Research Article

Optimal Regularity Properties of the Generalized Sobolev Spaces

G. E. Karadzhov^{1,2} and Qaisar Mehmood²

¹ Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

² Abdus Salam School of Mathematical Sciences, GC University, Lahore 68-B, Pakistan

Correspondence should be addressed to G. E. Karadzhov; geremika@yahoo.com

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We prove optimal embeddings of the generalized Sobolev spaces $W^k E$, where E is a rearrangement invariant function space, into the generalized Hölder-Zygmund space $\mathcal{C}H$ generated by a function space H .

1. Introduction

The classical Sobolev space W_p^k , $1 \leq p < \infty$, consists of all locally integrable functions f , defined on $\mathbf{R}^n$, $n \geq 1$, with the Lebesgue measure, such that the following norm is finite: $\|f\|_{W_p^k} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p$, where $\|f\|_p$ stands for the L^p -norm. In investigating the regularity of the function $f \in W_p^k$, we may assume, without any loss of generality, that $f \in L^1(\Omega)$, Ω is a domain in $\mathbf{R}^n$, and f is zero outside Ω . For simplicity we suppose that the Lebesgue measure of Ω equals one and that the origin lies in Ω . It is well known that in the supercritical case $k > n/p$,

$$W_p^k \hookrightarrow \mathcal{C}^{k-n/p}, \quad k > \frac{n}{p}, \quad (1)$$

where $\mathcal{C}^\gamma$, $\gamma > 0$, is the Hölder-Zygmund space (see [1]). In the critical case $k = n/p$ the function $f \in W_p^k$ may not be even continuous. The result (1) is not optimal. We prove that the optimal one is obtained if in (1) L^p is replaced by the Marcinkiewicz space $L^{p,\infty}$. In this paper we prove similar optimal results, when $L^{p,\infty}$ is replaced by a more general rearrangement invariant space E . The Sobolev space $W^k E$ consists of all $f \in W_1^k$ with a finite quasinorm $\|f\|_{W^k E} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_E$. More precisely, we consider quasinormed rearrangement invariant spaces E , consisting of functions $f \in L^1(\Omega)$, such that the quasinorm $\|f\|_E \approx \rho_E(f^*) < \infty$,

where ρ_E is a monotone quasinorm, defined on M^+ with values in $[0, \infty]$ and M^+ is the cone of all locally integrable functions $g \geq 0$ on $(0, 1)$ with the Lebesgue measure. Monotonicity means that $g_1 \leq g_2$ implies $\rho_E(g_1) \leq \rho_E(g_2)$. We suppose that $L^\infty(\Omega) \hookrightarrow E \hookrightarrow L^1(\Omega)$, which means continuous embeddings. Here f^* is the decreasing rearrangement of f , given by $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$, $t > 0$, and μ_f is the distribution function of f , defined by $\mu_f(\lambda) = |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}|_n$, $|\cdot|_n$ denoting Lebesgue n -measure. Note that $f^*(t) = 0$ for $t > 1$. Finally, $f^{**}(t) := (1/t) \int_0^t f^*(s) ds$.

Let α_E, β_E be the Boyd indices of E . For example, if $E = L^p$, then $\alpha_E = \beta_E = 1/p$ and the condition $k/n \geq 1/p$ means $k/n \geq \alpha_E > 0$. Note that for $k > n$ this is always satisfied. For these reasons we suppose that for the general E , $0 < \alpha_E = \beta_E \leq 1$ and the case $\min(k, n)/n > \alpha_E$ is called super-critical, while the case $\min(k, n)/n = \alpha_E$ -critical. In the super-critical case the function $f \in W^k E$ is always continuous, while the spaces in the critical case $\alpha_E = k/n$, $k < n$, can be divided into two subclasses: in the first subclass the functions $f \in W^k E$ may not be continuous—then the target space is rearrangement invariant, while these functions in the second subclass are continuous and the target space is the generalized Hölder-Zygmund space $\mathcal{C}H$ (see Definition 1). The separating space for these two subclasses is given by the Lorentz space $L^{n/k, 1}$, $k < n$. If $k \geq n$, then $W^k E$ consists of continuous functions (see the classical result of Stein [2]).

The main goal of this paper is to prove optimal embeddings of the Sobolev space $W^k E$ into the generalized Hölder-Zygmund space $\mathcal{C}H$. First we prove that this embedding for $k \leq n$ is equivalent to the continuity of the operator $R_k g(t) = \int_0^t u^{k/n-1} g(u) du$. The case $k > n$ is reduced to the continuity of R_n by using the lifting principle ([1]). Moreover, if, for example, $k \leq n$, then in the super-critical case, we can replace R_k by the operator of multiplication $t^{k/n} g(t)$. This implies a very simple characterization of both optimal target space H and optimal domain space E . Namely, the quasinorm in the optimal target space $H(E)$ is given by $\rho_E(t^{-k/n} g(t))$ and the quasinorm in the optimal domain space $E(H)$ is given by $\rho_H(t^{k/n} g(t))$. Note that we do not require ρ_E to be rearrangement invariant. In the critical case, the formula for the optimal target space is more complicated. In some cases it can be simplified. To this end, we apply the Σ^q -method of extrapolation ([3]) from the super-critical case. As a byproduct, we also characterize the embedding $W^k E \hookrightarrow C^j$, $j < k$, where C^j consists of all functions with bounded and uniformly continuous derivatives up to order j . Namely, this is equivalent to the embedding $E \hookrightarrow L^{n/(k-j),1}$ if $k \leq n$. The embedding $W^{n+j} E \hookrightarrow C^j$ is always true since $W^n E \hookrightarrow W_1^n \hookrightarrow C^0$.

The problem of the optimal target rearrangement invariant space for potential type operators is considered in [4] by using L^p -capacities. The problem of the mapping properties of the Riesz potential in optimal couples of rearrangement invariant spaces is treated in [5–7]. The optimal embeddings of generalized Sobolev type spaces into rearrangement invariant spaces are characterized in several papers [5, 8–21]. The characterization of the continuous embedding of the generalized Bessel potential spaces into the generalized Hölder-Zygmund spaces $\mathcal{C}H$, when H is a weighted Lebesgue space, is given in [22]. The optimal embeddings of Calderón spaces into the generalized Hölder-Zygmund spaces are characterized in [23].

The plan of the paper is as follows. In Section 2 we provide some basic definitions and known results. In Section 3 we characterize the embedding $W^k E \hookrightarrow \mathcal{C}H$. The optimal quasinorms are constructed in Section 4.

2. Preliminaries

We use the notations $a_1 \leq a_2$ or $a_2 \geq a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \leq a_2$ and $a_1 \geq a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

Let E be a quasinormed rearrangement invariant space as in the Introduction. There is an equivalent quasinorm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0, 1]$ that depends only on the space E (see [24]). We say that the quasinorm ρ_E satisfies Minkowski's inequality if for the equivalent quasinorm ρ_p ,

$$\rho_p^p\left(\sum g_j\right) \leq \sum \rho_p^p(g_j), \quad g_j \in M^+. \quad (2)$$

Usually we apply this inequality for functions $g_j \in M^+$ with some kind of monotonicity.

Recall the definition of the lower and upper Boyd indices α_E and β_E . Let $g_u(t) = g(t/u)$ if $t < u$ and $g_u(t) = 0$ if $t \geq u$, where $0 < t < 1$, $g \in M^+$, and let

$$h_E(u) = \sup \left\{ \frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+ \right\}, \quad (3)$$

$$g_u(t) := g\left(\frac{t}{u}\right), \quad u > 0,$$

be the dilation function generated by ρ_E . Suppose that it is finite. Then

$$\alpha_E := \sup_{0 < t < 1} \frac{\log h_E(t)}{\log t}, \quad \beta_E := \inf_{1 < t < \infty} \frac{\log h_E(t)}{\log t}. \quad (4)$$

The function h_E is submultiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \geq 1$; hence $0 \leq \alpha_E \leq \beta_E$. We suppose that $0 < \alpha_E = \beta_E \leq 1$.

If $\beta_E < 1$ we have by using Minkowski's inequality that $\rho_E(f^*) \approx \rho_E(f^{**})$. In particular, $\|f\|_E \approx \rho_E(f^{**})$ if $\beta_E < 1$. For example, consider the Gamma spaces $E = \Gamma^q(w)$, $0 < q \leq \infty$, w —positive weight, that is, a positive function from M^+ , with a quasinorm $\|f\|_{\Gamma^q(w)} := \rho_E(f^*)$, $\rho_E(g) := \rho_{w,q}\left(\int_0^1 g(tu) du\right)$, where

$$\rho_{w,q}(g) := \left(\int_0^1 [g(t) w(t)]^q \frac{dt}{t} \right)^{1/q}, \quad g \in M^+, \quad (5)$$

$$\left(\int_0^1 w^q(t) \frac{dt}{t} \right)^{1/q} < \infty.$$

Then $L^\infty(\Omega) \hookrightarrow \Gamma^q(w) \hookrightarrow L^1(\Omega)$. If $w(t) = t^{1/p}$, $1 < p < \infty$, we write as usual $L^{p,q}$ instead of $\Gamma^q(t^{1/p})$. Consider also the classical Lorentz spaces $\Lambda^q(w)$, $0 < q \leq \infty$; $f \in \Lambda^q(w)$ if $\|f\|_{\Lambda^q(w)} := \rho_{w,q}(f^*) < \infty$, $w(2t) \approx w(t)$. We suppose that $L^\infty(\Omega) \hookrightarrow \Lambda^q(w) \hookrightarrow L^1(\Omega)$.

Note that if $E = \Lambda^q(t^\alpha w)$, $0 < \alpha \leq 1$, where w is slowly varying, then $\alpha_E = \beta_E = \alpha$. Recall that $w \in M^+$ is slowly varying if for all $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to an increasing function and the function $t^{-\varepsilon} w(t)$ is equivalent to a decreasing function.

In order to introduce the Hölder-Zygmund class of spaces, we denote the modulus of continuity of order k by

$$\omega^k(t, f) = \sup_{|h| \leq t} \sup_{x \in \mathbb{R}^n} |\Delta_h^k f(x)|, \quad (6)$$

where $\Delta_h^k f$ are the usual iterated differences of f . When $k = 1$ we simply write $\omega(t, f)$.

Let H be a quasinormed space of locally integrable functions on the interval $(0, 1)$ with the Lebesgue measure, continuously embedded in $L^\infty(0, 1)$ and $\|g\|_H = \rho_H(|g|)$, where ρ_H is a monotone quasinorm on M^+ which satisfies Minkowski's inequality. The dilation function generated by ρ_H is given by

$$h_H(u) = \sup \left\{ \frac{\rho_H(g_{1/u})}{\rho_H(g)} : g \in M_m^+ \right\}, \quad (7)$$

where

$$M_m := \left\{ g \in M^+ : g(t) \text{ is increasing,} \right. \\ \left. t^{-m/n} g(t) \text{ is decreasing and } g(2t) \approx g(t) \right\}. \quad (8)$$

The choice of the space M_m is motivated by the fact that $\omega^m(t^{1/n}, f)$ is equivalent to a function $g \in M_m$. The function $h_H(u)$ is submultiplicative, increasing and $u^{-m/n} h_H(u)$ is decreasing and $h_H(1) = 1$, $h_H(u)h_H(1/u) \geq 1$. Therefore the Boyd indices of H are well defined

$$\alpha_H = \sup_{0 < t < 1} \frac{\log h_H(t)}{\log t}, \quad \beta_H = \inf_{1 < t < \infty} \frac{\log h_H(t)}{\log t}, \quad (9)$$

and they satisfy $\alpha_H \leq \beta_H \leq m/n$. In what follows, we suppose that $0 \leq \alpha_H = \beta_H \leq m/n$.

For example, let $H = L_*^q(b(t)t^{-\gamma/n})$. Here $0 \leq \gamma \leq m$ and b is a slowly varying function, and $L_*^q(w)$, or simply L_*^q if $w = 1$, is the weighted Lebesgue space with a quasinorm $\|g\|_{L_*^q(w)} = \rho_{w,q}(|g|)$, where $\rho_{w,q}$ is given by (5). It turns out that $\alpha_H = \beta_H = \gamma/n$.

Definition 1. Let $j = 0, 1, \dots$ and let C^j stand for the space of all functions f , defined on $\mathbf{R}^n$, that have bounded and uniformly continuous derivatives up to the order j , normed by $\|f\|_{C^j} = \sup \sum_{|\nu|=0}^j |P^\nu f(x)|$, where $P^\nu f(x) = \sum_{|\nu|=l} D^\nu f(x)$.

- (i) If $j/n < \alpha_H < (j+1)/n$ for $j \geq 1$ or $0 \leq \alpha_H < 1/n$ for $j = 0$, then $\mathcal{E}H$ is formed by all functions f in C^j having a finite quasinorm

$$\|f\|_{\mathcal{E}H} = \|f\|_{C^j} + \rho_H(\chi_{(0,1)}(t) t^{j/n} \omega(t^{1/n}, P^j f)). \quad (10)$$

- (ii) If $\alpha_H = (j+1)/n$, then $\mathcal{E}H$ consists of all functions f in C^j having a finite quasinorm

$$\|f\|_{\mathcal{E}H} = \|f\|_{C^j} + \rho_H(\chi_{(0,1)}(t) t^{j/n} \omega^2(t^{1/n}, P^j f)). \quad (11)$$

Here $\chi_{(a,b)}$, $0 < a < b < \infty$, is the characteristic function of the interval (a, b) .

In particular, if $H = L^\infty(t^{-\gamma/n})$, $\gamma > 0$, then $\mathcal{E}H$ coincides with the usual Hölder-Zygmund space $\mathcal{E}^\gamma$ (see [1]). Also, if $H = L^\infty$, then $\mathcal{E}H = C^0$.

We will use the following equivalent quasinorm.

Theorem 2 (equivalence [23]). *Let $0 \leq \alpha_H = \beta_H < m/n$. If $\rho_H(\chi_{(0,1)}(t)t^\alpha) < \infty$ for $\alpha > \alpha_H$, then for all such m ,*

$$\|f\|_{\mathcal{E}H} \approx \|f\|_{C^0} + \rho_H(\chi_{(0,1)}(t) \omega^m(t^{1/n}, f)). \quad (12)$$

Note that if $\rho_H(\chi_{(0,1)}(t)t^{m/n}) < \infty$, then $\mathcal{E}H$ is a K -interpolation space for the couple (C^0, C^m) , namely, $\mathcal{E}H = (C^0, C^m)_{H_1}$, where $\rho_{H_1}(g) = \rho_H(g(t^{m/n}))$. In particular, $\mathcal{E}L_*^1(t^{-j}) \hookrightarrow C^j \hookrightarrow \mathcal{E}L^\infty(t^{-j})$. By $\chi_{(a,b)}$, $0 \leq a < b \leq \infty$ we denote the characteristic function of the interval (a, b) .

Recall some basic definitions from the theory of interpolation spaces [21]. Let (A_0, A_1) be a couple of two quasinormed spaces, such that both are continuously embedded in some quasinormed space and let

$$K(t, f) = K(t, f; A_0, A_1) \\ = \inf_{f=f_0+f_1} \{ \|f_0\|_{A_0} + t\|f_1\|_{A_1} \}, \quad f \in A_0 + A_1, \quad (13)$$

be the K -functional of Peetre. By definition, the K -interpolation space $A_\Phi = (A_0, A_1)_\Phi$ has a quasinorm $\|f\|_{A_\Phi} = \|K(t, f)\|_\Phi$, where Φ is a quasinormed function space with a monotone quasinorm on $(0, \infty)$ with the Lebesgue measure and such that $\min\{1, t\} \in \Phi$. Then $A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1$. If

$$\|g\|_\Phi = \left(\int_0^\infty [w(t) t^{-\theta} g(t)]^q \frac{dt}{t} \right)^{1/q}, \quad (14)$$

$$0 \leq \theta \leq 1, \quad 0 < q \leq \infty, \quad w \in \mathcal{M}^+,$$

we write $(A_0, A_1)_{w t^{-\theta}, q}$ instead of $(A_0, A_1)_\Phi$. Also, if $w = 1$ then we write $(A_0, A_1)_{\theta, q}$. By definition,

$$\|f\|_{A_0 \cap A_1} = \|f\|_{A_0} + \|f\|_{A_1}, \\ \|f\|_{A_0 + A_1} = K(1, f; A_0, A_1). \quad (15)$$

Theorem 3 (lifting principle). *Let $\alpha_H > 0$ and let $\rho_{H_j}(g) := \rho_H(t^{-j/n} g(t))$, $j \geq 1$. Then*

$$\|f\|_{\mathcal{E}H_j} \approx \|\mathcal{D}^j f\|_{\mathcal{E}H}, \quad \mathcal{D}^j = \sum_{|\alpha| \leq j} D^\alpha. \quad (16)$$

Proof. Let $m/n > \alpha_H + j/n$. Since

$$\omega^m(t^{1/n}, f) \leq t^{j/n} \omega^{m-j}(t^{1/n}, P^j f), \quad (17)$$

it follows

$$\rho_{H_j}(\omega^m(t^{1/n}, f)) \leq \rho_H(\omega^{m-j}(t^{1/n}, \mathcal{D}^j f)). \quad (18)$$

Hence

$$\|f\|_{\mathcal{E}H_j} \leq \|\mathcal{D}^j f\|_{\mathcal{E}H}. \quad (19)$$

To prove the reverse, we use the formula (see [25, page 342])

$$\omega^m(t^{1/n}, D^\alpha f) \leq \int_0^t u^{-|\alpha|/n} \omega^m(u^{1/n}, f) \frac{du}{u}. \quad (20)$$

Then applying Minkowski's inequality and $\alpha_{H_j} = \beta_{H_j} = j/n + \alpha_H > j/n$, we get

$$\rho_H(\chi_{(0,1)}(t) \omega^m(t^{1/n}, D^\alpha f)) \leq \rho_{H_j}(\chi_{(0,1)}(t) \omega^m(t^{1/n}, f)), \\ |\alpha| \leq j. \quad (21)$$

Since $\mathcal{E}H_j \hookrightarrow C^j$, we derive

$$\|\mathcal{D}^j f\|_{\mathcal{E}H} \leq \|f\|_{\mathcal{E}H_j}. \quad (22)$$

□

Remark 4. The relation (19) is always true. But if $\alpha_H = 0$ then the reverse might not be true. For example, let $n = 1$, $H = L^\infty$, $j = 1$, and $f(x) = x \ln |2x|$ if $|x| \leq 1/2$ and $f(x) = 0$ if $|x| > 1/2$. Then $f \notin C^1$, but $\omega^2(t, f) = O(t)$.

It will be convenient to introduce the classes of the domain and target quasinorms, where the optimality is investigated. Let N_d consist of all domain quasinorms ρ_E that are monotone, satisfying Minkowski's inequality, $0 < \alpha_E = \beta_E \leq \min(k, n)/n$, $\rho_E(\chi_{(0,1)} t^{-\alpha}) < \infty$ if $\alpha < \alpha_E$ and the condition (30) below for $k \leq n$, and $E \hookrightarrow L^1$ for $k > n$. Let N_t consist of all target quasinorms ρ_H that are monotone, satisfy Minkowski's inequality, $0 \leq \alpha_H = \beta_H < \min(k, n)/n$, $\rho_H(\chi_{(0,1)}(t)t^\alpha) < \infty$ if $\alpha > \alpha_H$ and

$$\sup_{0 < t < 1} g(t) \leq \rho_H(g). \quad (23)$$

We use the following definitions.

Definition 5 (admissible couple). We say that the couple $\rho_E \in N_d$, $\rho_H \in N_t$ is admissible if $W^k E \hookrightarrow \mathcal{C}H$ when $k \leq n$, and if $\mathcal{D}^j(W^{n+j}E) \hookrightarrow \mathcal{C}H$ for $j \geq 1$. Moreover, $\rho_E(E)$ is called domain quasinorm (domain space), and $\rho_H(H)$ is called target quasinorm (target space).

Definition 6 (optimal target quasi-norm). Given the domain quasinorm $\rho_E \in N_d$, the optimal target quasi-norm, denoted by $\rho_{H(E)}$, is the strongest target quasi-norm; that is,

$$\rho_H(g) \leq \rho_{H(E)}(g), \quad g \in M_{\min(k,n)} \quad (24)$$

for any target quasinorm $\rho_H \in N_t$ such that the couple ρ_E , ρ_H is admissible. Since $\mathcal{C}H(E) \hookrightarrow \mathcal{C}H$, we call $\mathcal{C}H(E)$ the optimal Hölder-Zygmund space.

Definition 7 (optimal domain quasi-norm). Given the target quasinorm $\rho_H \in N_t$, the optimal domain quasi-norm, denoted by $\rho_{E(H)}$, is the weakest domain quasi-norm; that is,

$$\rho_{E(H)}(f^*) \leq \rho_E(f^*), \quad f \in L^1(\Omega), \quad (25)$$

for any domain quasinorm $\rho_E \in N_d$ such that the couple ρ_E , ρ_H is admissible.

Definition 8 (optimal couple). The admissible couple $\rho_E \in N_d$, $\rho_H \in N_t$ is said to be optimal if both ρ_E and ρ_H are optimal.

3. Admissible Couples

Here we give a characterization of all admissible couples $\rho_E \in N_d$, $\rho_H \in N_t$. We start with the main estimate. For $k = 1$, see also [26].

Theorem 9. Let $f \in W_1^k$ and $k \leq n$. Then

$$\omega^k(t^{1/n}, f) \leq \int_0^t u^{k/n-1} (\mathcal{D}^k f)^*(u) du, \quad (26)$$

where $\mathcal{D}^k f := \sum_{|\alpha| \leq k} D^\alpha f$.

Proof. We use the embedding

$$W^k L^{n/k,1} \hookrightarrow L^\infty, \quad (27)$$

whence

$$\begin{aligned} \omega^k(t^{1/n}, f) &\leq K(t^{k/n}, f; L^\infty, W_\infty^k) \\ &\leq K(t^{k/n}, f; W^k L^{n/k,1}, W_\infty^k). \end{aligned} \quad (28)$$

Then (26) follows from the basic formula [25, page 360]

$$K(t, f; W_1^k, W_\infty^k) \approx \int_0^t (\mathcal{D}^k f)^*(s) ds \quad (29)$$

and the reiteration formula of Holmstedt for the K -functional [25, page 310]. $\square$

Now we discuss the embedding $W^k E \hookrightarrow C^0$. For $k = 1$ more general results are proved in [27, Chapter 4].

Theorem 10. A necessary and sufficient condition for the embedding $W^k E \hookrightarrow C^0$, $k \leq n$, is the following one

$$\int_0^1 t^{k/n-1} g(t) dt \leq \rho_E(g), \quad g \in D_1, \quad (30)$$

where

$$D_1 = \{g \in M^+ : g \text{ is decreasing function}\}. \quad (31)$$

Proof. The conditions (30) and (26), (27) imply the embedding $W^k E \hookrightarrow L^\infty$ and $\lim_{t \rightarrow 0} \omega^k(t^{1/n}, f) = 0$ if $f \in W^k E$. On the other hand, by Marchaud's inequality (see [25], Theorem 5.4.4), we have

$$\omega(t^{1/n}, f) \leq t^{1/n} \int_t^\infty u^{-1/n} \omega^k(u^{1/n}, f) \frac{du}{u}. \quad (32)$$

It is easy to see that $\lim_{t \rightarrow 0} \omega(t^{1/n}, f) = 0$. Thus $W^k E \hookrightarrow C^0$.

Before proving the reverse, note that (30) is always satisfied if $k/n > \alpha_E$. Since

$$\rho_E(\chi_{(0,1)}) g(u) \leq \rho_E(g(tu)) \leq h_E\left(\frac{1}{u}\right) \rho_E(g), \quad (33)$$

we have

$$g(u) \leq h_E\left(\frac{1}{u}\right) \rho_E(g), \quad g \in D_1. \quad (34)$$

Hence for $0 < \varepsilon < k/n - \alpha_E$,

$$\int_0^1 u^{k/n-1} g(u) du \leq \rho_E(g) \int_0^1 u^{k/n-\alpha_E-\varepsilon} \frac{du}{u} \leq \rho_E(g). \quad (35)$$

It remains to prove that if $W^k E \hookrightarrow C^0$, $k \leq n$, then (30) is true for $\alpha_E = k/n$. To this end we choose a test function f as follows:

$$f(x) = \int_0^1 g(u) u^{k/n} \psi(xu^{-1/n}) \frac{du}{u}, \quad (36)$$

where $g \in D_1$ and ψ is in C_0^∞ such that $\psi(x) = 1$ if $|x| \leq 2^{-1}c^{-1/n}$ and $\psi(x) = 0$ if $|x| \geq c^{-1/n}$. Then $f(x) = 0$ for $|x| > c^{-1/n}$, whence $f^*(t) = 0$ for $t > 1$ for appropriate c and $|f(x)| \leq \int_{c|x|^n}^1 g(u)u^{k/n}du/u$, whence $|f(x)| \leq g(c|x|^n)$; therefore $f^*(t) \leq g(t)$. This implies

$$\rho_E(f^*) \leq \rho_E(g). \quad (37)$$

Analogously, since $|D^\alpha f(x)| \leq \int_{c|x|^n}^{2^n c|x|^n} g(u)du/u$, $1 \leq |\alpha| \leq k$, we have

$$\|f\|_{W^k E} \leq \rho_E(g). \quad (38)$$

Also $f(0) = \psi(0)/n \int_0^1 u^{k/n-1} g(u)du \leq \|f\|_{W^k E} \leq \rho_E(g)$. Thus (30) is proved. $\square$

Remark 11. Similar arguments show that $W^k E \hookrightarrow C^j$, $j < k \leq n$, if and only if $E \hookrightarrow L^{n/(k-j),1}$.

Theorem 12. The couple $\rho_E \in N_d$, $\rho_H \in N_t$ is admissible if and only if

$$\rho_H(\chi_{(0,1)} R_{\min(k,n)} g) \leq \rho_E(g), \quad g \in D_1, \quad (39)$$

where

$$R_k g(t) := \int_0^t u^{k/n-1} g(u) du. \quad (40)$$

Proof. Step 1 (sufficiency of (39)). If $k \leq n$ then it is clear that the embedding $W^k E \hookrightarrow \mathcal{CH}$ follows from (39), (26), and (27). Let now $k = n + j$, $j \geq 1$. Then (39) for $k = n$ implies $W^n E \hookrightarrow \mathcal{CH}$. Hence $\mathcal{D}^j(W^{n+j} E) \hookrightarrow \mathcal{CH}$ for $j \geq 1$.

Step 2 (necessity of (39) when $k \leq n$). Now we prove that the embedding $W^k E \hookrightarrow \mathcal{CH}$ implies (39) for $k \leq n$. To this end we choose the test function f as in (36).

Let $|h| = Ct^{1/n}$. We split $f = f_{1t} + f_{2t}$, $f_{1t}(x) = \int_0^t u^{k/n} g(u) \psi(xu^{-1/n}) du/u$, $f_{2t}(x) = \int_t^1 u^{k/n} g(u) \psi(xu^{-1/n}) du/u$, $0 < t < 1$. First we prove that for some large $C > 0$,

$$\omega^m(Ct^{1/n}, f_{1t}) \geq \psi(0) R_k g(t), \quad 0 < t < 1. \quad (41)$$

Indeed, we have $\omega^m(Ct^{1/n}, f_{1t}) \geq |(\Delta_h^m f_{1t})(0)|$ and $\psi(jCt^{1/n}u^{-1/n}) = 0$ for $u < t$, $1 \leq j \leq m$ if $C > 0$ is large enough. Hence (41) follows. Further,

$$\omega^m(t^{1/n}, f) \geq \omega^m(t^{1/n}, f_{1t}) - \omega^m(t^{1/n}, f_{2t}), \quad 0 < t < 1. \quad (42)$$

Since

$$\begin{aligned} \omega^m(t^{1/n}, f_{2t}) &\leq t^{m/n} \|P^m f_{2t}\|_{L^\infty} \\ &\leq t^{m/n} \int_t^1 u^{(k-m)/n} g(u) \frac{du}{u} \end{aligned} \quad (43)$$

and $R_k g(t) \geq t^{k/n} g(t)$ for $g \in D_1$, we get

$$\omega^m(t^{1/n}, f_{2t}) \leq t^{m/n} \int_t^1 u^{-m/n} R_k g(u) \frac{du}{u}. \quad (44)$$

Therefore

$$R_k g(t) \leq c_1 \omega^m(t^{1/n}, f) + ct^{m/n} \int_t^1 u^{-m/n} R_k g(u) \frac{du}{u}, \quad 0 < t < 1, \quad (45)$$

$$R_k g(t) \leq c_1 \omega^m(t^{1/n}, f) + ct^{m/n} \int_t^1 u^{-m/n} g(u) \frac{du}{u}, \quad 0 < t < 1. \quad (46)$$

To solve the integral inequality (45) for $p(t) := \chi_{(0,1)}(t) t^{-m/n} R_k g(t)$, we set $q(t) = c_1 t^{-m/n} \omega^m(t^{1/n}, f)$ and rewrite it as $p(t) \leq q(t) + c \int_t^1 p(u) du/u$. If $r(t) = \int_t^1 p(u) du/u$, then we get the differential inequality $0 \leq r'(t) + cr(t) + q(t)$. If $r(t) = t^{-c} v(t)$, then $0 \leq v'(t) + t^{c-1} q(t)$, whence $v(t) \leq \int_t^1 u^{c-1} q(u) du$. Therefore

$$\chi_{(0,1/2)}(t) R_k g(t) \leq t^{m/n-c} \int_t^1 u^{c-m/n} \omega^m(u^{1/n}, f) \frac{du}{u}. \quad (47)$$

Hence by using Minkowski's inequality and choosing m large enough, we obtain

$$\rho_H(\chi_{(0,1/2)} R_k g) \leq \rho_H(\omega^m(t^{1/n}, f)). \quad (48)$$

On the other hand, from (46), it follows that

$$\rho_H(\chi_{(1/2,1)} R_k g) \leq \rho_H(\omega^m(t^{1/n}, f)) + \int_0^1 g(u) du. \quad (49)$$

Hence, using also (30), we get

$$\rho_H(\chi_{(0,1)} R_k g) \leq \rho_H(\omega^m(t^{1/n}, f)) + \rho_E(g). \quad (50)$$

Thus, if $W^k E \hookrightarrow \mathcal{CH}$ is given, then (50), (38) imply (39).

Step 3 (necessity of (39) when $k = n + j$, $j \geq 1$). Now we prove that the embedding $\mathcal{D}^j(W^{n+j} E) \hookrightarrow \mathcal{CH}$, $j \geq 1$, implies (39) for $k = n$. To this end we choose the test function f in the form

$$f(x) = \int_0^1 g(u) u^{j/n} \psi(xu^{-1/n}) du, \quad (51)$$

where $g \in D_1$ and ψ is the same as in (36). Note that $\mathcal{D}^j \psi(0) = 1$. Then as before we get (37), and

$$|D^\alpha f(x)| \leq g(c|x|^n) \quad (52)$$

for $1 \leq |\alpha| \leq n + j$. Hence

$$\|f\|_{W^{n+j} E} \leq \rho_E(g). \quad (53)$$

On the other hand, using the arguments from Step 2 but for the function $\mathcal{D}^j f$, $j \geq 1$, we obtain

$$\rho_H(\chi_{(0,1)} R_n g) \leq \rho_H(\omega^m(t^{1/n}, \mathcal{D}^j f)) + \rho_E(g), \quad j \geq 1. \quad (54)$$

Thus, if $\mathcal{D}^j(W^{n+j}E) \hookrightarrow \mathcal{C}H$, $j \geq 1$ is given, then (54), (53) imply (39) for $k = n$. $\square$

Theorem 13. Let $\alpha_H > 0$. Then the couple $\rho_E \in N_d$, $\rho_H \in N_t$ is admissible for $k > n$ if and only if

$$W^{n+j}E \hookrightarrow \mathcal{C}H_j, \quad \rho_{H_j}(g) := \rho_H(t^{-j/n}g(t)), \quad j \geq 1. \quad (55)$$

Moreover, (55) is equivalent to

$$\rho_{H_j}(R_{n+j}g) \leq \rho_E(g), \quad g \in D_1. \quad (56)$$

Proof. Let $\rho_E \in N_d$, $\rho_H \in N_t$ be an admissible couple for $k > n$. Then $\mathcal{D}^j(W^{n+j}E) \hookrightarrow \mathcal{C}H$. Since

$$\omega^m(t^{1/n}, \mathcal{D}^j f) \leq \int_0^t (\mathcal{D}^{n+j} f)^*(u) du, \quad m \geq n, \quad (57)$$

we have, by applying (39) for $k = n$,

$$\rho_H(\omega^m(t^{1/n}, \mathcal{D}^j f)) \leq \rho_E((\mathcal{D}^{n+j} f)^*). \quad (58)$$

Using also (16) and $W^{n+j}E \hookrightarrow C^j$, we obtain $W^{n+j}E \hookrightarrow \mathcal{C}H_j$. Further, as in the proof of the previous theorem, this embedding implies (56). Finally,

$$\rho_{H_j}(R_{n+j}g) \approx \rho_H(R_n g), \quad \alpha_H > 0. \quad (59)$$

Indeed, $\rho_H(R_n g) = \rho_{H_j}(t^{j/n} R_n g(t))$ and

$$\begin{aligned} & \rho_{H_j}(t^{j/n} R_n g(t)) \\ & \leq \rho_{H_j} \left(t^{j/n} \int_0^t u^{(n+j)/n} (R_{n+j}g)(u) du \right). \end{aligned} \quad (60)$$

Applying Minkowski's inequality, we get, since $\alpha_{H_j} > j/n$,

$$\rho_H(R_n g) \leq \rho_{H_j}(R_{n+j}g). \quad (61)$$

For the reverse, we notice that $R_{n+j}g(t) \leq t^{j/n} R_n g(t)$. Then

$$\rho_{H_j}(R_{n+j}g) \leq \rho_H(R_n g). \quad (62) \quad \square$$

4. Optimal Quasinorms

Here we give a characterization of the optimal domain and optimal target quasinorms.

4.1. Optimal Domain Quasinorms. We can construct an optimal domain quasinorm $\rho_{E(H)}$ by Theorem 9 as follows.

Definition 14 (construction of an optimal domain quasinorm). For a given target quasinorm $\rho_H \in N_t$, we set

$$\rho_{E(H)}(g) := \rho_H(R_{\min(k,n)}g), \quad g \in M^+. \quad (63)$$

Note that $R_{\min(k,n)}(g_u) = u^{\min(k,n)/n} (\widetilde{R_{\min(k,n)}g})_{1/u}$ and $Sg \in L$ if $g \in D_1$. Hence $\alpha_{E(H)} = \beta_{E(H)} = \min(k,n)/n - \alpha_H$.

Theorem 15. The quasinorm $\rho_{E(H)}$ belongs to N_d , the couple $\rho_{E(H)}$, ρ_H is admissible, and the domain quasinorm $\rho_{E(H)}$ is optimal. Moreover, the target quasinorm ρ_H is also optimal and

$$\rho_{E(H)}(g) \approx \rho_H(t^{\min(k,n)/n}g), \quad g \in D_1 \text{ if } \alpha_H > 0. \quad (64)$$

Proof. It is easy to check that $\rho_{E(H)} \in N_d$. Further, the couple $\rho_{E(H)}$, ρ_H is admissible since $\rho_H(R_{\min(k,n)}g) = \rho_{E(H)}(g)$, $g \in D_1$. Moreover, $\rho_{E(H)}$ is optimal, since for any admissible couple $\rho_{E_1} \in N_d$, ρ_H , we have $\rho_H(\chi_{(0,1)} R_{\min(k,n)}g) \leq \rho_{E_1}(g)$, where $g \in D_1$. Therefore for $f \in L^1(\Omega)$,

$$\begin{aligned} \rho_{E(H)}(f^*) &= \rho_H(R_{\min(k,n)}f^*) \\ &\leq \rho_H(\chi_{(0,1)} R_{\min(k,n)}f^*) + \rho_{E_1}(f^*) \leq \rho_{E_1}(f^*). \end{aligned} \quad (65)$$

To prove that ρ_H is also optimal, let $\rho_{E(H)}$, $\rho_{H_1} \in N_t$ be an arbitrary admissible couple. Then

$$\rho_{H_1}(\chi_{(0,1)} R_{\min(k,n)}g) \leq \rho_{E(H)}(g), \quad g \in D_1. \quad (66)$$

It is enough to check that

$$\rho_{H_1}(\chi_{(0,1)}g) \leq \rho_H(g), \quad g \in M_{\min(k,n)}. \quad (67)$$

Let $k < n$. (The case $k \geq n$ is easier.) We introduce a better function $g_1(t) = g(t^{n/k})$. Then g_1 is quasiconcave; therefore $g_1(t) \approx \int_0^t h_1(u) du$ and $h_1 \in D_1$. By changing the variables, we get $g \approx R_k h$ with $h \in D_1$. Then

$$\begin{aligned} \rho_{H_1}(\chi_{(0,1)}g) &\leq \rho_{H_1}(\chi_{(0,1)}R_k h) \\ &\leq \rho_{E(H)}(h) \approx \rho_H(R_k h) \approx \rho_H(g). \end{aligned} \quad (68)$$

Thus (67) is proved. To prove the equivalence (64), we use $t^{\min(k,n)/n}g(t) \leq R_{\min(k,n)}g(t)$, $g \in D_1$ and Minkowski's inequality as follows:

$$\begin{aligned} \rho_H^p(R_{\min(k,n)}g) &\leq \sum_{j=-\infty}^0 h_H^p(2^j) \rho_H^p(t^{\min(k,n)/n}g(t)), \\ &g \in D_1, \alpha_H > 0, \end{aligned} \quad (69)$$

whence $\rho_{E(H)}(g) \leq \rho_H(t^{\min(k,n)/n}g(t))$, $g \in D_1$. $\square$

Remark 16. Let $k \geq n$ and $\chi_{(0,1)}(t)t \in H$. Then the couple $E = (L^1, L^\infty)_H$, H is optimal.

Example 17. Consider the space $H = L^1_*(v)$, where $\rho_H(g) = \int_0^\infty v(t)g(t)dt/t$ and $\rho_H \in N_t$. Using Theorem 15, we can construct an optimal domain E , where

$$\begin{aligned}\rho_E(g) &= \rho_H(R_{\min(k,n)}g) \\ &= \int_0^\infty t^{\min(k,n)/n} w(t) g(t) \frac{dt}{t}\end{aligned}\quad (70)$$

and $w(t) = \int_t^\infty v(u)du/u$. Hence $E = \Lambda^1(t^{\min(k,n)/n}w)$, and this couple is optimal. Also $\alpha_E = \beta_E = \min(k,n)/n$ if v is slowly varying. Note that if $k \geq n$, then $E = \Lambda^1(tw) = \Gamma^1(tv) = (L^1, L^\infty)_H$.

Example 18. Let $H = L^\infty(v)$, where $\rho_H(g) = \sup v(t)g(t)$ and $\rho_H \in N_t$ and let

$$\rho_E(g) = \sup v(t) \int_0^t u^{\min(k,n)/n} g(u) \frac{du}{u}. \quad (71)$$

Then by Theorem 15, the domain E is optimal and the couple is optimal. In particular, the couple $L^{n/\min(k,n),1}, C^0$ is optimal. If $k = n+j$, $j \geq 0$, this means that the embedding $W_1^{n+j} \hookrightarrow C^j$ is optimal.

Example 19. Let H be as in the previous example. Since

$$\begin{aligned}\rho_E(g) &\leq \sup t^{\min(k,n)/n} w(t) g(t), \\ \frac{1}{v(t)} &= \int_0^t \frac{1}{w(u)} \frac{du}{u},\end{aligned}\quad (72)$$

it follows that the couple $E = \Lambda^\infty(t^{\min(k,n)/n}w)$, $H = L^\infty_*(v)$ is admissible. In order to prove that ρ_H is optimal, take any $g \in M_0$, and define h from $t^{\min(k,n)/n} w(t)h(t) = \sup_{u \geq t} v(u)g(u)$. Then $h \in D_1$ and $\rho_{E_1}(h) \leq \rho_H(g)$. On the other hand

$$\begin{aligned}R_{\min(k,n)}h(t) &= \int_0^t \sup_{x \geq u} v(x) g(x) \frac{1}{w(u)} \frac{du}{u} \\ &\geq \sup_{u \geq t} v(u) g(u) \frac{1}{v(t)} \geq g(t).\end{aligned}\quad (73)$$

Hence $\rho_{H(E)}(g) \leq \rho_{E_1}(h) \leq \rho_H(g)$; therefore ρ_H is optimal.

Example 20 (case $k \geq n$). Let $H = L^q_*(v)$, $0 < q \leq \infty$, where $\rho_H(g) = (\int_0^\infty [v(t)g(t)]^q dt/t)^{1/q}$ and $\rho_H \in N_t$ and let $k \geq n$. Using Remark 16, we can construct an optimal domain $E = \Gamma^q(tv)$ and this couple is optimal. Also $\alpha_E = \beta_E = 1$ if v is slowly varying.

4.2. Optimal Target Quasinorms

Definition 21 (construction of the optimal target quasi-norm). For a given domain quasinorm $\rho_E \in N_d$, we set

$$\begin{aligned}\rho_{H(E)}(g) &:= \inf \{ \rho_E(h) : g \leq R_{\min(k,n)}h, h \in D_1 \}, \\ &g \in M_{\min(k,n)}.\end{aligned}\quad (74)$$

Note that $\alpha_{H(E)} = \beta_{H(E)} = \min(k,n)/n - \alpha_E$.

Theorem 22. The target quasinorm $\rho_{H(E)}$ belongs to N_t , the couple $\rho_E, \rho_{H(E)}$ is admissible, and the target quasinorm is optimal.

Proof. The property " $\rho_{H(E)}(g) = 0$ implies $g = 0$ " follows from (30). Also, since $\rho_E \in N_d$ it is easy to check that $\rho_{H(E)} \in N_t$. The couple is admissible since $\rho_{H(E)}(R_{\min(k,n)}h) \leq \rho_E(h)$, $h \in D_1$. Suppose that the couple $\rho_E, \rho_{H_1} \in N_t$ is admissible. Then $\rho_{H_1}(\chi_{(0,1)} R_{\min(k,n)}h) \leq \rho_E(h)$, $h \in D_1$. Therefore if $\chi_{(0,1)}g \leq R_{\min(k,n)}h$, $h \in D_1$, then $\rho_{H_1}(\chi_{(0,1)}g) \leq \rho_{H_1}(\chi_{(0,1)} R_{\min(k,n)}h) \leq \rho_E(h)$, whence $\rho_{H_1}(\chi_{(0,1)}g) \leq \rho_{H(E)}(g)$, $g \in M_{\min(k,n)}$. Hence $\rho_{H(E)}$ is optimal. $\square$

Theorem 23 (supercritical case). If $\alpha_E < \min(k,n)/n$ and $\rho_E(\chi_{(1,\infty)}(t)t^{-\min(k,n)/n}) < \infty$, then

$$\rho_{H(E)}(g) \approx \rho_E(t^{-\min(k,n)/n}g(t)), \quad g \in M_{\min(k,n)}. \quad (75)$$

Moreover, the couple $\rho_E, \rho_{H(E)}$ is optimal.

Proof. If $g \leq R_{\min(k,n)}h$, $h \in D_1$, then, by Minkowski's inequality and since $\min(k,n)/n > \alpha_E$, it follows

$$\begin{aligned}\rho_E(t^{-\min(k,n)/n}g(t)) \\ \leq \rho_E(t^{-\min(k,n)/n}R_{\min(k,n)}h(t)) \leq \rho_E(h).\end{aligned}\quad (76)$$

Hence, taking the infimum, we get $\rho_E(t^{-\min(k,n)/n}g(t)) \leq \rho_{H(E)}(g)$.

On the other hand, for $g \in M_{\min(k,n)}$, we have $g \leq R_{\min(k,n)}h$, $h(t) = t^{-\min(k,n)/n}g(t)$. Since $h \in D_1$ it follows $\rho_{H(E)}(g) \leq \rho_E(h) \leq \rho_E(t^{-\min(k,n)/n}g(t))$.

The domain quasinorm ρ_E is also optimal since for $f \in L^1(\Omega)$,

$$\begin{aligned}\rho_{H(E)}(f^*) &= \rho_{H(E)}(R_{\min(k,n)}f^*) \\ &\approx \rho_E(t^{-\min(k,n)/n}R_{\min(k,n)}f^*(t)) \geq \rho_E(f^*).\end{aligned}\quad (77)$$

$\square$

Example 24. Consider the space $E = \Lambda^q(w)$, $0 < q \leq \infty$, $\min(k,n)/n > \beta_E = \alpha_E > 0$. Then by Theorem 23 the couple E, H , $H = L^q_*(t^{-\min(k,n)/n}w)$ is optimal. In particular, using also Theorem 13, the embedding $W^k L^{p,\infty} \hookrightarrow \mathcal{C}^{k-n/p}$, $k > n/p$, $1 < p < \infty$, is optimal.

In the critical case we do not know how to simplify the optimal target quasi-norm, defined in (74). Instead, we can construct a large class of domain quasinorms and the corresponding optimal target quasinorms by using extrapolation from the super-critical case. Recall some basic definitions and results from the extrapolation theory [3]. Let (A_0, A_1) be a couple of quasi-Banach spaces. The sigma extrapolation space $\Sigma^q(M(\sigma)(A_0, A_1)_{a(t)t^{-\sigma,q}})$, a -positive weight, $0 < \sigma < \sigma_0$, $0 < q \leq \infty$, M -positive decreasing weight, consists of all

$f \in A_0 + A_1$ such that $f = \sum_{j=l}^{\infty} g_j$, $g_j \in A_j$, $A_j := (A_0, A_1)_{a(t)t^{-1/2^j}, q}$, with a quasinorm

$$\|f\|_{\Sigma^q(M(\sigma)(A_0, A_1)_{a(t)t^{-\sigma}, q})} = \inf \left(\sum_{j=l}^{\infty} \left[M(2^{-j}) \|g_j\|_{A_j} \right]^q \right)^{1/q}, \quad (78)$$

where the infimum is taken with respect to all representations $f = \sum_{j=l}^{\infty} g_j$.

This space can be characterized as an interpolation space.

Theorem 25 (see [3]). *Let $a(t) = t^{-\theta}b(t)$, b -slowly varying, $0 < \theta < 1$. Then*

$$\Sigma^q(M(\sigma)(A_0, A_1)_{a(t)t^{-\sigma}, q}) = (A_0, A_1)_{w, q}, \quad (79)$$

where

$$\frac{1}{w(t)} = \frac{1}{a(t)} \left(\int_0^{\sigma_0} \left[\frac{t^\sigma}{M(\sigma)} \right]^r \frac{d\sigma}{\sigma} \right)^{1/r} \quad (80)$$

and $1/r + 1/q = 1$ if $q > 1$, $r = \infty$ if $0 < q \leq 1$.

Our main result is the following one.

Theorem 26. *Let $E = \Lambda^q(t^{k/n}c(t)(1 - \ln t))$, $k < n$, c -slowly varying weight, $c(+0) = \infty$, $c(t^2) \approx c(t)$, $0 < q \leq \infty$, $H = L_*^q(c)$. We suppose that $\rho_E \in N_d$ and $\rho_H \in N_t$. Then this couple is admissible and the target quasinorm is optimal.*

Proof. Step 1 (admissibility). Since $\alpha_E = \beta_E = k/n < 1$, it will be enough to check that

$$\rho_H(R_k(g^{**})) \leq \rho_E(g^{**}), \quad (81)$$

where

$$\begin{aligned} \rho_E(g) &= \left(\int_0^1 [t^{k/n}c(t)(1 - \ln t)g(t)]^q \frac{dt}{t} \right)^{1/q}, \\ \rho_H(g) &= \left(\int_0^1 [c(t)g(t)]^q \frac{dt}{t} \right)^{1/q}. \end{aligned} \quad (82)$$

Applying Minkowski's inequality we obtain for $0 < \sigma < \sigma_0 < k/n$, b -slowly varying weight,

$$\sigma \|R_k(g^{**})\|_{L_*^q(b(t)t^{-\sigma})} \leq \|g\|_{\Gamma^q(t^{k/n-\sigma}b(t))}. \quad (83)$$

In order to extrapolate these inequalities, we write

$$\begin{aligned} \Gamma^q(t^{k/n-\sigma}b(t)) &= (L^1, L^\infty)_{b(t)t^{k/n-1-\sigma}, q}, \\ L_*^q(t^{-\sigma}b(t)) &= (L_*^q(t^{1/2}b(t)), L_*^q(t^{-1/2}b(t)))_{1/2+\sigma, q} \\ \sigma_0 &< \frac{1}{2}. \end{aligned} \quad (84)$$

This is true since

$$\begin{aligned} K(t, g; L_*^q(w_0), L_*^q(w_1)) \\ \approx \left(\int_0^1 [g(u) \min(w_0(u), tw_1(u))]^q \frac{du}{u} \right)^{1/q}, \end{aligned} \quad (85)$$

$0 < t < 1$.

Let $\sigma = 2^{-j}$ and $g = \sum g_j$ (convergence in L^1), where $g_j \in L^\infty$. Then $g^{**} \leq \sum g_j^{**}$, whence $R_k(g^{**}) \leq \sum R_k(g_j^{**})$ and for $M(\sigma) = \sigma^{-2}$, $p = \min(q, 1)$, we have

$$K^p(t, R_k(g^{**}); B_0, B_1) \leq C_\nu := \sum_{j \geq l} K^p(t, R_k(g_j^{**}); B_0, B_1), \quad (86)$$

where $B_0 = L_*^q(t^{1/2}b(t))$, $B_1 = L_*^q(t^{-1/2}b(t))$. We can write

$$\begin{aligned} C_\nu &= \sum_{j \geq l} \left[t^{-1/2-2^{-j}} 2^{-j} M(2^{-j}) K(t, R_k(g_j^{**}); B_0, B_1) \right]^p \\ &\quad \times \left[\frac{t^{1/2+2^{-j}}}{2^{-j} M(2^{-j})} \right]^p \end{aligned} \quad (87)$$

and using also Hölder's inequality if $q > 1$, we get

$$\begin{aligned} [\nu(t)]^p C_\nu \\ \leq \sum_{j \geq l} \left[t^{-1/2-2^{-j}} 2^{-j} M(2^{-j}) K(t, R_k(g_j^{**}); B_0, B_1) \right]^p, \end{aligned} \quad (88)$$

where

$$\frac{1}{\nu(t)} = \left(\sum_{j \geq l} \left[\frac{t^{1/2+2^{-j}}}{2^{-j} M(2^{-j})} \right]^r \right)^{1/r}. \quad (89)$$

Hence

$$\begin{aligned} \|R_k(g^{**})\|_{(B_0, B_1)_{\nu, q}} \\ \leq \left(\sum_{j \geq l} \left[2^{-j} M(2^{-j}) \|R_k(g_j^{**})\|_{(B_0, B_1)_{1/2+2^{-j}, q}} \right]^q \right)^{1/q}. \end{aligned} \quad (90)$$

Since

$$2^{-j} \|R_k(g_j^{**})\|_{(B_0, B_1)_{1/2+2^{-j}, q}} \leq \|g_j\|_{\Gamma^q(t^{k/n-2^{-j}}b(t))}, \quad (91)$$

we get

$$\|R_k(g^{**})\|_{(B_0, B_1)_{\nu, q}} \leq \|g\|_{\Sigma^q(M(\sigma)(L^1, L^\infty)_{b(t)t^{k/n-1-\sigma}, q})}, \quad (92)$$

whence

$$R_k : (L^1, L^\infty)_{w, q} \mapsto (L_*^q(t^{1/2}b(t)), L_*^q(t^{-1/2}b(t)))_{\nu, q}, \quad (93)$$

where w is given by (80) with $a(t) = b(t)t^{k/n-1}$ and $M(\sigma) = \sigma^{-2}$. It is easy to calculate these weights, see [3]. We have

$$w(t) \approx b(t)t^{k/n-1}(1 - \ln t)^2, \quad v(t) \approx t^{-1/2}(1 - \ln t), \quad (94)$$

$$0 < t < 1.$$

Then for $b(t) = c(t)(1 - \ln t)^{-1}$ we get

$$\begin{aligned} \Gamma^q \left(t^{k/n} c(t) (1 - \ln t) \right) &\hookrightarrow \left(L^1, L^\infty \right)_{w,q}, \\ \left(L^q_* \left(t^{1/2} b(t) \right), L^q_* \left(t^{-1/2} b(t) \right) \right)_{v,q} &\hookrightarrow L^q_*(c). \end{aligned} \quad (95)$$

Hence (81) is proved. $\square$

Step 2 (optimality of the target quasi-norm). We want to prove that ρ_H is an optimal target quasi-norm. It is sufficient to see that

$$\rho_{H(E)}(\chi_{(0,1)}g) \lesssim \rho_H(g), \quad g \in M_k, \quad (96)$$

where $\rho_{H(E)}$ is defined by (74). To this end for any such g we construct an $h \in D_1$ such that $\chi_{(0,1)}g \lesssim R_k h$ and $\rho_E(h) \lesssim \rho_H(g)$. Let $0 < q < \infty$. (The case $q = \infty$ is analogous, see Example 19.) Let $h^q(t) = \chi_{(0,2)}(t) \int_t^2 (1 - \ln u)^{-q} u^{-q} g^q(\sqrt{ue}) du/u$. Then $h \in D_1$ and $\rho_E(h) \lesssim \rho_H(g)$. On the other hand, for $0 < t < 1$,

$$R_k h(t) \geq \int_{t^2/e}^t \frac{g(\sqrt{ue})}{1 - \ln u} \frac{du}{u} \gtrsim \chi_{(0,1)}(t) g(t), \quad (97)$$

since $\int_{t^2/e}^t (1 - \ln u)^{-1} du/u = \ln 2$. Then by the definition of $\rho_{H(E)}$ we get

$$\rho_{H(E)}(\chi_{(0,1)}g) \lesssim \rho_E(h) \lesssim \rho_H(g). \quad (98)$$

In the limiting case $k = n$ we can use the fact that the weights $b(t) = c(t)(1 - \ln t)$ and $c(t)$, where c is decreasing and slowly varying, are Muckenhoupt's weights; that is,

$$\left(\int_t^1 c^q(u) \frac{du}{u} \right)^{1/q} \left(\int_0^t [b(u)]^{-r} \frac{du}{u} \right)^{1/r} \lesssim 1, \quad \frac{1}{q} + \frac{1}{r} = 1, \quad (99)$$

$$0 < t < 1.$$

Then the operator R_n is bounded from $L^q_*(b)$ to $L^q_*(c)$, $1 \leq q \leq \infty$ (see [28]). In this way we have the following result, with optimality being proved as in the case $k < n$.

Theorem 27. *Let $E = \Lambda^q(tc(t)(1 - \ln t))$, c -decreasing slowly varying weight, $c(+0) = \infty$, $c(t^2) \approx c(t)$, $1 \leq q \leq \infty$, $H = L^q_*(c)$. We suppose that $\rho_E \in N_d$ and $\rho_H \in N_t$. Then this couple is admissible and the target quasinorm is optimal.*

Authors' Contribution

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