this section, we develop a way of expressing functions in $A^2(u)$ as a linear combination of reproducing kernels. Similar problems have been studied in [8, 11]. We will use a similar reasoning as in [8], where the problem is studied in terms of the Bergman kernel. We will assume in the rest of this paper that $\mu := u dA$ is a probability measure.

Definition 8. Let $\varepsilon > 0$. A sequence $\{a_n\} \subset \mathbb{D}$ is called ε -separated if $\inf\{\rho(a_n, a_m) : n \neq m\} \geq \varepsilon > 0$. Here, ρ denotes the pseudohyperbolic metric: $\rho(z, w) := |z - w|/|1 - \bar{z}w|$.

Theorem 9 (see [8]). If u satisfies condition B_2 , then there exists an ε -separated sequence $\{a_n\}$ such that

$$\|f\|_{A^2(u)}^2 \sim \sum_k |f(a_n)|^2 \mu(D_r(a_n)), \quad (15)$$

for every function $f \in A^2(u)$.

Theorem 10. Suppose that u satisfies the condition B_2 . Then, there exists a sequence $\{a_n\} \subset \mathbb{D}$ which is ε -separated for some $\varepsilon > 0$, such that any $f \in A^2(u)$ has the form

$$f(z) = \sum_n c_n K_u(\cdot, a_n) \mu(D_r(a_n))^{1/2}, \quad (16)$$

for some sequence $\{c_n\} \in \ell^2$.

Proof. By Theorem 9, we know that there exists an ε -separated sequence $\{a_n\}$ such that $\|f\|_{A^2(u)}^2 \sim \sum_k |f(a_n)|^2 \mu(D_r(a_n))$. Consider the linear operator $R : A^2(u) \rightarrow l^2$ defined by

$$Rf := \left\{ f(a_n) \mu(D_r(a_n))^{1/2} \right\}. \quad (17)$$

Then, $\|Rf\|_{l^2}^2 = \sum_n |f(a_n)|^2 \mu(D_r(a_n)) \sim \|f\|_{A^2(u)}^2$, and so R is a bounded injective linear operator having closed range. Consequently, the dual mapping $R^* : l^2 \rightarrow A^2(u)$ is onto. Now, let $\{c_n\} \in l^2$; then, for $f \in A^2(u)$, we have

$$\begin{aligned} \langle R^* \{c_n\}, f \rangle_{A^2(u)} &= \langle \{c_n\}, Rf \rangle_{l^2} \\ &= \sum_n c_n \overline{f(a_n)} \mu(D_r(a_n))^{1/2} \\ &= \sum_n c_n \int_{\mathbb{D}} \overline{f(z)} K_u(z, a_n) d\mu(z) \mu(D_r(a_n))^{1/2} \\ &= \int_{\mathbb{D}} \overline{f(z)} \sum_n c_n K_u(z, a_n) \mu(D_r(a_n))^{1/2} d\mu(z) \\ &= \left\langle \sum_n c_n K_u(\cdot, a_n) \mu(D_r(a_n))^{1/2}, f \right\rangle_{A^2(u)}. \end{aligned} \quad (18)$$

Here, we justify the interchange of integration and summation as follows. Given a sequence $\{c_n\} \in l^2$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, l \geq N$, then

$$\sum_{n=m}^l |c_n|^2 < \frac{\varepsilon}{\|R^*\|}. \quad (19)$$

Thus,

$$\begin{aligned} &\sum_{n=m}^l c_n \int_{\mathbb{D}} \overline{f(z)} K_u(z, a_n) d\mu(z) \mu(D_r(a_n))^{1/2} \\ &= \int_{\mathbb{D}} \overline{f(z)} \sum_{n=m}^l c_n K_u(z, a_n) \mu(D_r(a_n))^{1/2} d\mu(z), \\ &\left\| \sum_{n=m}^l c_n K_u(z, a_n) \mu(D_r(a_n))^{1/2} \right\|_{A^2(u)} \\ &= \sup_{\|f\|_{A^2(u)}=1} \left| \left\langle \sum_{n=m}^l c_n K_u(z, a_n) \mu(D_r(a_n))^{1/2}, f \right\rangle_{A^2(u)} \right| \\ &= \sup_{\|f\|_{A^2(u)}=1} \left| \sum_{n=m}^l c_n \int_{\mathbb{D}} K_u(z, a_n) d\mu(z) \mu(D_r(a_n))^{1/2} \right| \\ &= \sup_{\|f\|_{A^2(u)}=1} \left| \langle R^* \{\hat{c}_n\}, f \rangle_A^2(u) \right| \leq \|R^*\| \|\{\hat{c}_n\}\|_{l^2} < \varepsilon, \end{aligned} \quad (20)$$

where

$$\hat{c}_n := \begin{cases} c_n, & \text{if } n \in \{m, \dots, l\} \\ 0, & \text{otherwise.} \end{cases} \quad (21)$$

Thus, $s_m := \sum_{n=1}^m c_n K_u(\cdot, a_n) \mu(D_r(a_n))^{1/2}$ forms a Cauchy sequence, and consequently it converges in the $A^2(u)$ -norm to

$$s := \sum_{n=1}^{\infty} c_n K_u(\cdot, a_n) \mu(D_r(a_n))^{1/2} \in A^2(u). \quad (22)$$

Therefore,

$$\int_{\mathbb{D}} f(z) \overline{s_m(z)} d\mu(z) \rightarrow \int_{\mathbb{D}} f(z) \overline{s(z)} d\mu(z). \quad (23)$$

Hence, $R^* \{c_n\} = \sum_{n=1}^{\infty} c_n K_u(\cdot, a_n) \mu(D_r(a_n))^{1/2}$, and since R^* is surjective, then the result holds. $\square$

3. Boundedness of Toeplitz Operators

Theorem 11. Suppose that u satisfies the B_2 condition and that the Toeplitz operator $T_\nu : A^2(u) \rightarrow A^2(u)$ is bounded. Then, the Berezin transform $\tilde{\nu}$ is bounded.

Proof. First, notice that for every finite sum $s_m := \sum_{n=1}^m \lambda_n K_u(\cdot, a_n)$, $a_n \in \mathbb{D}$; the following holds:

$$\begin{aligned} &\left\langle T_\nu \left(\sum_{n=1}^m \lambda_n K_u(\cdot, a_n) \right), \sum_{n=1}^m \lambda_n K_u(\cdot, a_n) \right\rangle_{A^2(u)} \\ &= \sum_{l=1}^m T_\nu \left(\sum_{n=1}^m \lambda_n K_u(\cdot, a_n) \right) (a_l) \\ &= \sum_{l=1}^m \lambda_l \int_{\mathbb{D}} K_u(a_l, z) \sum_{n=1}^m \lambda_n K_u(z, a_n) d\nu(z) \\ &= \left\| \sum_{n=1}^m \lambda_n K_u(\cdot, a_n) \right\|_{L^2(\nu)}^2 \\ &\leq \left\| \sum_{n=1}^m \lambda_n K_u(\cdot, a_n) \right\|_{A^2(u)}^2. \end{aligned} \quad (24)$$

So, if $\{s_m\}$ is a Cauchy sequence in $A^2(u)$, it is also a Cauchy sequence in $L^2(\nu)$, and so it converges in $L^2(\nu)$. Therefore, by Theorem 10, we have that if $s_m \rightarrow g$ in the $A^2(u)$ -norm, then $s_m \rightarrow g$ in $L^2(\nu)$, and consequently for any function $f \in A^2(u)$, $\langle f, s_m \rangle_{L^2(\nu)} \rightarrow \langle f, g \rangle_{L^2(\nu)}$. Moreover,

$$\begin{aligned} \langle T_\nu f, g \rangle_{A^2(u)} &= \lim_{m \rightarrow \infty} \langle T_\nu f, s_m \rangle_{A^2(u)} \\ &= \langle f, s_m \rangle_{L^2(\nu)}. \end{aligned} \quad (25)$$

Therefore,

$$\langle T_\nu f, g \rangle_{A^2(u)} = \langle f, g \rangle_{L^2(\nu)}. \quad (26)$$

Hence, for any $a \in \mathbb{D}$,

$$\begin{aligned} \langle T_\nu B_a, B_a \rangle_{A^2(u)} &= \langle B_a, B_a \rangle_{L^2(\nu)} \\ &= \tilde{\nu}(a) \|B_a\|_{A^2(u)}^2, \end{aligned} \quad (27)$$

and consequently, $\tilde{\nu}(a) \leq \|T_\nu\|$. $\square$

We will need the following lemma due to Constantin.

Lemma 12 (see [10]). *Suppose that u satisfies condition B_2 ; then,*

$$\|B_a\|_{A^2(u)}^2 \sim \frac{\mu(D_r(a))}{(1-|a|)^4}. \quad (28)$$

Proposition 13. *If $\tilde{\nu}$ is bounded on $\mathbb{D}$, then ν is a μ -Carleson measure, where $\mu = u dA$.*

Proof. Fix $0 < r < 1$; if $\tilde{\nu}$ is bounded, then there exists a constant $C > 0$ such that for every $a \in \mathbb{D}$

$$\frac{1}{\|B_a\|_{A^2(u)}^2} \int_{D_r(a)} \frac{1}{|1-\bar{a}z|^4} d\nu(z) \leq C, \quad (29)$$

and consequently

$$\begin{aligned} \nu(D_r(a)) &\leq C(1-|a|^2)^4 \|B_a\|_{A^2(u)}^2 \\ &\sim \frac{(1-|a|^2)^4 \mu(D_r(a))}{(1-|a|)^4} \\ &\leq \mu(D_r(a)), \end{aligned} \quad (30)$$

and by Theorem 2, we obtain the result. $\square$

Proposition 14. *Suppose that ν is a μ -Carleson measure; then, the Toeplitz operator $T_\nu : A^2(u) \rightarrow A^2(u)$ is bounded.*

Proof. For every function, $g \in A_2(u)$ define the linear operator $g^* : A^2(u) \rightarrow \mathbb{C}$ by: $g^*(f) := \langle f, g \rangle_{L^2(\nu)}$. Note that if ν is a μ -Carleson measure, then

$$\begin{aligned} |g^*(f)| &\leq \|f\|_{L^2(\nu)} \|g\|_{L^2(\nu)} \\ &\leq \|f\|_{A^2(u)} \|g\|_{L^2(\nu)}, \end{aligned} \quad (31)$$

so $g^* \in (A^2(u))^*$, $\|g^*\| \leq \|g\|_{L^2(\nu)}$, and there exists $\tilde{g} \in A^2(u)$ such that $g^*(f) = \langle f, \tilde{g} \rangle_{A^2(u)}$.

We have just proved that for every $g \in A^2(u)$, there exists $\tilde{g} \in A^2(u)$ such that

$$\langle f, g \rangle_{L^2(\nu)} = \langle f, \tilde{g} \rangle_{A^2(u)}, \quad \forall f \in A^2(u). \quad (32)$$

In particular, taking $f = K_u(\cdot, a)$, we have that for every $a \in \mathbb{D}$,

$$\tilde{g}(a) = \int g(z) K_u(a, z) d\nu(z) = T_\nu g(a). \quad (33)$$

Hence, $T_\nu(A^2(u)) \subset A^2(u)$. Moreover, since $\|\tilde{g}\|_{A^2(u)} = \|g^*\| \leq \|g\|_{L^2(\nu)} \leq \|g\|_{A^2(u)}$, then

$$\|T_\nu g\|_{A^2(u)} \leq \|g\|_{A^2(u)}, \quad (34)$$

and consequently T_ν is bounded. $\square$

Corollary 15. *For $T_\nu : A^2(u) \rightarrow A^2(u)$ bounded, the following equality holds for every f and g in $A^2(u)$:*

$$\langle T_\nu g, f \rangle_{A^2(u)} = \langle g, f \rangle_{L^2(\nu)}. \quad (35)$$

4. Compactness of Toeplitz Operators

In this section, we will characterize compactness of Toeplitz operators in terms of its Berezin transform. We will use the characterization for vanishing Carleson measures given by Constantin.

Definition 16. A positive Borel measure ν on $\mathbb{D}$ is a μ -vanishing Carleson measure if the inclusion operator $i : A^2(u) \rightarrow L^2(\nu)$ is compact.

Theorem 17 (see [10, Theorem 3.3]). *Suppose that u satisfies the B_2 condition. Then ν is a μ -vanishing Carleson measure if and only if for every $0 < r < 1$,*

$$\lim_{|a| \rightarrow 1^-} \frac{\nu(D_r(a))}{\mu(D_r(a))} = 0. \quad (36)$$

Theorem 18. *Suppose that ν is a positive Borel measure on $\mathbb{D}$. Then the following are equivalent:*

- (a) T_ν is compact.
- (b) $\tilde{\nu}(a) \rightarrow 0$ as $|a| \rightarrow 1^-$.
- (c) ν is a μ -vanishing Carleson.

Proof. (a) $\Rightarrow$ (b)

Notice that by Jensen's inequality, we have that

$$\mu(D_r(a)) \geq (1-|a|^2)^4 \left(\int_{D_r(a)} u^{-1}(z) dA(z) \right)^{-1}, \quad (37)$$

and consequently using Lemma 12, we get

$$|b_a(w)| \leq |1-\bar{a}z|^{-2} \left(\int_{D_r(a)} u^{-1}(z) dA(z) \right) \rightarrow 0, \quad (38)$$

as $|a| \rightarrow 1^-$. Thus, the family $\{b_a\}_{a \in \mathbb{D}}$ converges to zero weakly in $A^2(u)$ as $|a| \rightarrow 1^-$. We also know from the proof of Theorem 11 that $\tilde{\nu}(a) = \langle T_\nu b_a, b_a \rangle_{A^2(u)}$. Consequently, $\tilde{\nu}(a) \leq \|T_\nu b_a\|_{A^2(u)}^2$ which, since T_ν is compact, converges to zero as $|a| \rightarrow 1^-$.

(b) $\Rightarrow$ (c)

We have that $\tilde{\nu}(a) = \int_{\mathbb{D}} |b_a(z)|^2 d\nu(z)$, and by Lemma 12, we obtain that

$$\begin{aligned} \tilde{\nu}(a) &\geq \int_{D_r(a)} |b_a(z)|^2 d\nu(z) \\ &\sim \frac{\nu(D_r(a))}{(1-|a|^2)^4 \|B_a\|_{A^2(u)}^2} \sim \frac{\nu(D_r(a))}{\mu(D_r(a))}, \end{aligned} \quad (39)$$

then (c) follows from (b).

(c) $\Rightarrow$ (a)

By Corollary 15 and Cauchy-Schwarz inequality we have that:

$$\begin{aligned} \|T_\nu f\|_{A^2(u)} &= \sup \{ |\langle T_\nu f, g \rangle_{A^2(u)}| : \|g\|_{A^2(u)} = 1 \} \\ &= \sup \{ |\langle f, g \rangle_{L^2(\nu)}| : \|g\|_{A^2(u)} = 1 \} \\ &\leq \|f\|_{L^2(\nu)} \sup \{ \|g\|_{L^2(\nu)} : \|g\|_{A^2(u)} = 1 \}, \end{aligned} \quad (40)$$

and since ν is a μ -Carleson measure, then $\|g\|_{L^2(\nu)} \lesssim \|g\|_{A^2(u)}$ for every $g \in A^2(u)$, so

$$\|T_\nu f\|_{A^2(u)} \lesssim \|f\|_{L^2(\nu)}, \quad \forall f \in A^2(u). \quad (41)$$

Now, if f_n is a sequence in $A^2(u)$ that converges to zero weakly, then by the compactness of $i : A^2(u) \rightarrow L^2(\nu)$ we have that $\|f_n\|_{L^2(\nu)}$ converges to zero. Therefore, $\|T_\nu f_n\|_{A^2(u)}$ converges to zero, and so T_ν is compact on $A^2(u)$. $\square$

We have used the fact that the family of normalized Bergman kernels $\{b_a\}$ converges to zero weakly as $|a| \rightarrow 1^-$. This is also true for the family of normalized reproducing kernels of $A^2(u)$.

Proposition 19. *If u satisfies condition B_2 , then $K_u(\cdot, a)/\|K_u(\cdot, a)\|_{A^2(u)}$ converges to zero weakly in $A^2(u)$.*

Proof. We use the same notation as in the observation after Theorem 3. The operator $T : A^2(u) \rightarrow A^2(u^{-1})$ maps $K_u(\cdot, a)/\|K_u(\cdot, a)\|_{A^2(u)}$ to $B_a/\|K_u(\cdot, a)\|_{A^2(u)}$. Also, by Lemma 12, we have that

$$\frac{|B_a|}{\|K_u(\cdot, a)\|_{A^2(u)}} \sim \frac{|B_a|}{\|B_a\|_{A^2(u^{-1})}} \lesssim |B_a| \mu(D_r(a)), \quad (42)$$

which converges to zero uniformly on compact subsets of $\mathbb{D}$. Consequently, the family $\{B_a/\|K_u(\cdot, a)\|_{A^2(u)}\}$ is uniformly bounded on $A^2(u^{-1})$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Since $A^2(u^{-1})$ is a functional space, then $B_a/\|K_u(\cdot, a)\|_{A^2(u)}$ converges to zero weakly on $A^2(u^{-1})$.

Now, the operator $T^{-1} : A^2(u^{-1}) \rightarrow A^2(u)$ is bounded, so $K_u(\cdot, a)/\|K_u(\cdot, a)\|_{A^2(u)}$ converges to zero weakly in $A^2(u)$. $\square$

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