Research Article

Weak Type Inequalities for Some Integral Operators on Generalized Nonhomogeneous Morrey Spaces

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We prove weak type inequalities for some integral operators, especially generalized fractional integral operators, on generalized Morrey spaces of nonhomogeneous type. The inequality for generalized fractional integral operators is proved by using two different techniques: one uses the Chebyshev inequality and some inequalities involving the modified Hardy-Littlewood maximal operator and the other uses a Hedberg type inequality and weak type inequalities for the modified Hardy-Littlewood maximal operator. Our results generalize the weak type inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces and extend to some singular integral operators. In addition, we also prove the boundedness of generalized fractional integral operators on generalized non-homogeneous Orlicz-Morrey spaces.

1. Introduction

In this paper, we prove that the theory of generalized Morrey spaces can be staged on the nondoubling setting on $\mathbb{R}^d$, so that we assume that μ is a positive Borel measure on $\mathbb{R}^d$ satisfying the growth condition; that is, there exist $n \in [0, d]$ and $C_\mu > 0$ such that

$$\mu(B(a, r)) \leq C_\mu r^n \quad (1)$$

for any ball $B(a, r)$ centered at $a \in \mathbb{R}^d$ with radius $r > 0$ (see [1–4]). For $0 < n \leq d$ and a measurable function $\rho : (0, \infty) \rightarrow (0, \infty)$, we define the generalized fractional integral operator I_ρ by

$$I_\rho f(x) = \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y) \quad (2)$$

for any suitable function f on $\mathbb{R}^d$. This operator dates back to the book when $\rho(t) = t^\alpha$ for $0 < \alpha < n$ [5, Section 6.1]. Note that if μ is the Lebesgue measure, then $I_\rho = I_\alpha$ is the fractional integral operator introduced in [6, 7]. See also [8, 9] for exhaustive and comprehensive explanation about the operator. Below, we will always assume the Dini condition, that is, $\int_0^1 (\rho(t)/t) dt < \infty$ and we also assume that ρ satisfies the so-called growth condition; namely, there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{(r/2) < s \leq r} \rho(s) \leq C \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t} dt \quad (3)$$

for every $r > 0$. For convenience, write $\rho^*(r) = \int_{k_1 r}^{k_2 r} (\rho(t)/t) dt$. Note that if ρ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that $1/C \leq \rho(r)/\rho(s) \leq C$ whenever $1/2 \leq r/s \leq 2$, then ρ satisfies the growth

condition. See [10–13] for discussion about I_ρ , where ρ satisfies the doubling condition.

Now, we say that a function f belongs to the generalized nonhomogeneous Morrey space $L^{p,\phi}(\mu) = L^{p,\phi}(\mathbb{R}^d, \mu)$ for a function $\phi : (0, \infty) \rightarrow (0, \infty)$ and $1 \leq p < \infty$ if

$$\|f\|_{L^{p,\phi}(\mu)} := \sup_{B(a,r) \subseteq \mathbb{R}^d} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{B(a,r)} |f(y)|^p d\mu(y) \right)^{1/p} < \infty. \quad (4)$$

Note that this definition is a special case of [14, Definition 1.1], where different types of operators are considered. In this paper, we will assume the following two conditions.

- (1.a) The function ϕ is almost decreasing; that is, there exists a constant $C_1 > 0$ such that $\phi(r) \geq C_1 \phi(s)$ for every $r \leq s$.
- (1.b) The function $r \mapsto r^n \phi(r)^p$ is almost increasing; that is, there exists a constant $C_2 > 0$ such that $r^n \phi(r)^p \leq C_2 s^n \phi(s)^p$ for every $r \leq s$.

These two conditions imply that ϕ satisfies the doubling condition. Note that if $\phi(t) = t^{-n/p}$, then $L^{p,\phi}(\mu) = L^p(\mu)$ is the nonhomogeneous Lebesgue space.

The study of the boundedness of the fractional integral operator I_α on generalized Morrey spaces was initiated in [15, Theorem 3]. The following theorem presents the weak type inequalities for I_α on generalized nonhomogeneous Morrey spaces.

Theorem 1 (see [16, Theorem 2.4]). *Let $1 \leq p < q < \infty$. Suppose that $\inf_{r>0} \phi(r) = 0$, $\sup_{r>0} \phi(r) = \infty$, and there exist positive constants C' and C'' such that*

$$\int_r^\infty \frac{\phi(t)^p}{t} dt \leq C' \phi(r)^p, \quad (5)$$

$$r^\alpha \phi(r) + \int_r^\infty t^{\alpha-1} \phi(t) dt \leq C'' \phi(r)^{p/q}$$

for every $r > 0$. Then, there exists a constant $C > 0$ such that, for any function $f \in L^{p,\phi}(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$, one has

$$\mu(\{x \in B(a, r) : |I_\alpha f(x)| > \gamma\}) \leq C r^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^q, \quad (6)$$

for every $\gamma > 0$.

Remark 2. Note that we can obtain the weak type inequalities for I_α on nonhomogeneous Lebesgue spaces which are proved in [17, 18] by taking $\phi(r) = r^{-n/p}$ and $1/q = (1/p) - (\alpha/n)$ in Theorem 1. By substituting $1/q = 1 - (\alpha/(n - \lambda))$ for some $\lambda \in [0, n - \alpha]$ to $r^\alpha \phi(r) \leq C \phi(r)^{1/q}$, we have $\int_r^\infty t^{\alpha-1} \phi(t) dt \leq C r^{\lambda+\alpha-n}$ for every $r > 0$, which is one of the hypotheses in the weak type inequalities for I_α in [19].

The proof of Theorem 1 employs some inequalities involving the modified Hardy-Littlewood maximal operator M^n (see [8]), which is defined for any locally integrable function f by

$$M^n f(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)| d\mu(y) \quad (7)$$

and the Chebyshev inequality which is presented in the following theorem.

Theorem 3 (see [20]). *Let E be a measurable subset of $\mathbb{R}^d$. If f is an integrable function on E , then, for every $\gamma > 0$, one has*

$$\mu(\{x \in E : |f(x)| > \gamma\}) \leq \frac{1}{\gamma} \int_E |f(x)| d\mu(x). \quad (8)$$

One of the reasons why we are fascinated with the generalized fractional integral operators is that these operators appear naturally in the context of differential equations; see [21, Section 6.4] for a nice explanation in connection with the holomorphic calculus of operators and see [22, (4.3)] and [23, Lemma 2.5] for a detailed account that $(1 - \Delta)^{-\alpha/2}$ with $\alpha > 0$ satisfies the requirement of ρ in the present paper. In addition, investigating generalized Morrey spaces is not a mere quest to the abstract theory; it arises naturally in the context of Sobolev embedding. In [24], the following proposition is proved.

Proposition 4 (see [24, Theorem 5.1]). *Let $1 < p < \infty$ and $0 < \lambda < N$. Then, there exists a positive constant $C_{p,\lambda}$ such that*

$$\int_B |f(x)| dx \leq C_{p,\lambda} |B| (1 + |B|)^{-1/p} \log \left(e + \frac{1}{|B|} \right) \|(1 - \Delta)^{\lambda/2p} f\|_{L^{p,\lambda}} \quad (9)$$

holds for all $f \in L^{p,\lambda}(\mathbb{R}^N)$ with $(1 - \Delta)^{\lambda/2p} f \in L^{p,\lambda}(\mathbb{R}^N)$ and for all balls B , where $L^{p,\lambda}$ is the abbreviation of $L^{p,\phi}$ with $\phi(t) = t^{-\lambda}$.

Later Proposition 4 is strengthened by [25, Example 5]. An example in [24] as well as the necessary and sufficient condition obtained in [25, Theorem 1.3] implicitly shows that the log factor above is absolutely necessary.

In this paper, we will prove the weak type inequalities for I_ρ which is a generalization of Theorem 1. In Section 2, we will prove the weak type inequalities for I_ρ by using the Chebyshev inequality and some inequalities involving operator M^n . In Section 3, we will prove a Hedberg type inequality on generalized nonhomogeneous Morrey space by adapting the proof of a Hedberg type inequality on homogeneous setting in [25]. Through the weak type inequalities for M^n , we then prove the weak type inequalities for I_ρ on generalized nonhomogeneous Morrey spaces. In Section 4, we extend our results to the singular integral operators defined in [1]. Finally, in Section 5, we prove the boundedness of I_ρ

on generalized nonhomogeneous Orlicz-Morrey spaces. See [26–28] for related results.

Throughout the paper, C denotes a positive constant which is independent of the function f and the variable x and may have different values from line to line. We also denote by C_k ($k \in \mathbb{N}$) the fixed constants that satisfy certain conditions.

2. Weak Type Inequalities for I_ρ via the Chebyshev Inequality

Now, we give an inequality which is used in the proof of the weak type inequalities for I_ρ in the following lemma.

Lemma 5. *Let $1 \leq p < q < \infty$. If ρ and ϕ satisfy*

$$\int_0^r \frac{\rho(t)}{t} dt \leq C\phi(r)^{(p/q)-1} \quad (10)$$

for every $r > 0$, then, for any ball $B(x, R) \subseteq \mathbb{R}^d$ and every locally integrable function f , one has

$$\int_{B(x, R)} \frac{\rho(|x - y|)}{|x - y|^n} |f(y)| d\mu(y) \leq CM^n f(x) \phi(R)^{(p/q)-1}. \quad (11)$$

Proof. Let $I(X) = \int_{(B, r)} (\rho(|x - y|)/|x - y|^n) |f(y)| d\mu(y)$, for any ball $B(x, R) \subseteq \mathbb{R}^d$. By the dyadic decomposition of the ball $B(x, R)$ and the growth condition of ρ , we have

$$\begin{aligned} I(x) &= \sum_{j=-\infty}^{-1} \int_{B(x, 2^{j+1}R) \setminus B(x, 2^jR)} \frac{\rho(|x - y|)}{|x - y|^n} |f(y)| d\mu(y) \\ &\leq C \sum_{j=-\infty}^{-1} \frac{\rho^*(2^{j+1}R)}{(2^jR)^n} \int_{B(x, 2^{j+1}R)} |f(y)| d\mu(y) \\ &\leq CM^n f(x) \sum_{j=-\infty}^{-1} \int_{k_1 2^{j+1}R}^{k_2 2^{j+1}R} \frac{\rho(t)}{t} dt. \end{aligned} \quad (12)$$

Then, we use the overlapping property (see [29, 30]) to obtain

$$\begin{aligned} &\sum_{j=-\infty}^{-1} \int_{k_1 2^{j+1}R}^{k_2 2^{j+1}R} \frac{\rho(t)}{t} dt \\ &= \sum_{j=-\infty}^{-1} \int_0^{k_2 R} \chi_{[k_1 2^{j+1}R, k_2 2^{j+1}R]}(t) \frac{\rho(t)}{t} dt \\ &= \int_0^{k_2 R} \left(\sum_{j=-\infty}^{-1} \chi_{[k_1 2^{j+1}R, k_2 2^{j+1}R]}(t) \right) \frac{\rho(t)}{t} dt \\ &\leq \left(1 + \log_2 \left(\frac{k_2}{k_1} \right) \right) \int_0^{k_2 R} \frac{\rho(t)}{t} dt. \end{aligned} \quad (13)$$

If we use (10) and the doubling condition of ϕ , then we have

$$\sum_{j=-\infty}^{-1} \int_{k_1 2^{j+1}R}^{k_2 2^{j+1}R} \frac{\rho(t)}{t} dt \leq C\phi(k_2 R)^{(p/q)-1} \leq C\phi(R)^{(p/q)-1}. \quad (14)$$

Hence, $I(x) \leq CM^n f(x) \phi(R)^{(p/q)-1}$. $\square$

By letting $f \equiv 1$ or $f \equiv \chi_{B(a, r)}$, we have the following.

Corollary 6. *Let $1 \leq p < q < \infty$ and $B(x, R)$ be any ball in $\mathbb{R}^d$. If the functions ρ and ϕ satisfy inequality (10), then*

$$\int_{B(x, R)} \frac{\rho(|x - y|)}{|x - y|^n} d\mu(y) \leq C\phi(R)^{(p/q)-1} \quad (15)$$

and, for any ball $B(a, r) \subseteq \mathbb{R}^d$, one has

$$\begin{aligned} &\int_{B(x, R)} \frac{\rho(|x - y|)}{|x - y|^n} \chi_{B(a, r)}(y) d\mu(y) \\ &\leq CM^n \chi_{B(a, r)}(x) \phi(R)^{(p/q)-1}. \end{aligned} \quad (16)$$

Remark 7. These two inequalities will be used later to prove one of our main theorems. The next lemma presents an inequality involving the modified Hardy-Littlewood maximal operator M^n . This inequality is an important part of the proof of the weak type inequalities for I_α in [16, 19]. See [8] for similar results.

Lemma 8 (see [16]). *Let $1 \leq p < \infty$. If ϕ satisfies $\int_r^\infty (\phi(t)^p/t) dt \leq C\phi(r)^p$ for every $r > 0$, then, for any function $f \in L^{p, \phi}(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$, one has*

$$\int_{\mathbb{R}^d} |f(y)|^p M^n \chi_{B(a, r)}(y) d\mu(y) \leq Cr^n \phi(r)^p \|f\|_{L^{p, \phi}(\mu)}^p. \quad (17)$$

With Theorem 3, Corollary 6, and Lemma 8, we are now ready to prove the weak type inequalities for I_ρ on generalized nonhomogeneous Morrey spaces.

Theorem 9 (see [31]). *Let $1 \leq p < q < \infty$ and assume that $\sup_{r>0} \phi(r) = \infty$. If ρ and ϕ satisfy*

$$\int_r^\infty \frac{\phi(t)^p}{t} dt \leq C\phi(r)^p, \quad (18)$$

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C\phi(r)^{p/q}$$

for every $r > 0$, then, for any function $f \in L^{p, \phi}(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$, one has

$$\mu\left(\{x \in B(a, r) : |I_\rho f(x)| > \gamma\}\right) \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p, \phi}(\mu)}}{\gamma} \right)^q, \quad (19)$$

for every $\gamma > 0$.

Proof. Let $B(a, r)$ be any ball in $\mathbb{R}^d$. For every $x \in B(a, r)$ and $R > 0$, let

$$\begin{aligned} I_1(x) &= \int_{B(x, R)} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y), \\ I_2(x) &= \int_{\mathbb{R}^d \setminus B(x, R)} \frac{\rho(|x-y|)}{|x-y|^n} f(y) d\mu(y). \end{aligned} \quad (20)$$

Let $E_\gamma = \{x \in B(a, r) : |I_\rho f(x)| > \gamma\}$, for any $\gamma > 0$. Since $|I_\rho f(x)| \leq |I_1(x)| + |I_2(x)|$, we have

$$\begin{aligned} \mu(E_\gamma) &\leq \mu\left(\left\{x \in B(a, r) : |I_1(x)| > \frac{\gamma}{2}\right\}\right) \\ &\quad + \mu\left(\left\{x \in B(a, r) : |I_2(x)| > \frac{\gamma}{2}\right\}\right). \end{aligned} \quad (21)$$

By the dyadic decomposition of $\mathbb{R}^d \setminus B(x, R)$ and the growth condition of ρ , we have

$$\begin{aligned} |I_2(x)| &\leq \sum_{j=0}^{\infty} \int_{B(x, 2^{j+1}R) \setminus B(x, 2^jR)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| d\mu(y) \\ &\leq C \sum_{j=0}^{\infty} \frac{\rho^*(2^{j+1}R)}{(2^jR)^n} \int_{B(x, 2^{j+1}R)} |f(y)| d\mu(y). \end{aligned} \quad (22)$$

We use Hölder's inequality, the growth condition of μ , and the definition of $\|f\|_{L^{p,\phi}(\mu)}$ to obtain

$$\begin{aligned} |I_2(x)| &\leq C \sum_{j=0}^{\infty} \frac{\rho^*(2^{j+1}R)}{(2^jR)^n} \left(\int_{B(x, 2^{j+1}R)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\quad \times \left(\mu(B(x, 2^{j+1}R)) \right)^{1-(1/p)} \\ &\leq C \sum_{j=0}^{\infty} \rho^*(2^{j+1}R) \\ &\quad \times \left(\frac{1}{(2^{j+1}R)^n} \int_{B(x, 2^{j+1}R)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\phi}(\mu)} \sum_{j=0}^{\infty} \rho^*(2^{j+1}R) \phi(2^{j+1}R). \end{aligned} \quad (23)$$

By using the doubling condition of ϕ and the overlapping property, we have

$$\begin{aligned} &\sum_{j=0}^{\infty} \rho^*(2^{j+1}R) \phi(2^{j+1}R) \\ &= \sum_{j=0}^{\infty} \phi(2^{j+1}R) \int_{2k_1R}^{\infty} \chi_{[k_1 2^{j+1}R, k_2 2^{j+1}R]}(t) \frac{\rho(t)}{t} dt \\ &\leq C \sum_{j=0}^{\infty} \int_{2k_1R}^{\infty} \chi_{[k_1 2^{j+1}R, k_2 2^{j+1}R]}(t) \frac{\rho(t) \phi(t)}{t} dt \\ &= C \int_{2k_1R}^{\infty} \left(\sum_{j=0}^{\infty} \chi_{[k_1 2^{j+1}R, k_2 2^{j+1}R]}(t) \right) \frac{\rho(t) \phi(t)}{t} dt \\ &\leq C \left(1 + \log_2 \left(\frac{k_2}{k_1} \right) \right) \int_{2k_1R}^{\infty} \frac{\rho(t) \phi(t)}{t} dt. \end{aligned} \quad (24)$$

Now, we invoke the integral assumption on $\rho \cdot \phi$:

$$\sum_{j=0}^{\infty} \rho^*(2^{j+1}R) \phi(2^{j+1}R) \leq C \phi(2k_1R)^{p/q} \leq C \phi(R)^{p/q}. \quad (25)$$

Hence,

$$|I_2(x)| \leq C_3 \|f\|_{L^{p,\phi}(\mu)} \phi(R)^{p/q}. \quad (26)$$

Let $\tilde{\gamma} = (\gamma/2C_3 \|f\|_{L^{p,\phi}(\mu)})^{q/p}$. Remark that $\int_1^{\infty} (\phi(t)^p/t) dt \leq C \phi(1)^p$ implies that $\inf_{r>0} \phi(r) = 0$. Otherwise,

$$\infty = \inf_{r'>0} \phi(r')^p \int_1^{\infty} \frac{1}{t} dt \leq \int_1^{\infty} \frac{\phi(t)^p}{t} dt \leq C \phi(1)^p, \quad (27)$$

which is impossible. Now, from $\inf_{r>0} \phi(r) = 0 < \tilde{\gamma} < \infty = \sup_{r>0} \phi(r)$, we can find $k_0 \in \mathbb{Z}$ such that, for $R_1 = 2^{k_0}$ and $R_2 = 2^{k_0-1}$, we have

$$\phi(R_1) \leq \tilde{\gamma} \leq \phi(R_2). \quad (28)$$

Since $R_1/R_2 = 2$, there exists $C > 0$ such that $\phi(R_2) \leq C \phi(R_1)$. Hence,

$$\phi(R_1) \leq \tilde{\gamma} \leq C \phi(R_1). \quad (29)$$

Taking $R = R_1$, we obtain

$$|I_2(x)| \leq C_3 \|f\|_{L^{p,\phi}(\mu)} \phi(R_1)^{p/q} \leq C_3 \|f\|_{L^{p,\phi}(\mu)} \tilde{\gamma}^{p/q} \leq \frac{\gamma}{2}. \quad (30)$$

Consequently,

$$\mu(E_\gamma) \leq \mu\left(\left\{x \in B(a, r) : |I_1(x)| > \frac{\gamma}{2}\right\}\right). \quad (31)$$

We combine Hölder's inequality and the inequality (15) to obtain

$$\begin{aligned}
 |I_1(x)| &\leq \left(\int_{B(x, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\quad \times \left(\int_{B(x, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} d\mu(y) \right)^{1-(1/p)} \\
 &\leq C_4 \phi(R_1)^{((p/q)-1)(1-(1/p))} \\
 &\quad \times \left(\int_{B(x, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|^p d\mu(y) \right)^{1/p}.
 \end{aligned} \tag{32}$$

Finally, by using the last inequality and the Chebyshev inequality, we get

$$\begin{aligned}
 \mu(E_\gamma) &\leq \mu \left\{ x \in B(a, r) : \int_{B(x, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|^p d\mu(y) \right. \\
 &\quad \left. > \frac{(\gamma/2)^p}{C_4^p \phi(R_1)^{((p/q)-1)(p-1)}} \right\} \\
 &\leq \frac{2^p C_4^p \phi(R_1)^{((p/q)-1)(p-1)}}{\gamma^p} \\
 &\quad \times \int_{B(a, r)} \int_{B(x, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|^p d\mu(y) d\mu(x) \\
 &= \frac{C}{\gamma^p} \phi(R_1)^{((p/q)-1)(p-1)} \\
 &\quad \times \int_{\mathbb{R}^d} \int_{B(x, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|^p \chi_{B(a, r)}(x) \\
 &\quad \times (x) d\mu(y) d\mu(x).
 \end{aligned} \tag{33}$$

By virtue of the inequalities (16), (17), and (29) as well as the definition of $\tilde{\gamma}$, we get

$$\begin{aligned}
 \mu(E_\gamma) &\leq \frac{C}{\gamma^p} \phi(R_1)^{((p/q)-1)(p-1)} \int_{\mathbb{R}^d} |f(y)|^p \\
 &\quad \times \int_{B(y, R_1)} \frac{\rho(|x-y|)}{|x-y|^n} \chi_{B(a, r)}(x) d\mu(x) d\mu(y) \\
 &\leq \frac{C}{\gamma^p} \phi(R_1)^{((p/q)-1)p} \int_{\mathbb{R}^d} |f(y)|^p M^n \chi_{B(a, r)}(y) d\mu(y) \\
 &\leq \frac{C}{\gamma^p} \tilde{\gamma}^{(p^2/q)-p} r^n \phi(r)^p \|f\|_{L^{p, \phi}(\mu)}^p \\
 &\leq C r^n \phi(r)^p \left(\frac{\|f\|_{L^{p, \phi}(\mu)}}{\gamma} \right)^q,
 \end{aligned} \tag{34}$$

as desired. $\square$

Remark 10. Note that $\rho(t) = t^\alpha$, where $0 < \alpha < n$ satisfies the condition of Theorem 9 and, for this ρ , we obtain the weak type inequalities for I_α in Theorem 1.

3. Weak Type Inequalities for I_ρ via a Hedberg Type Inequality and Weak Type Inequalities for M^n

In this section, we will prove weak type inequalities for I_ρ using a different technique, namely, via a Hedberg type inequality and weak type inequalities for M^n . It turns out that some hypotheses can be removed. The Hedberg type inequality is presented in the following proposition.

Proposition 11 (see [25, 31]). *Let $1 \leq p < q < \infty$. If ρ and ϕ satisfy*

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t) \phi(t)}{t} dt \leq C \phi(r)^{p/q} \tag{35}$$

for every $r > 0$, then, for every $f \in L^{p, \phi}(\mu)$ and $x \in \mathbb{R}^d$, one has

$$|I_\rho f(x)| \leq C \left(M^n f(x)^{p/q} \|f\|_{L^{p, \phi}(\mu)}^{1-(p/q)} + \|f\|_{L^{p, \phi}(\mu)} \inf_{r>0} \phi(r)^{p/q} \right). \tag{36}$$

Proof. We adapt the proof of a Hedberg type inequality on generalized Morrey space in [25]. For every $x \in \mathbb{R}^d$ and $R > 0$, write $I_\rho f(x) = I_1(x) + I_2(x)$, where $I_1(x)$ and $I_2(x)$ are defined in the proof of Theorem 9. By using the inequalities (11) and (26), we get

$$|I_\rho f(x)| \leq C \left(M^n f(x) \phi(R)^{(p/q)-1} + \|f\|_{L^{p, \phi}(\mu)} \phi(R)^{p/q} \right). \tag{37}$$

Next, we separate the proof into the following two cases.

First Case ($M^n f(x) \leq 2C_\mu^{1-(1/p)} \|f\|_{L^{p, \phi}(\mu)} \inf_{r>0} \phi(r)$). In this case, we have

$$\begin{aligned}
 |I_\rho f(x)| &\leq C \left((2C_\mu^{1-(1/p)} \|f\|_{L^{p, \phi}(\mu)} \phi(R)) \phi(R)^{(p/q)-1} \right. \\
 &\quad \left. + \|f\|_{L^{p, \phi}(\mu)} \phi(R)^{p/q} \right) \\
 &\leq C \|f\|_{L^{p, \phi}(\mu)} \phi(R)^{p/q}
 \end{aligned} \tag{38}$$

for every $R > 0$. Hence, $|I_\rho f(x)| \leq C \|f\|_{L^{p, \phi}(\mu)} \inf_{r>0} \phi(r)^{p/q}$.

Second Case. $M^n f(x) > 2C_\mu^{1-(1/p)} \|f\|_{L^{p, \phi}(\mu)} \inf_{r>0} \phi(r)$. We use Hölder's inequality, the growth condition of μ , and the definition of $\|f\|_{L^{p, \phi}(\mu)}$ to obtain

$$\begin{aligned}
 &\frac{1}{R^n} \int_{B(x, R)} |f(y)| d\mu(y) \\
 &\leq \frac{1}{R^n} \left(\int_{B(x, R)} |f(y)|^p d\mu(y) \right)^{1/p} \mu(B(x, R))^{1-(1/p)} \\
 &\leq C_\mu^{1-(1/p)} \|f\|_{L^{p, \phi}(\mu)} \phi(R)
 \end{aligned} \tag{39}$$

for every $R > 0$. Hence,

$$M^n f(x) \leq C_\mu^{1-(1/p)} \|f\|_{L^{p,\phi}(\mu)} \sup_{r>0} \phi(r). \quad (40)$$

Since $\sup_{r>0} \phi(r) > 0$, we have

$$\inf_{r>0} \phi(r) < \frac{M^n f(x)}{2C_\mu^{1-(1/p)} \|f\|_{L^{p,\phi}(\mu)}} \leq \frac{1}{2} \sup_{r>0} \phi(r) < \sup_{r>0} \phi(r). \quad (41)$$

Thus, there exists $j_0 \in \mathbb{Z}$ such that

$$\phi(R_1) \leq \frac{M^n f(x)}{2C_\mu^{1-(1/p)} \|f\|_{L^{p,\phi}(\mu)}} \leq \phi(R_2) \quad (42)$$

for $R_1 = 2^{j_0}$ and $R_2 = 2^{j_0+1}$. Since $R_1/R_2 = 2$, there exists $C > 0$ such that

$$\phi(R_1) \leq \frac{M^n f(x)}{2C_\mu^{1-(1/p)} \|f\|_{L^{p,\phi}(\mu)}} \leq C\phi(R_1). \quad (43)$$

By choosing $R = R_1$ in the inequality (37) and using the inequality (43), we have

$$\begin{aligned} |I_\rho f(x)| &\leq C \left((2C_\mu^{1-(1/p)} \|f\|_{L^{p,\phi}(\mu)} C\phi(R_1)) \phi(R_1)^{(p/q)-1} \right. \\ &\quad \left. + \|f\|_{L^{p,\phi}(\mu)} \phi(R_1)^{p/q} \right) \\ &\leq C\phi(R_1)^{p/q} \|f\|_{L^{p,\phi}(\mu)} \\ &\leq C \left(\frac{M^n f(x)}{2C_\mu^{1-(1/p)} \|f\|_{L^{p,\phi}(\mu)}} \right)^{p/q} \|f\|_{L^{p,\phi}(\mu)} \\ &\leq CM^n f(x)^{p/q} \|f\|_{L^{p,\phi}(\mu)}^{1-(p/q)}. \end{aligned} \quad (44)$$

From these two cases, we obtain the inequality (36). $\square$

Sihwaningrum et al. [19] proved the weak type inequalities for M^n on generalized nonhomogeneous Morrey space by assuming that ϕ^p satisfies the integral condition; that is, $\int_r^\infty (\phi(t)^p/t)dt \leq C\phi(r)^p$ for every $r > 0$. In [19], the weak type inequalities for I_α are also proved by using the weak type inequalities for M^n . In this paper, we remove the integral condition of ϕ^p in the hypothesis of our proposition below. See [32, Theorem 2.3] and [33, Theorem 2.3] for such attempts.

Proposition 12 (see [31]). *Let $1 \leq p < \infty$; then, there exists a constant $C > 0$ such that, for any function $f \in L^{p,\phi}(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$, one has*

$$\mu \{x \in B(a, r) : M^n f(x) > \gamma\} \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^p, \quad (45)$$

for every $\gamma > 0$.

When $p > 1$, we have the strong boundedness; see [15, Theorem 1] and [34, Lemma 2.4] for the Lebesgue case and see [35, Theorem 4.3] for the strong $L^{p,\phi_1}(\mu)$ to $L^{p,\phi_2}(\mu)$ result and the weak $L^{p,\phi_1}(\mu)$ to $L^{p,\phi_2}(\mu)$ result with μ equal to the Lebesgue measure.

Proof. The proof is similar to that of strong boundedness of maximal operator on generalized nonhomogeneous Morrey spaces which is discussed in [34]. The difference is that in the final step we use the Chebyshev inequality, as we will see below. Consider the ball $B(a, r) \subseteq \mathbb{R}^d$. Let $x \in B(a, r)$ and let γ be any positive real number. For $y \in \mathbb{R}^d$, define $f_1(y) = \chi_{B(a, 2r)}(y)f(y)$ and $f_2(y) = f(y) - f_1(y)$. Note that

$$M^n f_2(x) = \sup_{R>0} \frac{1}{R^n} \int_{B(x, R)} (1 - \chi_{B(a, 2r)}(y)) |f(y)| d\mu(y). \quad (46)$$

Since $B(x, R) \subseteq B(a, 2r)$, for every $R < r$, we have

$$\frac{1}{R^n} \int_{B(x, R)} (1 - \chi_{B(a, 2r)}(y)) |f(y)| d\mu(y) = 0 \quad (47)$$

for every $R < r$. Hence,

$$M^n f_2(x) \leq \sup_{R>r} \frac{1}{R^n} \int_{B(x, R)} |f(y)| d\mu(y). \quad (48)$$

Observe that, for every $R > r$, we have

$$\begin{aligned} &\frac{1}{\phi(r) R^n} \int_{B(x, R)} |f(y)| d\mu(y) \\ &\leq \frac{C}{\phi(r) R^n} \left(\int_{B(x, R)} |f(y)|^p d\mu(y) \right)^{1/p} (\mu(B(x, R)))^{1-(1/p)} \\ &\leq \frac{C}{\phi(r)} \left(\frac{1}{R^n} \int_{B(x, R)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \|f\|_{L^{p,\phi}(\mu)}. \end{aligned} \quad (49)$$

Thus,

$$M^n f_2(x) \leq C\phi(r) \|f\|_{L^{p,\phi}(\mu)}. \quad (50)$$

Since $M^n f(x) \leq M^n f_1(x) + M^n f_2(x)$, we have

$$\begin{aligned} &\mu \{x \in B(a, r) : M^n f(x) > \gamma\} \\ &\leq \mu \left\{ x \in B(a, r) : M^n f_1(x) > \frac{\gamma}{2} \right\} \\ &\quad + \mu \left\{ x \in B(a, r) : M^n f_2(x) > \frac{\gamma}{2} \right\}. \end{aligned} \quad (51)$$

For the first term, we use the weak type inequalities for M^n on the nonhomogeneous Lebesgue space $L^p(\mu)$ (see [18]) to obtain

$$\begin{aligned} & \mu \left\{ x \in B(a, r) : M^n f_1(x) > \frac{\gamma}{2} \right\} \\ & \leq C \left(\frac{\|f_1\|_{L^p(\mu)}}{\gamma} \right)^p \\ & \leq \frac{C}{\gamma^p} \int_{B(a, 2r)} |f(y)|^p d\mu(y) \\ & \leq \frac{C}{\gamma^p} \phi(2r)^p (2r)^n \|f\|_{L^{p,\phi}(\mu)}^p \\ & \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^p. \end{aligned} \quad (52)$$

Meanwhile, for the second term, by using the Chebyshev inequality and the inequality (50), we have

$$\begin{aligned} & \mu \left\{ x \in B(a, r) : M^n f_2(x) > \frac{\gamma}{2} \right\} \\ & = \mu \left\{ x \in B(a, r) : M^n f_2(x)^p > \left(\frac{\gamma}{2} \right)^p \right\} \\ & \leq \frac{2^p}{\gamma^p} \int_{B(a, r)} M^n f_2(x)^p d\mu(x) \\ & \leq \frac{2^p}{\gamma^p} \int_{B(a, r)} C^p \phi(r)^p \|f\|_{L^{p,\phi}(\mu)}^p d\mu(x) \\ & \leq \frac{2^p}{\gamma^p} C^p \phi(r)^p \|f\|_{L^{p,\phi}(\mu)}^p \mu(B(a, r)) \\ & \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^p. \end{aligned} \quad (53)$$

Finally, by combining these two estimates, we obtain inequality (45). $\square$

With Propositions 11 and 12, we are now ready to prove the weak type inequalities for I_ρ on generalized nonhomogeneous Morrey spaces.

Theorem 13 (see [31]). *Let $1 \leq p < q < \infty$. If ρ and ϕ satisfy inequality (35), then, for any function $f \in L^{p,\phi}(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$, one has*

$$\mu \left(\left\{ x \in B(a, r) : |I_\rho f(x)| > \gamma \right\} \right) \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^q \quad (54)$$

for every $\gamma > 0$.

Proof. This proof is adapted from [25]. We replace γ by 2γ . Consider the ball $B(a, r) \subseteq \mathbb{R}^d$. By applying Proposition 11, we have

$$\begin{aligned} & \mu \left\{ x \in B(a, r) : |I_\rho f(x)| > 2\gamma \right\} \\ & \leq \mu \left\{ x \in B(a, r) : C \left(M^n f(x)^{p/q} \|f\|_{L^{p,\phi}(\mu)}^{1-(p/q)} \right. \right. \\ & \quad \left. \left. + \|f\|_{L^{p,\phi}(\mu)} \inf_{r>0} \phi(r)^{p/q} \right) > 2\gamma \right\} \\ & \leq \mu \left\{ x \in B(a, r) : CM^n f(x)^{p/q} \|f\|_{L^{p,\phi}(\mu)}^{1-(p/q)} > \gamma \right\} \\ & \quad + \mu \left\{ x \in B(a, r) : C \|f\|_{L^{p,\phi}(\mu)} \inf_{r>0} \phi(r)^{p/q} > \gamma \right\}. \end{aligned} \quad (55)$$

Observe that the second term in the most right-hand side of the above inequality vanishes, when

$$C \|f\|_{L^{p,\phi}(\mu)} \inf_{r>0} \phi(r)^{p/q} \leq \gamma. \quad (56)$$

So, to estimate the term, we can suppose that

$$C \|f\|_{L^{p,\phi}(\mu)} \inf_{r>0} \phi(r)^{p/q} > \gamma. \quad (57)$$

With this in mind, we calculate

$$\begin{aligned} & \mu \left\{ x \in B(a, r) : C \|f\|_{L^{p,\phi}(\mu)} \inf_{r>0} \phi(r)^{p/q} > \gamma \right\} \\ & = \mu(B(a, r)) \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^q. \end{aligned} \quad (58)$$

Meanwhile, by using Proposition 12, we have

$$\begin{aligned} & \mu \left\{ x \in B(a, r) : CM^n f(x)^{p/q} \|f\|_{L^{p,\phi}(\mu)}^{1-(p/q)} > \gamma \right\} \\ & \leq \mu \left\{ x \in B(a, r) : M^n f(x) > \left(\frac{\gamma}{C \|f\|_{L^{p,\phi}(\mu)}^{1-(p/q)}} \right)^{q/p} \right\} \\ & \leq Cr^n \phi(r)^p \|f\|_{L^{p,\phi}(\mu)}^p \left(\frac{\gamma}{C \|f\|_{L^{p,\phi}(\mu)}^{1-(p/q)}} \right)^{-q} \\ & \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^q. \end{aligned} \quad (59)$$

By summing the two previous estimates, we get the desired inequality. $\square$

Remarks 1. (i) Note that the hypotheses $\int_r^\infty (\phi(t)^p/t) dt \leq C\phi(r)^p$ in Theorem 9 are not included in Theorem 13, since we can prove the weak type inequalities for M^n without this condition.

(ii) The conditions on ϕ , namely, $\inf_{r>0} \phi(r) = 0$ and $\sup_{r>0} \phi(r) = \infty$, are not included in the hypotheses in Theorem 13. However, we have to use the weak type inequalities for M^n on generalized nonhomogeneous Morrey spaces and a Hedberg type inequality for I_ρ in the proof of Theorem 13.

4. Boundedness of Singular Integral Operators

Proposition 12 carries over to the singular integral operator whose definition is given in [1]. Recall that the singular integral operator T is a bounded linear operator on $L^2(\mu)$ for which there exists a function K that satisfies three properties listed below.

(4.a) There exists $C > 0$ such that $|K(x, y)| \leq C/|x - y|^n$ for all $x \neq y$.

(4.b) There exist $\varepsilon > 0$ and $C > 0$ such that

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq C \frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}} \quad (60)$$

if $|x - y| \geq 2|x - z|$ with $x \neq y$.

(4.c) If f is a bounded μ -measurable function with a compact support, then we have

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) d\mu(y) \quad \forall x \notin \text{supp}(f). \quad (61)$$

As for this singular integral operator T , the following result is due to Nazarov, Treil, and Volberg.

Proposition 14 (see [1, 2]). *The singular operator T is bounded on $L^p(\mu)$ for $1 < p < \infty$. Moreover, there exists a constant $C > 0$ such that*

$$\mu \{x \in \mathbb{R}^d : |Tf(x)| > \gamma\} \leq C \frac{\|f\|_{L^1(\mu)}}{\gamma} \quad (62)$$

for every $f \in L^1(\mu)$ and every $\gamma > 0$.

Theorem 15. *Let T be a singular integral operator. Let $1 \leq p < \infty$. In addition to the doubling condition, assume that*

$$\int_r^\infty \frac{\phi(t)}{t} dt \leq C\phi(r) \quad (63)$$

for every $r > 0$. Then, there exists a constant $C > 0$ such that, for any function $f \in L^{p,\phi}(\mu)$ and any ball $B(a, r) \subseteq \mathbb{R}^d$, one has

$$\mu \{x \in B(a, r) : |Tf(x)| > \gamma\} \leq Cr^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma} \right)^p \quad (64)$$

for every $\gamma > 0$.

Proof. The proof is a modification of that of Proposition 12. We decompose $f = f_1 + f_2$ as before. The treatment of f_1 is the same as that in Proposition 12 but by using the weak type inequality for T in Proposition 14. We need to take care of f_2 . By the condition (4.a), Hölder's inequality, and the growth condition of μ , we have

$$\begin{aligned} |Tf_2(x)| &\leq C \int_{\mathbb{R}^d \setminus B(x, r)} \frac{|f(y)|}{|x - y|^n} d\mu(y) \\ &\leq C \sum_{j=0}^{\infty} \frac{1}{(2^j r)^n} \int_{B(x, 2^{j+1} r)} |f(y)| d\mu(y) \\ &\leq C \sum_{j=0}^{\infty} \frac{1}{(2^j r)^n} \left(\int_{B(x, 2^{j+1} r)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\quad \times \left(\mu(B(x, 2^{j+1} r)) \right)^{1-(1/p)} \\ &\leq C \sum_{j=0}^{\infty} \left(\frac{1}{(2^{j+1} r)^n} \int_{B(x, 2^{j+1} r)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \sum_{j=0}^{\infty} \phi(2^{j+1} r) \|f\|_{L^{p,\phi}(\mu)} \\ &\leq C \|f\|_{L^{p,\phi}(\mu)} \int_r^\infty \frac{\phi(t)}{t} dt. \end{aligned} \quad (65)$$

If we use our integrability assumption, then we have a pointwise estimate:

$$|Tf_2(x)| \leq C\phi(r) \|f\|_{L^{p,\phi}(\mu)}. \quad (66)$$

So, we are done. $\square$

Remark 16. If we define the generalized weak Morrey space of nonhomogeneous type $wL^{p,\phi}(\mu)$ to be the set of all μ -measurable functions f such that

$$\begin{aligned} &\|f\|_{wL^{p,\phi}(\mu)} \\ &:= \sup_{B(a,r) \subseteq \mathbb{R}^d, \gamma > 0} \frac{\gamma (\mu \{x \in B(a, r) : |f(x)| > \gamma\})^{1/p}}{r^{n/p} \phi(r)} < \infty, \end{aligned} \quad (67)$$

then the inequality (64) amounts to the boundedness of T from $L^{p,\phi}(\mu)$ to $wL^{p,\phi}(\mu)$. Similarly, our previous results can be translated into this language. In the following section, we will use these notations for convenience.

5. Generalized Nonhomogeneous Orlicz-Morrey Spaces

Our results above can be carried over to generalized nonhomogeneous Orlicz-Morrey spaces. We first formulate our main results and then prove them later in Sections 5.1–5.3.

Recall that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function, if Φ is bijective and convex. We define the Φ -average of f over a ball $B(a, r)$ as follows:

$$\|f\|_{\Phi, B(a, r)} = \inf \left\{ \lambda > 0 : \frac{1}{r^n} \int_{B(a, r)} \Phi \left(\frac{|f(y)|}{\lambda} \right) d\mu(y) \leq 1 \right\}. \quad (68)$$

The generalized nonhomogeneous Orlicz-Morrey space $L^{\Phi, \phi}(\mu) = L^{\Phi, \phi}(\mathbb{R}^d, \mu)$ is the set of all $f \in L^1_{\text{loc}}(\mu)$ for which the norm

$$\|f\|_{L^{\Phi, \phi}(\mu)} = \sup_{B(a, r) \subseteq \mathbb{R}^d} \frac{1}{\phi(r)} \|f\|_{\Phi, B(a, r)} \quad (69)$$

is finite. Note that if $\Phi(t) = t^p$, then $L^{\Phi, \phi}(\mu) = L^{p, \phi}(\mu)$. About the structure of this function space, we have the following.

Theorem 17. *Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a Young function and let ϕ satisfy the two conditions (1.a) and (1.b) as usual. Then, $L^{\Phi, \phi}(\mu)$ is a Banach space.*

We define the generalized weak Orlicz-Morrey spaces of nonhomogeneous type as follows. For a Young function Φ , the generalized weak Orlicz-Morrey space of nonhomogeneous type $wL^{\Phi, \phi}(\mu) = wL^{\Phi, \phi}(\mathbb{R}^d, \mu)$ is the set of all μ -measurable functions f for which the norm

$$\|f\|_{wL^{\Phi, \phi}(\mu)} = \sup_{B(a, r) \subseteq \mathbb{R}^d, \gamma > 0} \frac{\gamma}{\phi(r)} \|\chi_{\{|f| > \gamma\}}\|_{\Phi, B(a, r)} \quad (70)$$

is finite. Write $wL^{p, \phi}(\mu) = wL^{\Phi, \phi}(\mu)$, when $\Phi(t) = t^p$. It is not so hard to prove

$$\|f\|_{wL^{\Phi, \phi}(\mu)} \leq \|f\|_{L^{\Phi, \phi}(\mu)} \quad (71)$$

for all μ -measurable functions f from the inequality

$$\gamma \chi_{\{|f| > \gamma\}} \leq |f|. \quad (72)$$

By taking $f(x) = (1/|x|)\chi_{[-1, 1]}(x)$ and $\phi(t) = (1/t) \log(3 + t)$ and μ is the Lebesgue measure on $\mathbb{R}$, we see that $f \in wL^{1, \phi}(\mathbb{R}, \mu) \setminus L^{1, \phi}(\mathbb{R}, \mu)$, showing that $wL^{1, \phi}(\mathbb{R}, \mu)$ is a proper superset of $L^{1, \phi}(\mathbb{R}, \mu)$.

Theorem 18. *Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a Young function and let ϕ satisfy the two conditions (1.a) and (1.b) as usual. Then, $wL^{\Phi, \phi}(\mu)$ is a quasi-Banach space. More precisely,*

- (1) $\|f\|_{wL^{\Phi, \phi}(\mu)} = 0$ if and only if $f = 0$;
- (2) $\|cf\|_{wL^{\Phi, \phi}(\mu)} = |c| \|f\|_{wL^{\Phi, \phi}(\mu)}$ for all $c \in \mathbb{C}$ and $f \in wL^{\Phi, \phi}(\mu)$;
- (3) if $\{f_k\}_{k=1}^{\infty}$ is a sequence in $wL^{\Phi, \phi}(\mu)$ such that

$$\lim_{k_1, k_2 \rightarrow \infty} \|f_{k_1} - f_{k_2}\|_{wL^{\Phi, \phi}(\mu)} = 0. \quad (73)$$

Then, there exists $g \in wL^{\Phi, \phi}(\mu)$ such that

$$\lim_{k \rightarrow \infty} \|g - f_k\|_{wL^{\Phi, \phi}(\mu)} = 0. \quad (74)$$

We prove the following boundedness result on generalized nonhomogeneous Orlicz-Morrey spaces.

Theorem 19. *Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a Young function and let ϕ satisfy the two conditions (1.a) and (1.b) as usual. Then, the maximal operator M is bounded from $L^{\Phi, \phi}(\mu)$ to $wL^{\Phi, \phi}(\mu)$. If we assume that*

$$\int_R^\infty \frac{\phi(t)}{t} dt \leq C\phi(R) \quad (R > 0) \quad (75)$$

and that Φ satisfies the doubling condition, then the singular integral operator T is bounded from $L^{\Phi, \phi}(\mu)$ to $wL^{\Phi, \phi}(\mu)$.

Theorem 20. *Let $\rho : (0, \infty) \rightarrow (0, \infty)$ and, for some $b \in (0, 1]$, $\phi : (0, \infty) \rightarrow (0, \infty)$, satisfy*

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_0^r \frac{\phi(t) \rho(t)}{t} dt \leq C\phi(r)^b \quad (76)$$

for every $r > 0$. Suppose that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function with the doubling condition. Set

$$\psi(t) = \phi(t)^b, \quad \Psi(t) = \Phi(t^{1/b}), \quad t \in [0, \infty). \quad (77)$$

Then,

$$\|I_\rho f\|_{wL^{\Psi, \psi}(\mu)} \leq C \|f\|_{L^{\Phi, \phi}(\mu)}. \quad (78)$$

5.1. Proof of Theorems 17 and 18. We start with a lemma.

Lemma 21. *Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a Young function with the doubling property:*

$$\Phi(2t) \leq C\Phi(t), \quad t \in [0, \infty). \quad (79)$$

For μ -measurable functions f and g and a ball $B(a, r)$, one has

$$\|f + g\|_{\Phi, B(a, r)} \leq \|f\|_{\Phi, B(a, r)} + \|g\|_{\Phi, B(a, r)}. \quad (80)$$

Proof. If $f = 0$ μ -a.e. or $g = 0$ μ -a.e., then we have the equality trivially; so let us assume that $f \neq 0$ μ -a.e. and $g \neq 0$ μ -a.e. Then, by virtue of the convexity, we have

$$\begin{aligned}
 & \frac{1}{r^n} \int_{B(a,r)} \Phi \left(\frac{|f(x) + g(x)|}{\|f\|_{\Phi, B(a,r)} + \|g\|_{\Phi, B(a,r)}} \right) d\mu(x) \\
 & \leq \frac{\|f\|_{\Phi, B(a,r)}}{\|f\|_{\Phi, B(a,r)} + \|g\|_{\Phi, B(a,r)}} \\
 & \quad \times \frac{1}{r^n} \int_{B(a,r)} \Phi \left(\frac{|f(x)|}{\|f\|_{\Phi, B(a,r)}} \right) d\mu(x) \\
 & \quad + \frac{\|g\|_{\Phi, B(a,r)}}{\|f\|_{\Phi, B(a,r)} + \|g\|_{\Phi, B(a,r)}} \\
 & \quad \times \frac{1}{r^n} \int_{B(a,r)} \Phi \left(\frac{|g(x)|}{\|g\|_{\Phi, B(a,r)}} \right) d\mu(x) \\
 & \leq \frac{\|f\|_{\Phi, B(a,r)}}{\|f\|_{\Phi, B(a,r)} + \|g\|_{\Phi, B(a,r)}} \\
 & \quad + \frac{\|g\|_{\Phi, B(a,r)}}{\|f\|_{\Phi, B(a,r)} + \|g\|_{\Phi, B(a,r)}} = 1.
 \end{aligned} \tag{81}$$

From the definition of the quantity $\|f + g\|_{\Phi, B(a,r)}$, we obtain the inequality. $\square$

Lemma 22. If $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function, then

$$\frac{1}{r^n} \int_{B(a,r)} |f(y)| d\mu(y) \leq C \|f\|_{\Phi, B(a,r)} \tag{82}$$

for any ball $B(a, r)$ and μ -measurable function f .

Proof. A normalization allows us to assume that $\|f\|_{\Phi, B(a,r)} = 1$; our target will be to prove

$$\frac{1}{r^n} \int_{B(a,r)} |f(y)| d\mu(y) \leq C. \tag{83}$$

In view of the growth condition, we may suppose that $|f|$ assumes its value in $\{0\} \cup [1, \infty)$. Since Φ is a Young function, we have

$$\Phi(t) \geq \Phi(1)t, \quad t \in \{0\} \cup [1, \infty). \tag{84}$$

Therefore,

$$\begin{aligned}
 & \Phi(1) \frac{1}{r^n} \int_{B(a,r)} |f(y)| d\mu(y) \\
 & \leq \frac{1}{r^n} \int_{B(a,r)} \Phi(|f(y)|) d\mu(y) \\
 & = \lim_{\varepsilon \downarrow 0} \frac{1}{r^n} \int_{B(a,r)} \Phi \left(\frac{|f(y)|}{\|f\|_{\Phi, B(a,r)} + \varepsilon} \right) d\mu(y) \leq 1.
 \end{aligned} \tag{85}$$

So, we are done. $\square$

Now, we are ready for the proof of Theorem 17.

Proof of Theorem 17. In view of Lemma 21, $L^{\Phi, \phi}(\mu)$ is a normed space. So, we need to prove the completeness. To this end, we choose a sequence $\{f_k\}_{k=1}^{\infty}$ of μ -measurable function such that

$$\sum_{k=1}^{\infty} \|f_k\|_{L^{\Phi, \phi}(\mu)} < \infty. \tag{86}$$

Denote by O the origin. Then, we have

$$\sum_{k=1}^{\infty} \frac{1}{\phi(L)} L^n \|f_k\|_{L^1(B(O,L), \mu)} \leq C \sum_{k=1}^{\infty} \|f_k\|_{L^{\Phi, \phi}(B(O,L), \mu)} < \infty \tag{87}$$

from Lemma 22. This implies that $\sum_{k=1}^{\infty} |f_k(x)|$ is finite μ -a.e. on $B(O, L)$ for all $L \in \mathbb{N}$. Hence, $\sum_{k=1}^{\infty} |f_k(x)|$ is finite μ -a.e. on $\mathbb{R}^d$. With this in mind, let us set $g(x) = \sum_{k=1}^{\infty} f_k(x)$ whenever the series is absolutely convergent; otherwise set $g(x) = 0$.

We fix a ball $B(a, r)$. Then, we have

$$\begin{aligned}
 & \|g - f_1 - f_2 - \cdots - f_k\|_{\Phi, B(a,r)} \\
 & = \inf \left\{ \lambda > 0 \right. \\
 & \quad : \int_{B(a,r)} \Phi \left(\frac{|g(x) - f_1(x) - f_2(x) - \cdots - f_k(x)|}{\lambda} \right) \\
 & \quad \times d\mu(x) \leq r^n \Big\} \\
 & \leq \inf \left\{ \lambda > 0 \right. \\
 & \quad : \int_{B(a,r)} \liminf_{K \rightarrow \infty} \Phi \left(\frac{1}{\lambda} \left| \sum_{j=k+1}^K f_j(x) \right| \right) d\mu(x) \leq r^n \Big\} \\
 & \leq \inf \left\{ \lambda > 0 \right. \\
 & \quad : \liminf_{K \rightarrow \infty} \int_{B(a,r)} \Phi \left(\frac{1}{\lambda} \sum_{j=k+1}^K |f_j(x)| \right) d\mu(x) \leq r^n \Big\} \\
 & \leq \phi(r) \sum_{j=k+1}^{\infty} \|f_j\|_{L^{\Phi, \phi}(\mu)}.
 \end{aligned} \tag{88}$$

As a result, $g \in L^{\Phi, \phi}(\mu)$ and

$$\|g - f_1 - f_2 - \cdots - f_k\|_{L^{\Phi, \phi}(\mu)} \leq \sum_{j=k+1}^{\infty} \|f_j\|_{L^{\Phi, \phi}(\mu)}. \tag{89}$$

So $\sum_{j=1}^{\infty} f_j$ converges to g in $L^{\Phi, \phi}(\mu)$. $\square$

The proof of Theorem 18 is similar; we use the embedding

$$\|f\|_{wL^1(B(a,r), \mu)} \leq C \|f\|_{wL^{\Phi, \phi}(\mu)} \tag{90}$$

which follows from Lemma 22.

5.2. Proof of Theorem 19. We first concentrate on the maximal operator; we modify the argument to prove the boundedness of singular integral operators later.

The proof hinges upon the decomposition in Proposition 12, keeping the same notation as before. As for f_2 , we have a pointwise estimate, so that a small modification works. Also, we normalize $\|f\|_{L^{\Phi,\phi}(\mu)} = 1$.

Let us concentrate on f_1 . Let us establish

$$\gamma \|\chi_{\{Mf_1 > 2\gamma\}}\|_{\Phi, B(a,r)} \leq C \quad (91)$$

for any $\gamma > 0$, where the constant $C > 0$ is independent of γ and f . Write $\Lambda = \|\chi_{\{Mf_1 > 2\gamma\}}\|_{\Phi, B(a,r)}$. Then, we have

$$\begin{aligned} & \frac{\mu\{Mf_1 > 2\gamma\}}{r^n} \Phi\left(\frac{1}{\Lambda}\right) \\ &= \frac{1}{r^n} \int_{B(a,r)} \Phi\left(\frac{\chi_{\{Mf_1 > 2\gamma\}}(y)}{\Lambda}\right) d\mu(y) = 1 \end{aligned} \quad (92)$$

by virtue of the dominated convergence theorem. So, we have

$$1 \leq C \frac{1}{r^n \gamma} \Phi\left(\frac{1}{\Lambda}\right) \|f\chi_{\{|f| > \gamma\}}\|_{L^1(B(a,2r), \mu)}. \quad (93)$$

So

$$r^n \leq C \int_{B(a,2r)} \Phi\left(\frac{1}{\Lambda}\right) \frac{|f(x)| \chi_{\{|f| > \gamma\}}(x)}{\gamma} d\mu(x). \quad (94)$$

Since

$$\frac{|f(x)| \chi_{\{|f| > \gamma\}}(x)}{\gamma} \in \{0\} \cup [1, \infty), \quad (95)$$

we have (by the convexity of Φ)

$$r^n \leq C \int_{B(a,2r)} \Phi\left(\frac{|f(x)| \chi_{\{|f| > \gamma\}}(x)}{\gamma \Lambda}\right) d\mu(x). \quad (96)$$

In view of the doubling property, we are done with the maximal operator.

As for the singular integral operator, we combine the above proof and that of Theorem 15. We mimic the argument above for f_1 , while we use estimate (66) obtained in the proof of Theorem 15. We omit the further details.

5.3. Proof of Theorem 20. We start with the proof of a Hedberg type inequality. Let $R > 0$. Then, as in (37), we have

$$\begin{aligned} |I_\rho f(x)| &\leq \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| d\mu(y) \\ &\leq \int_{B(x,R)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| d\mu(y) \\ &\quad + \int_{\mathbb{R}^d \setminus B(x,R)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| d\mu(y) \\ &\leq C \left(M^n f(x) \phi(R)^{b-1} + \|f\|_{L^{\Phi,\phi}(\mu)} \phi(R)^b \right). \end{aligned} \quad (97)$$

So we are led to

$$|I_\rho f(x)| \leq C \left(M^n f(x)^b \|f\|_{L^{\Phi,\phi}(\mu)}^{1-b} + \|f\|_{L^{\Phi,\phi}(\mu)} \inf_{r>0} \phi(r)^b \right), \quad (98)$$

as we did in Proposition 11.

So we have to prove

$$\begin{aligned} & \frac{\gamma}{\psi(r)} \left\| \chi_{\{x \in B(a,r) : \|f\|_{L^{\Phi,\phi}(\mu)} \inf_{r>0} \phi(r)^b > \gamma\}} \right\|_{\Psi, B(a,r)} \leq C \|f\|_{L^{\Phi,\phi}(\mu)}, \\ & \frac{\gamma}{\psi(r)} \left\| \chi_{\{x \in B(a,r) : M^n f(x)^b \|f\|_{L^{\Phi,\phi}(\mu)}^{1-b} > \gamma\}} \right\|_{\Psi, B(a,r)} \leq C \|f\|_{L^{\Phi,\phi}(\mu)}. \end{aligned} \quad (99)$$

As for the first inequality, we use the following observation:

$$\begin{aligned} & \frac{\gamma}{\psi(r)} \left\| \chi_{\{x \in B(a,r) : \|f\|_{L^{\Phi,\phi}(\mu)} \inf_{r'>0} \phi(r')^b > \gamma\}} \right\|_{\Psi, B(a,r)} \\ & \leq \frac{1}{\psi(r)} \|f\|_{L^{\Phi,\phi}(\mu)} \inf_{r'>0} \phi(r')^b. \end{aligned} \quad (100)$$

In view of the definition of ψ , we are done with the estimate.

As for the second inequality, we proceed as follows:

$$\begin{aligned} & \frac{\gamma}{\psi(r)} \left\| \chi_{\{x \in B(a,r) : M^n f(x)^b \|f\|_{L^{\Phi,\phi}(\mu)}^{1-b} > \gamma\}} \right\|_{\Psi, B(a,r)} \\ &= \frac{\gamma}{\phi(r)^b} \left(\left\| \chi_{\{x \in B(a,r) : M^n f(x) \|f\|_{L^{\Phi,\phi}(\mu)}^{1/b-1} > \gamma^{1/b}\}} \right\|_{\Phi, B(a,r)} \right)^b \\ &= \left(\frac{\gamma^{1/b}}{\phi(r)} \left\| \chi_{\{x \in B(a,r) : M^n f(x) \|f\|_{L^{\Phi,\phi}(\mu)}^{1/b-1} > \gamma^{1/b}\}} \right\|_{\Phi, B(a,r)} \right)^b \\ &\leq C \|f\|_{L^{\Phi,\phi}(\mu)}. \end{aligned} \quad (101)$$

Here, for the last inequality, we used Theorem 19.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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