Research Article

Generalized Quasilinearization for the System of Fractional Differential Equations

Peiguang Wang¹ and Ying Hou²

¹ College of Electronic and Information Engineering, Hebei University, Baoding 071002, China
² College of Mathematics and Computer Science, Hebei University, Baoding 071002, China

Correspondence should be addressed to Peiguang Wang; pgwang@mail.hbu.edu.cn

Received 9 December 2012; Accepted 8 February 2013

Academic Editor: Yongsheng S. Han

Copyright © 2013 P. Wang and Y. Hou. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper considers the initial value problems of the system of fractional differential equations and constructs two monotone sequences of upper and lower solutions. By using quasilinearization technique, monotone sequences of approximate solutions that converge quadratically to a solution are obtained.

1. Introduction

In this paper, we consider the system of Caputo fractional differential equations:

\[ ^cD^q x(t) = f(t, x), \quad x(t_0) = x_0, \quad (1) \]

where \( 0 < q < 1, f \in C[\Omega, \mathbb{R}^n], \Omega = \{(t, x) : \alpha(t) \leq x \leq \beta(t), \alpha, \beta \in C^1[J, \mathbb{R}], t \in J\}, \) and \( J = [t_0, T]. \)

The theories and properties of fractional differential equations have received attention from some researchers because many mathematical modeling appeared in the fields of physics, chemistry, engineering and biological sciences and so on. For examples and details, we can refer to the, monographs of Miller and Ross [1], Podlubny [2], Kilbas et al. [3], and West et al [4] and the papers of Debnath [5], Rossikhin and Shitikova [6], and Ferreira et al [7]. There are many results on the basic theory of initial value and boundary value problems for fractional differential equations, which can be found in [8–10]. Meanwhile, there are some qualitative and numerical solutions for various fractional equations with delay and impulsive effects. For details, see some recent papers [II–18] and the references therein.

It is well known that the monotone iterative technique is an ingenious method providing a constructive approach to find solutions for the nonlinear problem via linear iterates. Lakshmikantham and Vatsala [19], and McRae [20] investigated the existence of minimal and maximal solutions of fractional differential equations by establishing a comparison result and using the monotone method, respectively; Benchohra and Hamani [21] used a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of solutions to impulsive fractional differential inclusions.

Quasilinearization [22] provides an elegant and easier approach to obtain a sequence of approximate solutions with quadratic convergence, and the method has been extended to fractional differential equations in [23, 24]. However, to the best of our knowledge, there are few results for the system of fractional differential equations, especially results on the convergence of the system. In the present paper, we will discuss the approximate solutions of the system of fractional differential equations through the application of quasilinearization. The significance of this work lies in the fact that the system of fractional differential equations can also obtain a monotone sequence of approximate solutions converging uniformly to the solution of the problem and possessing quadratic convergence.

The nonhomogeneous linear system of Caputo fractional differential equations is given by

\[ ^cD^q x(t) = Bx(t) + g(t, y), \quad x(t_0) = x_0, \quad (2) \]

where \( B \) is an \( n \)th order matrix over complex field, and \( g \) is an \( n \)-dimensional locally integrable column vector function on \( J \).
Using the method of successive approximations, we get the solution of (2) as
\[ x(t) = x_0 E_q \left( B(t-t_0)^q \right) + (t-t_0)^{q-1} E_{q,q} \]
\[ \times \left( B(t-t_0)^q \right)^k g(t,y), \quad t \in J, \]
where
\[ E_q(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(qk+1)}, \quad E_{q,q}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(qk+q+1)} \]
are Mittag-Leffler functions of one parameter and two parameters, respectively.

**2. Preliminaries**

Now, we present the following definition and lemma which help to prove our main result.

**Definition 1.** Let \( \alpha, \beta \in C^q[I, R^n] \) be lower and upper solutions of (1) if they satisfy the inequalities
\[ \begin{align*}
\beta &\leq f(t, \alpha), & \alpha_0 \leq x_0, \\
\alpha &\leq f(t, \beta), & \beta_0 \leq x_0,
\end{align*} \]
respectively, for \( t \in J \).

**Lemma 2.** Suppose that \( \alpha, \beta \in C^q[I, R^n] \) are lower and upper solutions of (1), and
\[ (H_1) f \in C[\Omega, R^n] \]
are quasimonotone nondecreasing in \( x \) for each \( t \in J \) and for each \( i \in \{1,2,\ldots,n\} \),
\[ f_i(t, x) - f_i(t, y) \leq L \sum_{j=1}^{n} (x_j - y_j), \]
where \( x \geq y, \ L \geq 0 \) is a constant.

Then, \( \alpha(t_0) \leq \beta(t_0) \) implies that \( \alpha(t) \leq \beta(t) \).

**Proof.** Firstly, suppose that
\[ \begin{align*}
\beta &> f(t, \alpha), & \alpha \leq f(t, \beta), & t \in J
\end{align*} \]
and \( \alpha(t_0) < \beta(t_0) \). We will prove that \( \alpha(t) < \beta(t), \ t \in J \).

Suppose the conclusion is not true, then the set
\[ Z = \bigcup_{i=1}^{n} [t \in J, \beta_i(t) \geq \alpha_i(t)] \]
is nonempty.

Let \( t_1 = \inf Z \). Certainly, \( t_1 > 0 \). Since the set \( Z \) is closed, \( t_1 \in Z \) and consequently there exists a \( j \) such that \( \beta_j(t_1) = \alpha_j(t_1) \). Moreover, \( \beta_j(t) \geq \alpha_j(t) \), for \( i \neq j \), and
\[ \alpha_i(t) < \beta_j(t), \quad t \in [0, t_1). \]

Hence, it easily follows that
\[ \begin{align*}
\beta_j(t_1) &\geq f_j(t_1, \alpha_j(t_1)), \\
\alpha_j(t_1) &\geq f_j(t_1, \beta_j(t_1)), \quad t \in J,
\end{align*} \]
This together with the quasimonotonicity of \( f \) yields
\[ f_j(t_1, \alpha_j(t_1)) \geq f_j(t_1, \beta_j(t_1)) \geq f_j(t_1, \alpha(t_1)), \]
which leads to a contradiction.

In order to prove the case of nonstrict inequalities, consider the functions
\[ \tilde{\beta}_j(t) = \beta_j(t) + e(t)^{n+1} \]
where \( e > 0 \) is sufficiently small constant. Then using (6), we have
\[ \begin{align*}
\beta_j(t) &\geq f_j(t, \beta_j(t)) + e(t)^{n+1} L \]
\[ \geq f_j(t, \beta_j(t)) + e(t)^{n+1} L \]
\[ > f_j(t, \tilde{\beta}_j(t)). \]
Also \( \tilde{\beta}_j(t_0) > \alpha_j(t_0) \). Now using the result corresponding to strict inequalities, we get
\[ \alpha_j(t) < \tilde{\beta}_j(t), \quad t \in J. \]

Letting \( e \to 0 \), we obtain the required result and the proof is complete.

**Corollary 3.** The function \( f(t, \alpha) = \sigma(t) \alpha \) where \( \sigma(t) \leq L \) is admissible in Lemma 2 to yield \( \alpha(t) \leq 0 \), \( t \in J \).

**3. Main Result**

**Theorem 4.** Assume that \( \alpha_0, \beta_0 \in C^q[I, R^n] \) are lower and upper solutions of (1) such that \( \alpha_0(t) \leq \beta_0(t) \), \( t \in J \), and
\[ (H_2) \quad \begin{align*}
\beta_j(t, \alpha_0, \beta_0) &\geq 0, \quad i \neq j, \quad \text{where} \quad A(t, \alpha_0, \beta_0) = \left[ \alpha_{ij}(t, \alpha_0, \beta_0) \right] \text{is an} \ n \times n \text{matrix given by} \\
A &\equiv f_x(t, \alpha_0) + \phi_x(t, \alpha_0) - \phi_x(t, \beta_0) \\
&= F_x(t, \alpha_0) - \phi_x(t, \beta_0),
\end{align*} \]
where \( F_x = f_x + \phi_x \).

Then, there exist monotone sequences \( \alpha_n(t) \), \( \beta_n(t) \) which converge uniformly to the solution of (1) and the convergence is quadratic.

**Proof.** \( (H_1) \) in Theorem 4 implies for any \( x \geq y, \ x, y \in \Omega \),
\[ f(t, x) \geq f(t, y) + [f_x(t, x, y) + \phi_x(t, y)] \]
\[ \times (x - y) - [\phi(t, x) - \phi(t, y)]. \]
And for any
\[ a_i (t) \leq y \leq x \leq \beta_0 (t), \]  
we have
\[ a_i (t, y, x) \geq 0, \quad i \neq j, \]  
(17)
since \( a_i (t, y, x) \geq a_j (t, \alpha_0, \beta_0) \) by assumption (H1). It is also clear that for \( a_0 (t) \leq y \leq x \leq \beta_0 (t) \),
\[ f_i (t, x_1, \ldots, x_n) - f_i (t, y_1, \ldots, y_n) \leq L_i \sum_{i=1}^n (x_i - y_i). \]  
(19)
Let \( \alpha_i, \beta_1 \) be the solutions of IVPs:
\[ \begin{align*}
\dot{\alpha}^D \alpha_1 &= f (t, \alpha_0) + A (t, \alpha_0, \beta_0) (\alpha_1 - \alpha_0), \\
&= A (t, \alpha_0, \beta_0) p, \\
\dot{\beta}^D \beta_1 &= f (t, \beta_0) + A (t, \alpha_0, \beta_0) (\beta_1 - \beta_0), \\
&= A (t, \alpha_0, \beta_0) p,
\end{align*} \]  
(20)
where \( \alpha_0 (0) \leq x_0 \leq \beta_0 (0) \). We will prove that \( \alpha_0 \leq \alpha_1, \; t \in J \). To do this, let \( p = \alpha_0 - \alpha_1 \), so that \( p (0) \leq 0 \). Then using (20), we obtain
\[ \begin{align*}
\dot{\alpha}^D p &= \dot{\beta}^D \beta_1 - \dot{\beta}^D \alpha_1, \\
&\leq f (t, \alpha_0) - [f (t, \alpha_0) + A (t, \alpha_0, \beta_0) (\alpha_1 - \alpha_0)], \tag{21}
\end{align*} \]

Since \( A (t, \alpha_0, \beta_0) \) is quasimonotone nondecreasing by assumption (H1) to (H2) and it follows from Corollary 3 that \( p (t) \leq 0, \; t \in J \), proving that \( \alpha_0 \leq \alpha_1, \; t \in J \).

Now we let \( \alpha_1 = \beta_0 \) and note that \( p (0) \leq 0 \). Also,
\[ \begin{align*}
\dot{\alpha}^D \beta_1 &= \dot{\beta}^D \beta_1 - \dot{\beta}^D \beta_0, \\
&\leq f (t, \alpha_0) + A (t, \alpha_0, \beta_0) (\beta_1 - \beta_0) - f (t, \beta_0), \tag{22}
\end{align*} \]
since \( \beta_0 \geq \alpha_0 \), using (16), we get
\[ \begin{align*}
f (t, \beta_0) \geq f (t, \alpha_0) + F_x (t, \alpha_0) (\beta_0 - \alpha_0) - [\phi (t, \beta_0) - \phi (t, \alpha_0)]. \\
\end{align*} \]  
(23)
In view of (H1), we have
\[ \phi (t, \beta_0) - \phi (t, \alpha_0) \leq \phi_x (t, \beta_0) (\beta_0 - \alpha_0), \]  
(24)
which yields
\[ f (t, \beta_0) \geq f (t, \alpha_0) + [F_x (t, \alpha_0) - \phi_x (t, \beta_0)] (\beta_0 - \alpha_0). \]  
(25)
Hence, we obtain
\[ \begin{align*}
\dot{\beta}^D \beta_0 &\leq A (t, \alpha_0, \beta_0) p, \tag{26}
\end{align*} \]
this implies that \( \alpha_i (t) \leq \beta_0 (t), \; t \in J \), using Corollary 3. As a result, we have
\[ \alpha_0 (t) \leq \alpha_1 (t) \leq \beta_0 (t), \; t \in J. \]  
(27)
In a similar way, we can prove that
\[ \alpha_0 (t) \leq \beta_1 (t) \leq \beta_0 (t), \; t \in J. \]  
(28)
To show \( \alpha_i (t) \leq \beta_i (t) \), we use (16), (19), and (H1):
\[ \begin{align*}
\dot{\alpha}^D \alpha_1 &= f (t, \alpha_0) + [F_x (t, \alpha_0) - \phi_x (t, \beta_0)] (\alpha_1 - \alpha_0), \\
&\leq f (t, \alpha_1) + [F_x (t, \alpha_0) - \phi_x (t, \beta_0)] (\alpha_1 - \alpha_0), \\
&\leq f (t, \alpha_1) + [\int_0^1 \phi_x (t, s \alpha_0 + (1-s) \alpha_0) \; ds] \tag{29}
\end{align*} \]
by (19), we have \( f \) is Lipschitzian in \( x \) on \( \Omega \). This proves that
\[ \alpha_0 (t) \leq \alpha_1 (t) \leq \beta_1 (t) \leq \beta_0 (t), \; t \in J. \]  
(30)
Now assume that for some \( k > 0 \),
\[ \alpha_{k+1} (t) \leq \alpha_k (t) \leq \beta_k (t) \leq \beta_{k+1} (t), \; t \in J. \]  
(33)
We now aim to show that
\[ \alpha_k (t) \leq \alpha_{k+1} (t) \leq \beta_{k+1} (t) \leq \beta_k (t), \; t \in J, \]  
(34)
where \( \alpha_{k+1} (t) \) and \( \beta_{k+1} (t) \) are the solutions of linear IVPs:
\[ \begin{align*}
\dot{\alpha}^D \alpha_{k+1} &= f (t, \alpha_k) + [F_x (t, \alpha_k) - \phi_x (t, \beta_k)] \times (\alpha_{k+1} - \alpha_k), \tag{35}
\end{align*} \]
\[ \begin{align*}
\dot{\beta}^D \beta_{k+1} &= f (t, \beta_k) + [F_x (t, \alpha_k) - \phi_x (t, \beta_k)] \times (\beta_{k+1} - \beta_k), \tag{36}
\end{align*} \]
Now, set \( p = \alpha_k - \alpha_{k+1} \) so that
\[ \begin{align*}
\dot{\alpha}^D p &= \dot{\beta}^D \beta_{k+1} - \dot{\beta}^D \beta_k, \\
&\leq f (t, \alpha_k) - [f (t, \alpha_k) + (F_x (t, \alpha_k) - \phi_x (t, \beta_k))] \times (\alpha_{k+1} - \alpha_k), \\
&= F_x (t, \alpha_k) - \phi_x (t, \beta_k) p, \\
&= A (t, \alpha_k, \beta_k) p.
\end{align*} \]
and \( p(0) = 0 \). It follows from Corollary 3 and using (16) that
\[
\alpha_k(t) \leq \alpha_{k+1}(t), \quad t \in J. \tag{37}
\]
On the other hand, letting \( p = \alpha_{k+1} - \beta_k \) yields
\[
\mathcal{D}^3 p = \mathcal{D}^3 \alpha_{k+1} - \mathcal{D}^3 \beta_k \\
\leq f(t, \alpha_k) + \left[ F_x(t, \alpha_k) - \phi_x(t, \beta_k) \right] (\alpha_{k+1} - \alpha_k) \\
\times \left( \alpha_{k+1} - \alpha_k \right) \times f(t, \beta_k).
\]
Since \( \beta_k \geq \alpha_k \), (16) and (H1) give, as before,
\[
f(t, \beta_k) \geq f(t, \alpha_k) + \left[ F_x(t, \alpha_k) - \phi_x(t, \beta_k) \right] (\beta_k - \alpha_k), \tag{39}
\]
which shows that
\[
\mathcal{D}^3 p \leq \left[ F_x(t, \alpha_k) - \phi_x(t, \beta_k) \right] = A(t, \alpha_k, \beta_k) p.
\]
This proves that \( p(t) \leq 0 \) using Corollary 3 and (16), since \( p(0) = 0 \). Hence, we get
\[
\alpha_k(t) \leq \alpha_{k+1}(t) \leq \beta_k(t), \quad t \in J. \tag{41}
\]
Similarly, we can prove that
\[
\alpha_k(t) \leq \beta_{k+1}(t) \leq \beta_k(t), \quad t \in J. \tag{42}
\]
Also, by (16), (35), and the fact that \( \alpha_{k+1} \geq \alpha_k \), we obtain
\[
\mathcal{D}^3 \alpha_{k+1} \leq f(t, \alpha_{k+1}) + \phi(t, \alpha_{k+1}) - \phi(t, \alpha_k) \\
- \phi_x(t, \beta_k) (\alpha_{k+1} - \alpha_k) \\
\leq f(t, \alpha_{k+1}) + \left[ \int_0^1 \phi_x(t, s \alpha_{k+1} + (1-s) \alpha_k) ds \right] \\
\times (\alpha_{k+1} - \alpha_k) - \phi_x(t, \beta_k) (\alpha_{k+1} - \alpha_k) \\
\leq f(t, \alpha_{k+1}) + \phi_x(t, \alpha_{k+1}) (\alpha_{k+1} - \alpha_k) \\
- \phi_x(t, \beta_k) (\alpha_{k+1} - \alpha_k) \\
\leq f(t, \alpha_{k+1}). \tag{43}
\]
Using a similar argument, we have, as before, \( \mathcal{D}^3 \beta_{k+1} \geq f(t, \beta_{k+1}) \), and hence Lemma 2 shows that
\[
\alpha_{k+1}(t) \leq \beta_{k+1}(t), \quad t \in J. \tag{44}
\]
By (19), we have that \( f \) is Lipschitzian in \( x \) on \( \Omega \) and is quasimonotone nondecreasing in \( x \). This proves (34). Therefore, by induction, we have for all \( n \):
\[
\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in J. \tag{45}
\]
Employing the standard procedure, it is now easy to prove that the sequences \( \{\alpha_n(t)\} \) and \( \{\beta_n(t)\} \) converge uniformly and monotonically to the unique solution of (1) on \( J \).

We will now show that the convergence of \( \{\alpha_n(t)\} \) and \( \{\beta_n(t)\} \) to \( x(t) \) is quadratic. First set
\[
\rho_{n+1} = x - \alpha_{n+1}, \quad \rho_{n+1} = \beta_{n+1} - x, \quad t \in J, \tag{46}
\]
so that \( \rho_{n+1}(0) = \rho_{n+1}(0) = 0 \). Then using integral mean value theorem and the fact that \( x \geq \alpha_{n+1} \) and (H1), we get
\[
\mathcal{D}^3 p_{n+1} = \mathcal{D}^3 x - \mathcal{D}^3 \alpha_{n+1} \\
= f(t, x) - \left[ f(t, \alpha_n) + \left( F_x(t, \alpha_n) - \phi_x(t, \beta_n) \right) \right] \\
\times (\alpha_{n+1} - \alpha_n) \\
= f(t, x) - f(t, \alpha_n) - \left[ f_x(t, \alpha_n) - \phi_x(t, \beta_n) \right] \\
\times (\alpha_{n+1} - \alpha_n) \tag{47}
\]
So, we get
\[
\mathcal{D}^3 p_{n+1} \leq \left[ \int_0^1 f_x(t, s \alpha_n + (1-s) \alpha_n) ds \right] p_n \\
+ \left( r_n + p_n \right) \left[ \int_0^1 \phi_x(t, s \beta_n + (1-s) \alpha_n) ds \right] \\
+ f(t, \beta_n) p_{n+1} \\
\leq M p_{n+1} + W (p_n, p_n) + N (r_n + p_n) p_n \\
\leq M p_{n+1} + W p_n + N r_n p_n \\
\leq M p_{n+1} + W + N r_n p_n. \tag{48}
\]
where \( \left| p_n \right|^2 = \left( |p_{n1}|^2, |p_{n2}|^2, \ldots, |p_{nm}|^2 \right), \quad \left| r_n \right|^2 = \left( |r_{n1}|^2, |r_{n2}|^2, \ldots, |r_{nm}|^2 \right), \quad f_x \leq M, \quad \sum_{i=1}^n f_{x_i} \leq W \quad \text{and} \quad \sum_{i=1}^n g_{x_i} \leq N \quad \text{in} \quad \Omega, \quad M, W, \text{and} \quad N \quad \text{are} \quad n \times n \quad \text{positive matrices}. \quad \text{Using (2) to (4), we can get}
\[
\rho_{n+1}(t) \leq \left[ \left( W + \frac{3N}{2} \right) |p_n|^2 + \frac{N}{2} |r_n|^2 \right] \\
\times (T - t_0)^{\alpha - 1} E_{t_0}(M(T - t_0)^{\alpha - 1}) \tag{49}
\]
\[
\leq V \left[ \left( W + \frac{3N}{2} \right) |p_n|^2 + \frac{N}{2} |r_n|^2 \right], \tag{50}
\]
where \( V = (T - t_0)^{\alpha - 1} E_{t_0}(M(T - t_0)^{\alpha - 1}) \). Thus, we have
\[
\left| \rho_{n+1} \right| \leq V \left[ \left( W + \frac{3N}{2} \right) |p_n|^2 + \frac{N}{2} |r_n|^2 \right]. \tag{50}
\]
Similarly, we have
\[
|r_{n+1} - r_n| \leq V \left[ \frac{W}{2} |p_n| + \left( \frac{3W}{2} + \frac{5N}{2} \right) |r_n| \right].
\] (51)

The proof is complete. \( \square \)

**Acknowledgments**

The authors would like to thank the reviewers for their valuable suggestions and comments. This work is supported by the National Natural Science Foundation of China (11271106) and the Natural Science Foundation of Hebei Province of China (A2013201232).

**References**


Submit your manuscripts at
http://www.hindawi.com