Research Article

L^p Approximation Strategy by Positive Linear Operators

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Received 31 March 2013; Accepted 28 June 2013

Academic Editor: Satit Saejung

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Two important techniques to achieve the Jackson type estimation by Kantorovich type positive linear operators in L^p spaces are introduced in the present paper, and three typical applications are given.

1. Introduction

It is well known that Kantorovich type operators are usually used for approximation in L^p spaces. Let $f(x) \in L^p_{[0,1]}$, the class of all p power integrable functions on the interval $[0, 1]$, and $\mathcal{K}_{n,j}(x)$ ($j = 0, 1, 2, \dots, n$, or $j = 1, 2, \dots, n$) be given kernels satisfying

$$\sum_{j=1}^n \mathcal{K}_{n,j}(x) = 1 \quad \left(\text{or } \sum_{j=0}^n \mathcal{K}_{n,j}(x) = 1 \right), \quad (1)$$

$$\mathcal{K}_{n,j}(x) \geq 0, \quad x \in [0, 1].$$

This paper discusses how well the function $f(x) \in L^p_{[0,1]}$ can be approximated by the discrete Kantorovich-type operators such as

$$L_n(f, x) = (n+1) \sum_{j=0}^n \mathcal{K}_{n,j}(x) \int_{j/(n+1)}^{(j+1)/(n+1)} f(t) dt \quad (2)$$

or

$$L_n(f, x) = n \sum_{j=1}^n \mathcal{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} f(t) dt, \quad (3)$$

and the L^p approximation is characterized by

$$\|f - L_n(f)\|_{L^p} \leq C\omega(f, \epsilon_n)_{L^p}, \quad (4)$$

where ϵ_n , a positive number depending on n only (such as $n^{-1/2}$, n^{-1}), is called *Jackson order* of L^p approximation; and

$C > 0$, a constant depending sometimes upon p as well as the kernels $\{\mathcal{K}_{n,j}(x)\}$ (e.g., the kernels of the Shepard operators; it contains a parameter λ ; see (10)), is called *Jackson constant*. Since the kernel $\{\mathcal{K}_{n,j}(x)\}$ can have two types, decided by the summation indices, respectively (see (1)), the Kantorovich type operators are therefore defined by (2) or (3). However, with similar arguments, we can only investigate the positive linear operators of the form of (3).

Sometimes, we need to write $Q_n(x, t) = \mathcal{K}_{n,j}(x)$ for $(j-1)/n \leq t < j/n$, $j = 1, 2, \dots, n$; then the operators (3) can be written as an integral form:

$$L_n(f, x) = n \int_0^1 Q_n(x, t) f(t) dt; \quad (5)$$

hence,

$$\int_0^1 Q_n(x, t) dt = n^{-1}. \quad (6)$$

This means, for Kantorovich type operators, there does not exist any difference whether (3) or (5) is taken. In particular, we give the following examples.

When the kernel function $Q_n(x, t)$ satisfies

$$\frac{\partial}{\partial x} Q_n(x, t) = \frac{n}{\phi(x)} Q_n(x, t) (t - x), \quad (7)$$

where $\phi(x) > 0$ satisfying $\phi(0) = \phi(1) = 0$ is a polynomial with a degree at most 2, then the operators $L_n(f, x)$ have exponential type, which have been studied deeply in [1–3], for example.

When the kernel function $\mathcal{K}_{n,j}(x)$ in (3) is taken as

$$R_j(x) = \frac{P_j(x)}{\sum_{k=1}^n P_k(x)}, \quad (8)$$

$$P_k(x) = x^{\lambda_k} \prod_{l=1}^k \left(\frac{l}{n}\right)^{-\Delta\lambda_l},$$

where $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}, \dots\}$ is a Müntz system satisfying that $\Delta\lambda_n = \lambda_n - \lambda_{n-1} \geq Mn$ (M is a given positive number), then the operator $L_n(f, x)$ becomes the rational Müntz operator:

$$M_n(f, x) = n \sum_{k=1}^n R_k(x) \int_{(k-1)/n}^{k/n} f(t) dt; \quad (9)$$

readers can refer to [4, 5].

When the kernel function $\mathcal{K}_{n,j}(x)$ of (2) is taken as

$$r_j(x) = \frac{|x - j/n|^{-\lambda}}{\sum_{k=0}^n |x - k/n|^{-\lambda}} \quad (\lambda > 1), \quad (10)$$

then the operator $L_n(f, x)$ is the well-known Shepard operator

$$S_{n,\lambda}(f, x) = (n+1) \sum_{k=0}^n r_k(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt; \quad (11)$$

one can check [6, 7], for example.

When the kernel $\mathcal{K}_{n,j}(x)$ in (2) is taken as

$$P_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad (12)$$

then we obtain the Kantorovich-Bernstein operator

$$B_n(f, x) = (n+1) \sum_{k=0}^n P_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt, \quad (13)$$

which has been studied most widely among the positive linear operators of the form (2) (see [8–23]). Interested readers could also refer to the related papers for the other similar operators.

This paper will take the above three typical operators (9), (11), and (13) as examples to illustrate two quantitative methods on L^p approximation. Over discussion, we find out that the Jackson order in L^p spaces to approximate $f(x) \in L^p_{[0,1]}$ by the operators in (3) or (5) is decided completely by the kernels $\{\mathcal{K}_{n,j}(x)\}_{j=1}^n$, or by the kernel function $Q_n(x, t)$. Therefore, on applying this idea, we need only to investigate the properties of the kernels to obtain the magnitude of the Jackson order of the corresponding operators, which seems to be a different approach from the past L^p approximating methods.

2. Notations and Terminologies

In this section, we give all preliminary notations and terminologies. For $f(x) \in L^p_{[0,1]}$, the usual L^p norm is defined by

$$\|f\|_{L^p_{[a,b]}} = \left\{ \int_a^b |f(x)|^p dx \right\}^{1/p}, \quad \|f\|_{L^p} = \|f\|_{L^p_{[0,1]}}, \quad (14)$$

and the L^p modulus of continuity of $f \in L^p_{[0,1]}$ is defined by

$$\begin{aligned} \omega(f, \delta)_{L^p} &= \omega(f, \delta)_{L^p_{[0,1-\delta]}} \\ &= \sup_{0 < t \leq \delta} \|f(x+t) - f(x)\|_{L^p_{[0,1-\delta]}}. \end{aligned} \quad (15)$$

To understand clearly L^p approximation by the positive linear operators, we need to make analysis of the kernels corresponding to the operators. Hence, some new terminologies on the kernels will be given and explained by some examples. For convenience, C always indicates an absolute positive constant and C_x indicates a positive constant depending upon at most x . C or C_x may have different values at different occurrences even at the same line. Sometimes we write $\mathcal{K}_n = \mathcal{K}_n(x) = \{\mathcal{K}_{n,j}(x)\}_{j=1}^n$.

Definition 1. For any $x \in [0, 1]$, if there exists a real sequence $\{\Phi(k)\}_{k=1}^\infty$ with $0 < \Phi(k) < 1$, $k = 1, 2, \dots$, satisfying

$$\mathcal{K}_{n,r}(x) \leq C\Phi(|[nx] - r| + 1), \quad r = 1, 2, \dots, n, \quad (16)$$

then $\{\Phi(k)\}_{k=1}^\infty$ is called a *global domination* of $\mathcal{K}_n(x)$, or we say $\mathcal{K}_n(x)$ is globally dominated by $\{\Phi(k)\}_{k=1}^\infty$, where $[x]$ indicates the greatest integer not exceeding x . In particular, if $\Phi(k) = q^k$, $0 < q < 1$, $k = 1, 2, \dots$, then the kernel sequence $\mathcal{K}_n(x)$ is called a *globally dominated geometrical sequence*, or being dominated by a global geometrical sequence, or having (global) *geometrical order*. If $\Phi(k) = k^{-\rho}$, $\rho > 0$, $k = 1, 2, \dots$, then the kernel sequence $\mathcal{K}_n(x)$ is called a *globally dominated arithmetic sequence with power ρ* , or being dominated by a global arithmetic sequence with power ρ , or having (global) *ρ -arithmetic order*.

Definition 2. For any $x \in [0, 1]$, if there exist a sequence $\{\Phi(k)\}_{k=1}^\infty$ with $0 < \Phi(k) < 1$, $k = 1, 2, \dots$ and an integer subset $N_x \subset \{1, 2, \dots, n\}$ satisfying that

$$\mathcal{K}_{n,r}(x) \leq C\Phi(|[nx] - r| + 1), \quad r \in N_x, \quad (17)$$

then $\{\Phi(k)\}_{k=1}^\infty$ is called a *local domination* of $\mathcal{K}_n(x)$. Like in Definition 1, a *locally dominated geometrical sequence* of $\mathcal{K}_n(x)$ and a *locally dominated ρ -arithmetic sequence* can be defined similarly.

The conceptions of Definitions 1 and 2 will be illustrated by the following examples.

Example 3. The kernel functions $R_k(x)$ of the rational Müntz operators defined by (9) satisfy

$$R_k(x) \leq C_M \exp(-C_M |[nx] - k|), \quad k = 0, 1, 2, \dots, n \quad (18)$$

(see [4] or [5, Lemma 1]). The kernel functions $R_k(x)$ then have geometrical order.

Example 4. The kernels $r_k(x)$ of the Shepard operators $S_{n,\lambda}(f, x)$ ($\lambda > 1$) defined by (11) satisfy

$$r_k(x) \leq \left(\frac{2}{|[(n+1)x] - k| + 1} \right)^\lambda, \quad k = 0, 1, 2, \dots, n \quad (19)$$

(see [12, Lemma 1]). The kernels $r_k(x)$ have λ -arithmetic order.

Example 5. The kernels $P_{n,k}(x)$ of the Kantorovich-Bernstein operators (13) satisfy

$$P_{n,k}(x) \approx (2\pi n x(1-x))^{-1/2} \cdot \exp\left(-\frac{n}{2x(1-x)}\left(\frac{k}{n} - x\right)^2\right), \quad (20)$$

where $x \in (0, 1)$, and k satisfies

$$\left|\frac{k}{n} - x\right| \leq n^{-\alpha} \quad (21)$$

for real number $\alpha > 1/3$ (see [15, Theorem 1.5.2]). That is to say, if the set of all k satisfying (21) is written as N_x , then, while $k \in N_x$, $P_{n,k}(x)$ has asymptotic expression (20). Hence, the kernels $P_{n,k}(x)$ of Bernstein operators, from Definition 2, have local geometrical order.

Definition 6. For $r = 0, 1, \dots$, and any $x \in [0, 1]$, denote

$$M^r(\mathcal{K}_n) := \sup_{n \geq 1} \sum_{j=1}^n |nx - j|^r \mathcal{K}_{n,j}(x), \quad (22)$$

particularly,

$$\begin{aligned} M(\mathcal{K}_n) &= M^1(\mathcal{K}_n), \\ D(\mathcal{K}_n) &= M^0(\mathcal{K}_n). \end{aligned} \quad (23)$$

Definition 7. For $r = 1, 2, \dots$, write

$$DM^r(\mathcal{K}_n) := \sup_{n \geq 1} \sum_{k=1}^n k^r \Phi(k), \quad (24)$$

where $\{\Phi(k)\}_{k=1}^\infty$ is the global domination of $\mathcal{K}_n(x)$ in Definition 1. In particular,

$$DM(\mathcal{K}_n) = DM^1(\mathcal{K}_n). \quad (25)$$

Example 8. For any $\epsilon \in (0, 1)$, from Example 3, we have

$$D(R_k^\epsilon(x)) \leq C_M \sum_{k=0}^\infty \exp(-C_M \epsilon k) < \infty, \quad (26)$$

where the kernels $R_k(x)$ (8), as well as $\Phi(k) = e^{-C_M k}$, from Definitions 6 and 7, satisfy $M^r(\mathcal{K}_n^\epsilon) < \infty$ and $DM^r(\mathcal{K}_n^\epsilon) < \infty$ for all $r = 1, 2, \dots$ and any given $0 < \epsilon \leq 1$.

Example 9. Propose that $\lambda > 1$, write $\epsilon_\lambda = (\lambda + 1)/2$. In view of Example 4, we get

$$D(r_k^{\epsilon_\lambda}(x)) \leq 2^{\lambda(\lambda+1)/2} \sum_{k=1}^\infty k^{-\lambda(\lambda+1)/2} < \infty. \quad (27)$$

Thus, from Definitions 6 and 7, the kernels $r_k(x)$ (10), as well as $\Phi(k) = k^{-\lambda}$, satisfy $M^r(\mathcal{K}_n^{\epsilon_\lambda}) < \infty$ and $DM^r(\mathcal{K}_n^{\epsilon_\lambda}) < \infty$ for some r .

Remark 10. We make a brief discussion on the above definitions.

- (1) $M^r(\mathcal{K}_n^\epsilon) < \infty$ or $DM^r(\mathcal{K}_n^\epsilon) < \infty$ for some $0 < \epsilon < 1$ obviously yields that $M^r(\mathcal{K}_n) < \infty$ or $DM^r(\mathcal{K}_n) < \infty$; the converse may not be true.
- (2) The kernels having geometrical order satisfy $M^r(\mathcal{K}_n^\epsilon) < \infty$ and $DM^r(\mathcal{K}_n^\epsilon) < \infty$ for all r and $0 < \epsilon \leq 1$.
- (3) The kernels having ρ -arithmetic order satisfy $D(\mathcal{K}_n^\epsilon) < \infty$ for $\rho > 1$.

3. Elementary Approximation Technique (I)

This section gives one of the approximation techniques in L^p spaces by the Kantorovich type operators (3). We mainly apply K -functional and maximum principle to obtain the Jackson type estimation in L^p spaces.

Theorem 11. Let $\{\delta_n\}$ be a positive null sequence. For any $f(x) \in L_{[0,1]}^p$, $1 < p < \infty$, the Kantorovich operators defined by (3) satisfy

- (i) $\|L_n(f)\|_{L^p}$ which is uniformly bounded;
- (ii) $L_n(|t - x|, x) = O(\delta_n)$.

Then, the estimate

$$\|L_n(f) - f\|_{L^p} \leq C_p \omega(f, \delta_n)_{L^p} \quad (28)$$

holds, where C_p is a positive constant depending upon p only.

Proof. For any function $f(x) \in L_{[0,1]}^p$, from the definition of the K -functional

$$K(f, h)_{L^p} = \inf_{g \in AC_{[0,1]}, g' \in L^p} (\|f - g\|_{L^p} + h \|g'\|_{L^p}), \quad (29)$$

where $AC_{[0,1]}$ indicates the set of all absolute continuous functions on the interval $[0, 1]$, we know that $K(f, h)_{L^p}$ and $\omega(f, h)_{L^p}$ are equivalent (see [24, Theorem 2.1]); that is,

$$\omega(f, h)_{L^p} \approx K(f, h)_{L^p}, \quad (30)$$

where $A_h \approx B_h$ we mean there is a constant $C > 0$ independent of h such that $C^{-1}A_h \leq B_h \leq CA_h$.

For given $f(x) \in L_{[0,1]}^p$, the Hardy-Littlewood maximum function is defined by

$$M(f, x) = \sup_{\delta > 0} \frac{1}{\delta} \int_0^\delta |f(x+u)| du, \quad (31)$$

where if $x+u \notin [0, 1]$, we simply set $f(x+u) = 0$. It is well known that (see [25])

$$\|M(f)\|_{L^p} \leq C_p \|f\|_{L^p}, \quad p > 1. \quad (32)$$

Since for any $g(x) \in AC_{[0,1]}$, $g'(x) \in L_{[0,1]}^p$, $p > 1$, we have

$$\begin{aligned} \|L_n(f) - f\|_{L^p} &\leq \|L_n(f - g) - (f - g)\|_{L^p} \\ &\quad + \|L_n(g) - g\|_{L^p}, \end{aligned} \quad (33)$$

then the condition (i) of Theorem 11 induces that

$$\|L_n(f) - f\|_{L^p} \leq C\|f - g\|_{L^p} + \|L_n(g) - g\|_{L^p}. \quad (34)$$

It is easy to deduce from definition (3) that

$$\begin{aligned} & |L_n(g, x) - g(x)| \\ & \leq n \sum_{j=1}^n \mathcal{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} |g(t) - g(x)| dt \\ & = n \sum_{j=1}^n \mathcal{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} \left| \int_x^t g'(u) du \right| dt \\ & \leq nM(g', x) \sum_{j=1}^n \mathcal{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} |t - x| dt \\ & = M(g', x) L_n(|t - x|, x). \end{aligned} \quad (35)$$

With the condition (ii) in Theorem 11, we obtain

$$|L_n(g, x) - g(x)| \leq CM(g', x) \delta_n. \quad (36)$$

Combining (32) and (36) leads to

$$\|L_n(g) - g\|_{L^p} \leq C_p \delta_n \|g'\|_{L^p}. \quad (37)$$

Together with (34) and (37), we get

$$\|L_n(f) - f\|_{L^p} \leq C\|f - g\|_{L^p} + C_p \delta_n \|g'\|_{L^p}. \quad (38)$$

Finally, from the definition of the K -functional we achieve that

$$\|L_n(f) - f\|_{L^p} \leq C_p K(f, \delta_n)_{L^p}. \quad (39)$$

Therefore, in view of (30) and (39), we have completed Theorem 11. $\square$

4. Elementary Approximation Technique (II)

In Section 3, by applying the K -functional, we obtain the Jackson type estimation in L^p spaces for $p > 1$. However, the Jackson constant in that case must depend upon p , and thus we cannot establish corresponding result in L^1 space! In this section, we will exhibit another efficient technique in L^p spaces which will be used to obtain Jackson constant independent of p !

Theorem 12. Let $f(x) \in L^p_{[0,1]}$, $1 \leq p < \infty$, an ϵ be given with $0 < \epsilon < 1$ and the positive linear operators $L_n(f, x)$ defined by (3). If the kernels $\mathcal{K}_n(x)$ with (1) are dominated globally by $\{\Phi(k)\}_{k=1}^\infty$, and for some $0 < \epsilon < 1$ satisfy the following conditions:

- (i) $D(\mathcal{K}_n^{1-\epsilon}) < \infty$,
- (ii) $DM(\mathcal{K}_n^\epsilon) < \infty$,

then, the estimate

$$\|L_n(f) - f\|_{L^p_{[0,1]}} \leq C\omega(f, 1/n)_{L^p_{[0,1]}} \quad (40)$$

holds, where $C > 0$ is an absolute constant.

To prove Theorem 12, we first give two lemmas.

Lemma 13. Given h with $0 < h < 1$ and $f(x) \in L^p_{[0,1]}$, write the Steklov function of $f(x)$ as

$$f_h(x) = h^{-1} \int_0^h f(x+u) du, \quad (41)$$

where $x+u \in [0, 1]$. Then, the following results hold:

$$(i) \quad f'_h(x) = \frac{f(x+h) - f(x)}{h}; \quad (42)$$

$$(ii) \quad \|f - f_h\|_{L^p_{[0,1-h]}} \leq \omega(f, h)_{L^p}. \quad (43)$$

Proof. Equations (42) and (43) can be directly verified from the definition of the Steklov function (41). $\square$

Lemma 14. Propose that $0 \leq a, b \leq 1$, $0 < \delta < 1$, $a + \delta \leq 1$, and $b + \delta \leq 1$, then for any $f(x) \in L^p_{[0,1]}$, one has

$$\int_a^{a+\delta} \int_b^{b+\delta} |f(x) - f(y)|^p dx dy \leq 2\delta \omega^p(f, |a-b| + \delta)_{L^p}. \quad (44)$$

Proof. This Lemma is proved in [7]; we give a sketch here for self-completeness. Due to the symmetries on a and b , as well as on x and y , we need only to prove the lemma under $b \geq a$. By calculations,

$$\begin{aligned} & \int_a^{a+\delta} \int_b^{b+\delta} |f(x) - f(y)|^p dx dy \\ & = 2 \int_a^{a+\delta} \left(\int_{b-a+x}^{b+\delta} |f(x) - f(y)|^p dy \right) dx \\ & = 2 \int_a^{a+\delta} \left(\int_{b-a}^{b+\delta-x} |f(x) - f(x+y)|^p dy \right) dx \\ & = 2 \int_{b-a}^{b-a+\delta} \left(\int_a^{b+\delta-y} |f(x) - f(x+y)|^p dx \right) dy \\ & \leq 2\delta \omega^p(f, b-a+\delta)_{L^p}. \end{aligned} \quad (45)$$

Lemma 14 is done. $\square$

Proof of Theorem 12. Take $h = 1/n$ in the Steklov function (41); check

$$\begin{aligned} & \|L_n(f) - f\|_{L^p_{[1-h,1]}} \\ & = \left\{ \int_{1-h}^1 \left(n \sum_{r=1}^n \mathcal{K}_{n,r}(x) \int_{(r-1)/n}^{r/n} (f(t) - f(x)) dt \right)^p dx \right\}^{1/p} \end{aligned} \quad (46)$$

by applying Minkowski inequality, we get

$$\begin{aligned} & \|L_n(f) - f\|_{L^p_{[1-h,1]}} \\ & \leq n \sum_{r=1}^n \left\{ \int_{1-1/n}^1 \mathcal{K}_{n,r}^p(x) \left(\int_{(r-1)/n}^{r/n} |f(t) - f(x)| dt \right)^p dx \right\}^{1/p}. \end{aligned} \quad (47)$$

Then, apply Hölder inequality:

$$\begin{aligned} & \|L_n(f) - f\|_{L^p_{[1-h,1]}} \\ & \leq n^{1/p} \sum_{r=1}^n \left\{ \int_{1-1/n}^1 \mathcal{K}_{n,r}^p(x) \int_{(r-1)/n}^{r/n} |f(t) - f(x)|^p dt dx \right\}^{1/p}. \end{aligned} \quad (48)$$

When $x \in [1 - 1/n, 1]$, $[nx] = n - 1$. From the definition of the domination, we know

$$\mathcal{K}_{n,r}(x) \leq C\Phi(|[nx] - r| + 1) = C\Phi(|n - r - 1| + 1). \quad (49)$$

Then,

$$\begin{aligned} & \|L_n(f) - f\|_{L^p_{[1-h,1]}} \\ & \leq Cn^{1/p} \sum_{r=1}^n \Phi(|n - r - 1| + 1) \\ & \quad \times \left\{ \int_{1-1/n}^1 \int_{(r-1)/n}^{r/n} |f(t) - f(x)|^p dt dx \right\}^{1/p}. \end{aligned} \quad (50)$$

Applying Lemma 14, we have

$$\begin{aligned} & \|L_n(f) - f\|_{L^p_{[1-h,1]}} \\ & \leq C \sum_{r=1}^n \Phi(|n - r - 1| + 1) \omega\left(f, \frac{n - r + 1}{n}\right)_{L^p} \\ & \leq C\omega\left(f, \frac{1}{n}\right)_{L^p} \sum_{r=1}^n \Phi(|n - r - 1| + 1) (n - r + 1) \\ & \leq C\omega\left(f, \frac{1}{n}\right)_{L^p} \sum_{r=1}^n (|n - r - 1| + 1) \Phi(|n - r - 1| + 1). \end{aligned} \quad (51)$$

Applying (ii), we see that

$$\begin{aligned} & \sum_{r=1}^n (|n - r - 1| + 1) \Phi(|n - r - 1| + 1) \\ & \leq \sum_{r=1}^n (|n - r - 1| + 1) \Phi^\epsilon(|n - r - 1| + 1) \\ & = DM(\mathcal{K}_n^\epsilon) < \infty, \end{aligned} \quad (52)$$

so that we obtain

$$\|L_n(f) - f\|_{L^p_{[1-1/n,1]}} \leq C\omega\left(f, \frac{1}{n}\right)_{L^p}. \quad (53)$$

Now we verify the case when $x \in [0, 1 - 1/n]$. It is not difficult to see from Lemma 13 that $n \int_{r/n}^{(r+1)/n} f(t) dt = f_h(r/n)$. By the definition of Kantorovich type operators, rewrite

$$\begin{aligned} & L_n(f, x) - f_h(x) \\ & = n \sum_{r=1}^n \mathcal{K}_{n,r}(x) \int_{(r-1)/n}^{r/n} f(t) dt - f_h(x) \\ & = \sum_{r=1}^n \mathcal{K}_{n,r}(x) \left(f_h\left(\frac{r-1}{n}\right) - f_h(x) \right) \\ & = \sum_{r=1}^n \mathcal{K}_{n,r}(x) \int_x^{(r-1)/n} f'_h(u) du. \end{aligned} \quad (54)$$

This leads to

$$\begin{aligned} & \|L_n(f) - f_h\|_{L^p_{[0,1-n^{-1}]}}^p \\ & = \int_0^{1-1/n} |L_n(f, x) - f_h(x)|^p dx \\ & = \int_0^{1-1/n} \left| \sum_{r=1}^n \mathcal{K}_{n,r}(x) \int_x^{(r-1)/n} f'_h(u) du \right|^p dx \\ & \leq \int_0^{1-1/n} \left(\sum_{r=1}^n \mathcal{K}_{n,r}^{p(1-\epsilon)/(p-1)}(x) \right)^{p-1} \\ & \quad \cdot \sum_{j=1}^n \mathcal{K}_{n,j}^{p\epsilon}(x) \left| \int_x^{(j-1)/n} f'_h(u) du \right|^p dx \\ & \leq \int_0^{1-1/n} \left(\sum_{r=1}^n \mathcal{K}_{n,r}^{1-\epsilon}(x) \right)^p \\ & \quad \cdot \sum_{j=1}^n \mathcal{K}_{n,j}^{p\epsilon}(x) \left| \frac{j-1}{n} - x \right|^{p-1} \\ & \quad \cdot \left| \int_x^{(j-1)/n} |f'_h(u)|^p du \right| dx. \end{aligned} \quad (55)$$

Since $D(\mathcal{K}_n^{1-\epsilon}) < \infty$,

$$\begin{aligned} & \|L_n(f) - f_h\|_{L^p_{[0,1-n^{-1}]}}^p \\ & \leq C^p \int_0^{1-1/n} \sum_{j=1}^n \mathcal{K}_{n,j}^{p\epsilon}(x) \left| \frac{j-1}{n} - x \right|^{p-1} \\ & \quad \times \left| \int_x^{(j-1)/n} |f'_h(u)|^p du \right| dx \\ & = C^p \sum_{l=1}^{n-1} \int_{(l-1)/n}^{l/n} \sum_{j=1}^n \mathcal{K}_{n,j}^{p\epsilon}(x) \left| \frac{j-1}{n} - x \right|^{p-1} \\ & \quad \times \left| \int_x^{(j-1)/n} |f'_h(u)|^p du \right| dx \end{aligned}$$

$$\begin{aligned}
&= C^p \sum_{l=1}^{n-1} \sum_{j=1}^n \int_0^{1/n} \mathcal{K}_{n,j}^{pe} \left(t + \frac{l-1}{n} \right) \left| \frac{j-l}{n} - t \right|^{p-1} \\
&\quad \times \left| \int_{t+((l-1)/n)}^{(j-1)/n} |f'_h(u)|^p du \right| dt \\
&\leq C^p n^{-p} \sum_{l=1}^{n-1} \sum_{j=1}^n \Phi^{pe} (|l-j-1|+1) \\
&\quad \times (|l-j-1|+1)^{p-1} \\
&\quad \times \left| \int_{\tau}^{(j-1)/n} |f'_h(u)|^p du \right|,
\end{aligned} \tag{56}$$

where

$$\tau = \begin{cases} \frac{l-1}{n}, & \text{for } j > l, \\ \frac{l}{n}, & \text{for } j \leq l. \end{cases} \tag{57}$$

Note that

$$\begin{aligned}
&\sum_{l=1}^{n-1} \sum_{j=1}^n \Phi^{pe} (|l-j-1|+1) (|l-j-1|+1)^{p-1} \\
&\quad \times \left| \int_{\tau}^{(j-1)/n} |f'_h(u)|^p du \right| \\
&= \sum_{m=1}^n \sum_{\substack{|l-j-1|+1=m \\ 1 \leq l \leq n-1, 1 \leq j \leq n}} \Phi^{pe}(m) m^{p-1} \\
&\quad \times \left| \int_{\tau}^{(j-1)/n} |f'_h(u)|^p du \right| \\
&\leq \sum_{m=1}^n \Phi^{pe}(m) m^{p-1} \cdot 2m \cdot \int_0^{1-h} |f'_h(u)|^p du \\
&\leq 2 \int_0^{1-h} |f'_h(u)|^p du \cdot \left(\sum_{m=1}^n m \Phi^e(m) \right)^p.
\end{aligned} \tag{58}$$

Furthermore, condition (ii) implies that

$$\sum_{m=1}^n m \Phi^e(m) \leq DM(\mathcal{K}_n^e) < \infty. \tag{59}$$

This means

$$\|L_n(f) - f_h\|_{L^p_{[0,1-1/n]}}^p \leq C^p n^{-p} \int_0^{1-1/n} |f'_h(u)|^p du. \tag{60}$$

However, from Lemma 13,

$$\begin{aligned}
&\int_0^{1-1/n} |f'_h(u)|^p du \\
&= n^p \int_0^{1-1/n} \left| f\left(u + \frac{1}{n}\right) - f(u) \right|^p du \\
&\leq n^p \omega^p\left(f, \frac{1}{n}\right)_{L^p},
\end{aligned} \tag{61}$$

which leads to

$$\|L_n(f) - f_h\|_{L^p_{[0,1-1/n]}} \leq C \omega\left(f, \frac{1}{n}\right)_{L^p}. \tag{62}$$

Combining (62) with (43), we get

$$\|L_n(f) - f\|_{L^p_{[0,1-1/n]}} \leq C \omega\left(f, \frac{1}{n}\right)_{L^p}. \tag{63}$$

This, with (53), finishes Theorem 12. $\square$

For L^1 space, we have the following result while conditions of Theorem 12 can be loosed.

Theorem 15. Let $f(x) \in L^1_{[0,1]}$, $L_n(f, x)$ be defined by (3). If the kernels with (1) are dominated globally by $\{\Phi(k)\}$ and satisfy $DM(\mathcal{K}_n) < \infty$, then the estimate

$$\|L_n(f) - f\|_{L^1} \leq C \omega\left(f, \frac{1}{n}\right)_{L^1} \tag{64}$$

holds.

Proof. The argument of proof is similar, and we can just repeat the corresponding parts of the proof of Theorem 12. $\square$

Remark 16. (1) If the kernels possess good properties, the conditions of Theorem 12 can be easily verified on the terminology of domination. For instance, if the kernels have geometric order, then the corresponding conditions of Theorem 12 are obviously satisfied (see the next section).

(2) There exists essential difference between Theorem 11 and Theorem 12. Theorem 11 requires weaker conditions than Theorem 12 does, but the latter obtains stronger result (the Jackson order is complete up to $1/n$, and the Jackson constant is independent of p !); we will make further illustrations in the coming section.

5. Applications

This section illustrates how to apply Theorems 11 and 12 to estimate L^p approximation. To check the efficiency of two techniques on L^p approximation by Kantorovich type positive linear operators, three examples will be exhibited. Those positive linear operators come from three different categories: rational Müntz operators from rational Müntz systems; the Shepard operators from general real rational function systems; and Bernstein polynomials from the polynomial system. Moreover, in our point of view, they represent three different types: positive linear operators with kernels of geometric order, positive linear operators of arithmetical order, and positive linear operators of local geometric order. It is because the kernels have different domination properties or different speeds of $\{\Phi(n)\}$ that the L^p approximations by the corresponding positive linear operators possess different Jackson orders.

To show the key role of the global (or local) domination on the kernels, the condition (ii) of Theorem 11 will be further explicated to the following lemma.

Lemma 17. For any kernel $\mathcal{K}_n(x)$, one has

$$L_n(|t-x|, x) = O(n^{-1})(M(\mathcal{K}_n) + 1). \quad (65)$$

Especially, if the kernels $\mathcal{K}_n(x)$ are globally dominated, then

$$L_n(|t-x|, x) = O(n^{-1})DM(\mathcal{K}_n). \quad (66)$$

Proof. From definition (3),

$$L_n(|t-x|, x) = n \sum_{r=1}^n \mathcal{K}_{n,r}(x) \int_{(r-1)/n}^{r/n} |t-x| dt. \quad (67)$$

For any $x \in [0, 1]$, there exists an integer m such that $(m-1)/n \leq x < m/n$; then $\int_{(r-1)/n}^{r/n} |t-x| dt$ will be calculated according to the following three cases, respectively.

(a) When $1 \leq r < m$,

$$\begin{aligned} \int_{(r-1)/n}^{r/n} |t-x| dt &= \int_{(r-1)/n}^{r/n} (x-t) dt \\ &= n^{-2} \left(nx - r + \frac{1}{2} \right). \end{aligned} \quad (68)$$

(b) When $m+1 \leq r \leq n$,

$$\begin{aligned} \int_{(r-1)/n}^{r/n} |t-x| dt &= \int_{(r-1)/n}^{r/n} (t-x) dt \\ &= n^{-2} \left(r - nx - \frac{1}{2} \right). \end{aligned} \quad (69)$$

(c) When $r = m$,

$$\int_{(m-1)/n}^{m/n} |t-x| dt \leq \int_{(m-1)/n}^{m/n} n^{-1} dt = n^{-2}. \quad (70)$$

Combining (a), (b), and (c), we obtain that

$$\int_{(r-1)/n}^{r/n} |t-x| dt \leq n^{-2} (|[nx] - r| + 1). \quad (71)$$

From Definition 6, (67) and (71) yield (65). At the same time, since the kernels of $L_n(f, x)$ are globally dominated, we obtain by Definition 1 that

$$\mathcal{K}_{n,r}(x) \leq C\Phi(|[nx] - r| + 1), \quad r = 1, 2, \dots, n. \quad (72)$$

Thus,

$$L_n(|t-x|, x) \leq Cn^{-1} \sum_{r=1}^n (|[nx] - r| + 1) \Phi(|[nx] - r| + 1); \quad (73)$$

that is, from Definition 7, (66) holds.

Lemma 17 is done. $\square$

5.1. Rational Müntz Approximation. Rational Müntz Approximation has been researched in [5], shows the application of Theorem 12, and simplifies the proof of [5].

Write

$$\Lambda_n = \{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}, \quad (74)$$

where $\text{span}\{x^{\lambda_k}\}$, $\lambda_k \in \Lambda_n$, is the class of all linear combinations of $\{x^{\lambda_k}\}_{k=1}^n$. For $f(x) \in L_{[0,1]}^p$, define

$$R_n(f, \Lambda) = \min_{r \in R(\Lambda_n)} \|f - r\|_{L^p}. \quad (75)$$

Corollary 18. Given $f(x) \in L_{[0,1]}^p$, $1 \leq p < \infty$, the rational Müntz operators are defined by (9). If $\Delta\lambda_n \geq Mn$, $n = 1, 2, 3, \dots$, where $M > 0$ is an absolute constant, one has

$$R_n(f, \Lambda) \leq \|M_n(f) - f\|_{L^p} \leq C_M \omega\left(f, \frac{1}{n}\right)_{L^p}, \quad (76)$$

where C_M is an absolute constant depending only upon M (independent of p !).

Proof. We verify that $M_n(f, x)$ satisfies all the requirements of Theorem 12. Take $\Phi(k) = e^{-C_M k}$, where $C_M > 0$ is the constant appeared in [4], or [5]; see the following inequality (77). Given a fixed ϵ with $0 < \epsilon < 1$, we verify that

(i) $D(R_k^{1-\epsilon}(x)) < \infty$.

From [4], or [5],

$$R_k(x) \leq C_M \exp(-C_M |[nx] - k|). \quad (77)$$

Then,

$$\begin{aligned} \sum_{k=1}^n R_k^{1-\epsilon}(x) &\leq C_M \sum_{k=1}^n \exp(-C_M |[nx] - k| (1-\epsilon)) \\ &\leq C_M \sum_{k=0}^{\infty} \exp(-C_M k (1-\epsilon)) < \infty. \end{aligned} \quad (78)$$

(ii) $DM(R_k^\epsilon(x)) < \infty$.

Due to Example 3, we have $\Phi(k) = \exp(-C_M k)$. From Definition 7,

$$\begin{aligned} DM(R_k^\epsilon) &= \sup_{n>1} \sum_{k=1}^n k \exp(-C_M k \epsilon) \\ &\leq \sum_{k=0}^{\infty} k \exp(-C_M k \epsilon) < \infty. \end{aligned} \quad (79)$$

Therefore, Corollary 18 can be deduced from Theorem 12. $\square$

By the same argument of (ii), we obviously can obtain $DM(R_k) < \infty$. From (66), we know $M_n(|t-x|, x) = O(n^{-1})$. Moreover, it is simple to verify $\|M_n(f)\|_{L^p} < \infty$ (see, e.g., [7]). Hence, applying Theorem 11, we get

Corollary 19. For rational Müntz operator $M_n(f, x)$ (9), one has

$$R_n(f, \Lambda)_{L^p} \leq C_{M,p} \omega\left(f, \frac{1}{n}\right)_{L^p}, \quad (80)$$

where $C_{M,p}$ is depending on M and p .

Remark 20. Since the kernels $R_k(x)$ (8) have geometrical order (see Example 3), they certainly satisfy the conditions of both Theorems 11 and 12. Hence, L^p approximation by these rational Müntz operators can always reach the Jackson order by applying both techniques of Theorems 12 and 11. However, for these operators, the conclusion of Corollary 18 surely contains Corollary 19 and makes the latter trivial.

5.2. The Shepard Operators. The approximation of the Shepard operators in the continuous function space $C_{[0,1]}$ has been studied very deeply (see [13, 26–32]). The L^p approximation by the Shepard operators is investigated in [6, 7].

Corollary 21. *Proposed that $f(x) \in L^p_{[0,1]}$, $p > 1$, the Shepard operators are defined by (11). Then,*

$$\|S_{n,\lambda}(f) - f\|_{L^p} \leq C_{p,\lambda} \omega(f, \epsilon_n)_{L^p}, \quad (81)$$

where

$$\epsilon_n = \begin{cases} n^{-1}, & \text{if } \lambda > 2, \\ n^{-1} \log n, & \text{if } \lambda = 2, \\ n^{1-\lambda}, & \text{if } 1 < \lambda < 2. \end{cases} \quad (82)$$

Proof. By applying Theorem 11, we verify this result.

- (i) $\|S_{n,\lambda}(f)\|_{L^p}$ are uniformly bounded (see [6]).
- (ii) $S_{n,\lambda}(|t - x|, x) = \epsilon_n$ (Lemma 17, Inequality (66)).

Due to Example 4, we know that the global dominated sequence is $\Phi(k) = k^{-\lambda}$. Then, from Definition 7,

$$DM(r_k(x)) = \sup_{n>1} \sum_{k=1}^n k \Phi(k) = \sup_{n>1} \sum_{k=1}^n k^{1-\lambda}. \quad (83)$$

When $\lambda > 2$, $\sum_{k=1}^{\infty} k^{1-\lambda} < \infty$.

When $\lambda = 2$, $\sum_{k=1}^{n+1} k^{1-\lambda} \leq 2 \log n$.

When $1 < \lambda < 2$, $\sum_{k=1}^{n+1} k^{1-\lambda} \leq C_\lambda n^{2-\lambda}$.

By Theorem 11, Corollary 21 holds. $\square$

Corollary 22. *Propose that $f(x) \in L^p_{[0,1]}$, $1 \leq p < \infty$, the Shepard operators are defined by (11). If $\lambda > 3$, then*

$$\|S_{n,\lambda}(f) - f\|_{L^p} \leq C_\lambda \omega\left(f, \frac{1}{n+1}\right)_{L^p}, \quad (84)$$

where C_λ depends on λ only.

Proof. Theorem 12 will be applied to prove this result. The details can be referred in [7]. When $\lambda > 3$, let $\epsilon = (\lambda + 1)/2\lambda$, then $0 < \epsilon < 1$, $\lambda\epsilon > 2$ and $\lambda(1 - \epsilon) > 1$. Evidently, the Shepard kernels have λ -arithmetic order (see Example 4). We check the corresponding conditions of Theorem 12:

- (i) $DM(r_k^{1-\epsilon}(x)) < \infty$.

From Definition 6 and Example 4,

$$\begin{aligned} \sum_{k=1}^n r_k^{1-\epsilon}(x) &\leq C_\lambda \sum_{k=0}^n (|(n+1)x - k| + 1)^{-\lambda(1-\epsilon)} \\ &\leq C_\lambda \sum_{k=1}^{\infty} k^{-\lambda(1-\epsilon)} < \infty. \end{aligned} \quad (85)$$

Then,

$$(ii) DM(r_k^\epsilon(x)) < \infty.$$

Note that the present dominated sequence $\Phi(k) = k^{-\lambda}$; then from Definition 7

$$DM(r_k^\epsilon(x)) = \sup_{n>1} \sum_{k=1}^n k \Phi^\epsilon(k) \leq \sup_{n>1} \sum_{k=1}^{\infty} k^{1-\lambda\epsilon} < \infty. \quad (86)$$

Corollary 22 is completed from Theorem 12 as (i) and (ii) hold. $\square$

Remark 23. The difference between L^p approximation techniques (I) and (II) with respect to Theorems 11 and 12 is fully exhibited on the Shepard operators by Corollaries 21 and 22. Stronger requirements by applying technique (II) than by applying technique (I) are needed. However, if the conditions are satisfied, the former can obtain essentially better result (the Jackson constant is independent of p !). On this particular case, we can obtain the Jackson type estimation by applying technique (I) for $\lambda > 1$, while achieve the corresponding result by applying technique (II) only for $\lambda > 3$. We still do not know how to deal with the cases when $1 < \lambda \leq 3$ by applying technique (II).

5.3. Bernstein Operators. There are many results on L^p approximation by Bernstein polynomials; interested readers may refer to [8–23]. These results can be classified into the following categories:

- (1) uniform convergence; see [8, 12, 15, 18];
- (2) quantitative estimations; see [10, 11, 14, 19];
- (3) equivalence theorems; see [16, 17];
- (4) saturation problems; see [9, 16, 20–23].

Here, we apply our technique (I) to test the corresponding Jackson type estimation as an example. The following proof is simpler than the past proof.

Corollary 24. *Given $f(x) \in L^p_{[0,1]}$, $p > 1$, one has the following estimation:*

$$\|B_n(f) - f\|_{L^p} \leq C_p \omega\left(f, \frac{1}{\sqrt{n}}\right)_{L^p}. \quad (87)$$

Proof. $\|B_n(f)\|_{L^p}$ is uniformly bounded which is known from [33]. We only need to evaluate $B_n(|t - x|, x)$ from Theorem 11. It is known from [15, page 15] and Definition 6 that

$$M(P_{n,k}(x)) = \sup_{n \geq 1} \sum_{k=0}^n |nx - k| P_{n,k}(x) \leq C \sqrt{n}. \quad (88)$$

From (65) of Lemma 17 and (88),

$$B_n(|t-x|, x) = O(n^{-1}) M(P_{n,k}(x)) = O(n^{-1/2}) \quad (89)$$

which, from Theorem 11, leads to Corollary 24. $\square$

Remark 25. The approximation order of Corollary 24 is sharp which shows that (88) cannot be improved. In other words, $M(P_{n,k}(x))$ is unbounded. Hence, the approximation technique (II) or Theorem 12 cannot be applied in this case. That is to say, we can never obtain the Jackson constant independent of p for Bernstein polynomials in L^p spaces. Furthermore, Corollary 24 also exhibits that Kantorovich type operators (2) or (3) cannot reach Jackson type estimation with the Jackson constant independent of p unless their kernels possess good properties such as having globally geometric domination (rational Müntz approximation case) or having global p -arithmetic domination for sufficiently large p (the Shepard operators with $\lambda > 3$).

6. Conclusions

On the above discussions, the positive linear operators used in L^p approximation can be classified according to the properties of their kernels. We have three categories: kernels with geometrical order (such as the rational Müntz operators), kernels with arithmetic order (such as the Shepard operators), and kernels with local arithmetic order (such as the Bernstein operators). In another word, for characterizing the Jackson type estimate in L^p spaces by the Kantorovich type operators (3) or (5), it always plays an essential role how well the kernels of the operators under study behave.

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