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Research Article

Global Existence for Functional Differential Equations with State-Dependent Delay

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Our aim in this work is to study the existence of solutions of a functional differential equation with state-dependent delay. We use Schauder's fixed point theorem to show the existence of solutions.

1. Introduction

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, and functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay [1–5]. In 1806, Poisson [6] published one of the first papers on functional differential equations and studied a geometric problem leading to an example with a state-dependent delay (see also [7]). An extensive theory is developed for evolution equations [8, 9]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [10–12]. However, complicated situations in which the delay depends on the unknown functions have been considered in recent years. These equations are frequently called equations with state-dependent delay: see, for instance, [3, 13–15]. Existence results were derived recently for functional differential equations when the solution is depending on the delay for impulsive problems. We refer the reader to the papers by Abada et al. [16], Ait Dads and Ezzinbi [17], Anguraj et al. [18], Hartung et al. [19, 20], Hernández et al. [21], and Li et al. [22]. Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as in classical electrodynamics [23], in population models [24–27], in models of commodity price fluctuations [28, 29], in models of blood cell productions [30–33], and in drilling [34].

In this work, we prove the existence of solutions of a class of functional differential equations. Our investigations will be situated in the Banach space of real functions which are defined, continuous, and bounded on the real axis $\mathbb{R}$. We will use Schauder's fixed point theorem combined with the semigroup theory to have the existence of solutions of the following functional differential equation with state-dependent delay:

$$\begin{gathered} y'(t) = Ay(t) + f\left(t, y_{\rho(t,y_t)}\right), \quad \text{a.e. } t \in J := [0, +\infty) \\ y(t) = \phi(t), \quad t \in (-\infty, 0], \end{gathered} \tag{1}$$

where $f : J \times \mathscr{B} \to E$ is a given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J, \mathscr{B}$ is the phase space to be specified later, $\phi \in \mathscr{B}$, $\rho : J \times \mathscr{B} \to (-\infty, +\infty)$, and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathscr{B}$ defined by

$y_t(\theta) = y(t+\theta), \theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $-\infty$ up to the present time t. We assume that the histories y_t belong to some abstract phase space $\mathscr{B}$, to be specified later. To our knowledge, the literature on the global existence of evolution equations with delay is very limited, so the present paper can be considered as a contribution to this question.

2. Preliminaries

In this section, we present briefly some notations, a definition and a theorem which are used throughout this work.

In this paper, we will employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato in [1] and follow the terminology used in [3]. Thus, $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E and satisfying the following axioms.

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathscr{B}$, then for every $t \in J$ the following conditions hold:

(i) $y_t \in \mathscr{B}$;

(ii) there exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathscr{B}}$;

(iii) there exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded and M locally bounded such that

$$\|y_t\|_{\mathscr{B}} \le L(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathscr{B}}. \quad (2)$$

(A_2) For the function y in (A_1), y_t is a $\mathscr{B}$-valued continuous function on J.

(A_3) The space $\mathscr{B}$ is complete.

Denote

$$l = \sup\{L(t) : t \in J\}, \qquad m = \sup\{M(t) : t \in J\}. \quad (3)$$

Remark 1. (1)(A_1) (ii) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathscr{B}}$ for every $\phi \in \mathscr{B}$.

(2) Since $\|\cdot\|_{\mathscr{B}}$ is a seminorm, two elements $\phi, \psi \in \mathscr{B}$ can verify $\|\phi - \psi\|_{\mathscr{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.

(3) From the equivalence in the first remark, we can see that, for all $\phi, \psi \in \mathscr{B}$ such that $\|\phi - \psi\|_{\mathscr{B}} = 0$. We necessarily have that $\phi(0) = \psi(0)$.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

By $\mathrm{BC} := \mathrm{BC}(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{\mathrm{BC}} = \sup_{t \in (-\infty, +\infty)} |y(t)|. \quad (4)$$

Finally, by $\mathrm{BC}' := \mathrm{BC}'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{\mathrm{BC}'} = \sup_{t \in [0, +\infty)} |y(t)|. \quad (5)$$

Definition 2. A map $f : J \times \mathscr{B} \to E$ is said to be Carathéodory if

(i) $t \to f(t, y)$ is measurable for all $y \in \mathscr{B}$;

(ii) $y \to f(t, y)$ is continuous for almost each $t \in J$.

Theorem 3 (see Schauder fixed point [35]). *Let B be a closed, convex, and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B. That is, there exists an $y \in B$ such that $Ny = y$.*

Lemma 4 (see Corduneanu [36]). *Let $D \subset BC([0, +\infty), E)$. Then D is relatively compact if the following conditions hold.*

(a) *D is bounded in BC.*

(b) *The function belonging to D is almost equicontinuous on $[0, +\infty)$, that is, equicontinuous on every compact of $[0, +\infty)$.*

(c) *The set $D(t) := \{y(t) : y \in D\}$ is relatively compact on every compact of $[0, +\infty)$.*

(d) *The function from D is equiconvergent that is, given $\epsilon > 0$, it corresponds a $T(\epsilon) > 0$ such that $|u(t) - \lim_{t \to +\infty} u(t)| < \epsilon$, for any $t \ge T(\epsilon)$ and $u \in D$.*

3. Existence of Mild Solutions

Now we give our main existence result for problem (1). Before starting and proving this result, we give the definition of the mild solution.

Definition 5. We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (1) if $y(t) = \phi(t), t \in (-\infty, 0]$, and the restriction of $y(\cdot)$ to the interval $[0, +\infty)$ is continuous and satisfies the following integral equation:

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s) f\left(s, y_{\rho(s, y_s)}\right) ds, \quad t \in J. \quad (6)$$

Set

$$\mathscr{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathscr{B}, \rho(s, \phi) \le 0\}. \quad (7)$$

We always assume that $\rho : J \times \mathscr{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) the function $t \to \phi_t$ is continuous from $\mathscr{R}(\rho^-)$ into $\mathscr{B}$, and there exists a continuous and bounded function $\mathscr{L}^\phi : \mathscr{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\| \le \mathscr{L}^\phi(t)\|\phi\| \quad \text{for every } t \in \mathscr{R}(\rho^-). \quad (8)$$

Remark 6. The condition (H_ϕ) is frequently verified by functions continuous and bounded. For more details, see, for instance, [3].

Lemma 7 ([21, Lemma 2.4]). *If $y : (-\infty, +\infty) \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathscr{B}} \le \left(M + \mathscr{L}^{\phi}\right)\|\phi\|_{\mathscr{B}} + l \sup\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathscr{R}(\rho^-) \cup J, \tag{9}$$

where $\mathscr{L}^{\phi} = \sup_{t\in\mathscr{R}(\rho^-)}\mathscr{L}^{\phi}(t)$.

Let us introduce the following hypotheses.

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, which is compact for $t > 0$ in the Banach space E. Let $M' = \sup\{\|T\|_{B(E)} : t \ge 0\}$.

(H_2) The function $f : J \times \mathscr{B} \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that

$$|f(t,u) - f(t,v)| \le k(t)\|u - v\|_{\mathscr{B}}, \quad t \in J,\ u, v \in \mathscr{B}, \qquad k^* := \sup_{t\in J}\int_0^t k(s)\,ds < \infty. \tag{10}$$

(H_4) The function $t \to f(t, 0) = f_0 \in L^1(J, [0, +\infty))$ with $F^* = \|f_0\|_{L^1}$.

Theorem 8. *Assume that (H_1)–(H_4), (H_ϕ) hold. If $k^* M' l < 1$, then the problem* (1) *has at least one mild solution on BC.*

Proof. Transform problem (1) into a fixed point problem. Consider the operator $N : \mathrm{BC} \to \mathrm{BC}$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s) \times f\left(s, y_{\rho(s,y_s)}\right) ds, & \text{if } t \in J. \end{cases} \tag{11}$$

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\phi(0), & \text{if } t \in J. \end{cases} \tag{12}$$

Then $x_0 = \phi$. For each $z \in \mathrm{BC}$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0]; \\ z(t), & \text{if } t \in J. \end{cases} \tag{13}$$

If y satisfies $y(t) = N(y)(t)$, we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t T(t-s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds, \quad t \in J. \tag{14}$$

Set

$$\mathrm{BC}_0' = \left\{z \in \mathrm{BC}' : z(0) = 0\right\}, \tag{15}$$

and let

$$\|z\|_{\mathrm{BC}_0'} = \sup\{|z(t)| : t \in J\}, \quad z \in \mathrm{BC}_0'. \tag{16}$$

BC_0' is a Banach space with the norm $\|\cdot\|_{\mathrm{BC}_0'}$. We define the operator $\mathscr{A} : \mathrm{BC}_0' \to \mathrm{BC}_0'$ by

$$\mathscr{A}(z)(t) = \int_0^t T(t-s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds, \quad t \in J. \tag{17}$$

We will show that the operator $\mathscr{A}$ satisfies all conditions of Schauder's fixed point theorem. The operator A maps BC_0' into BC_0'; indeed, the map $\mathscr{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in \mathrm{BC}_0'$, and for each $t \in J$ we have

$$\begin{aligned} |\mathscr{A}(z)(t)| &\le M' \int_0^t \left|f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) - f(s,0) + f(s,0)\right| ds \\ &\le M' \int_0^t |f(s,0)|\, ds + M' \int_0^t k(s) \left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|_{\mathscr{B}} ds \\ &\le M'F^* + M' \int_0^t k(s) \times \left(l|z(s)| + \left(m + \mathscr{L}^{\phi} + lM'H\right)\|\phi\|_{\mathscr{B}}\right) ds. \end{aligned} \tag{18}$$

Set

$$C := \left(m + \mathscr{L}^{\phi} + lM'H\right)\|\phi\|_{\mathscr{B}}. \tag{19}$$

Then, we have

$$\begin{aligned} |\mathscr{A}(z)(t)| &\le M'F^* + M'C\int_0^t k(s)\,ds + M'\int_0^t l|z(s)|k(s)\,ds \\ &\le M'F^* + M'Ck^* + M'l\|z\|_{\mathrm{BC}_0'}k^*. \end{aligned} \tag{20}$$

Hence, $\mathscr{A}(z) \in \mathrm{BC}_0'$.

Moreover, let $r > 0$ be such that

$$r \ge \frac{M'F^* + M'Ck^*}{1 - M'k^*l}, \tag{21}$$

and let B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $z \in B_r$, and let $t \in [0, +\infty)$. Then,

$$|\mathscr{A}(z)(t)| \le M'F^* + M'Ck^* + M'k^*lr. \tag{22}$$

Thus,

$$\|\mathscr{A}(z)\|_{BC_0'} \le r, \tag{23}$$

which means that the operator $\mathscr{A}$ transforms the ball B_r into itself.

Now we prove that $\mathscr{A} : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1. ($\mathscr{A}$ is continuous in B_r). Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}$, $s \in J$.

If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have

$$\begin{aligned}
&\left\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\right\|_{\mathscr{B}} \\
&\le \left\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\right\|_{\mathscr{B}} + \left\|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\right\|_{\mathscr{B}} \\
&\le L\|z_n - z\|_{\mathscr{B}} + \left\|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\right\|_{\mathscr{B}},
\end{aligned} \tag{24}$$

which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathscr{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, if $\rho(s, z_s) < 0$, we get

$$\left\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\right\|_{\mathscr{B}} = \left\|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\right\|_{\mathscr{B}} = 0, \tag{25}$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathscr{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned}
&\left|\mathscr{A}(z_n)(t) - \mathscr{A}(z)(t)\right| \\
&\quad\le M' \int_0^t \left|f\left(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}\right)\right. \\
&\qquad\left. - f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| ds \\
&\quad\le M' \int_0^t \left|f\left(s, z_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}\right)\right. \\
&\qquad\left. - f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| ds.
\end{aligned} \tag{26}$$

Then by (H_2), we have

$$\begin{aligned}
f\left(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}\right) &\longrightarrow f\left(s, z_{\rho(s,z_s+x_s)}\right. \\
&\left. + x_{\rho(s,z_s+x_s)}\right), \\
&\text{as } n \longrightarrow \infty,
\end{aligned} \tag{27}$$

and by the Lebesgue dominated convergence theorem we get

$$\|\mathscr{A}(z_n) - \mathscr{A}(z)\|_{BC_0'} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{28}$$

Thus, $\mathscr{A}$ is continuous.

Step 2. $\mathscr{A}(B_r) \subset B_r$ which is clear.

Step 3. $\mathscr{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$; we have

$$\begin{aligned}
&\left|\mathscr{A}(z)(\tau_2) - \mathscr{A}(z)(\tau_1)\right| \\
&\quad\le \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} \\
&\qquad\times \left|f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| ds \\
&\qquad+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} \\
&\qquad\times \left|f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| ds \\
&\quad\le \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} \\
&\qquad\times \left|f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) - f(s, 0)\right| ds \\
&\qquad+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} |f(s, 0)| ds \\
&\qquad+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} \\
&\qquad\times \left|f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) - f(s, 0)\right| ds \\
&\qquad+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s, 0)| ds \\
&\quad\le C \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} k(s)\, ds \\
&\qquad+ rL \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} k(s)\, ds \\
&\qquad+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} |f(s, 0)| ds \\
&\qquad+ C \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s)\, ds \\
&\qquad+ rL \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s)\, ds \\
&\qquad+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s, 0)| ds.
\end{aligned} \tag{29}$$

When $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero; since $T(t)$ is a strongly continuous operator, and the compactness of $T(t)$ for $t > 0$ implies the continuity

in the uniform operator topology (see [37]), this proves the equicontinuity.

Step 4. $\mathscr{A}(B_r)(t)$ is relatively compact on every compact interval of $t \in [0,\infty)$. Let $t \in [0,b]$ for $b > 0$, and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in B_r$, we define

$$\mathscr{A}_\varepsilon(z)(t) = T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon) \times f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds. \tag{30}$$

Note that the set

$$\left\{\int_0^{t-\varepsilon} T(t-s-\varepsilon) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds : z \in B_r\right\} \tag{31}$$

is bounded. Since $T(t)$ is a compact operator for $t > 0$, the set

$$\{\mathscr{A}_\varepsilon(z)(t) : z \in B_r\} \tag{32}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\begin{aligned} &\left|\mathscr{A}(z)(t) - \mathscr{A}_\varepsilon(z)(t)\right| \\ &\quad \le \int_{t-\varepsilon}^{t} T(t-s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) ds \\ &\quad \le M'F^*\varepsilon + M'C\int_{t-\varepsilon}^{t} k(s)\,ds + rM'\int_{t-\varepsilon}^{t} lk(s)\,ds, \\ &\quad \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \tag{33}$$

Therefore, the set $\{\mathscr{A}(z)(t) : z \in B_r\}$ is precompact, that is, relatively compact.

Step 5 ($\mathscr{A}(B_r)$ is equiconvergent). Let $t \in [0,+\infty)$ and $z \in B_r$; we have

$$\begin{aligned} |\mathscr{A}(z)(t)| &\le M'\int_0^t \left|f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| ds \\ &\le M'F^* + M'C\int_0^t k(s)\,ds + M'r\int_0^t Lk(s)\,ds \\ &\le M'F^* + M'C\int_0^t k(s)\,ds + M'rl\int_0^t k(s)\,ds. \end{aligned} \tag{34}$$

Then by (37), we have

$$|\mathscr{A}(z)(t)| \longrightarrow M'F^*, \quad \text{as } t \longrightarrow +\infty. \tag{35}$$

Hence,

$$|\mathscr{A}(z)(t) - \mathscr{A}(z)(+\infty)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{36}$$

As a consequence of Steps 1–4, with Lemma 4, we can conclude that $\mathscr{A} : B_r \to B_r$ is continuous and compact. From Schauder's theorem, we deduce that $\mathscr{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of problem (1). □

4. An Example

Consider the following functional partial differential equation:

$$\begin{aligned} &\frac{\partial}{\partial t} z(t,x) - \frac{\partial^2}{\partial x^2} z(t,x) \\ &= e^{-t}\int_{-\infty}^{0} z\left(s - \sigma_1(t)\sigma_2\left(\int_0^\pi a(\theta)\left|z(t,\theta)\right|^2 d\theta\right), x\right) ds, \\ &\qquad x \in [0,\pi], \quad t \in [0,+\infty) \\ &z(t,0) = z(t,\pi), \quad t \in [0,+\infty), \\ &z(\theta,x) = z_0(\theta,x), \quad t \in (-\infty,0], \ x \in [0,\pi], \end{aligned} \tag{37}$$

where $z_0 \not\equiv 0$. Set

$$\begin{aligned} f(t,\psi)(x) &= \int_{-\infty}^{0} e^{-t}\psi(s,x)\,ds, \\ \rho(t,\psi) &= t - \sigma_1(t)\sigma_2\left(\int_0^\pi a_2(\theta)\left|\psi(t,\theta)\right|^2 d\theta\right), \end{aligned} \tag{38}$$

where $\sigma_i : \mathbb{R}^+ \to \mathbb{R}^+, i = 1,2$, and $a : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$\begin{aligned} D(A) = \{&\omega \in E, \omega, \omega' \text{ are absolutely continuous}, \\ &\omega'' \in E, \quad \omega(0) = \omega(\pi) = 0\}. \end{aligned} \tag{39}$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \quad \omega \in D(A), \tag{40}$$

where $\omega_n(s) = \sqrt{2/\pi}\sin ns, n = 1,2,\ldots$, is the orthogonal set of eigenvectors in A. It is well known (see [37]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \ge 0$, in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp\left(-n^2 t\right)(\omega,\omega_n)\omega_n, \quad \omega \in E. \tag{41}$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \le M. \tag{42}$$

Let $\mathscr{B} = \mathrm{BCU}(\mathbb{R}^-; E)$, and let $\phi \in \mathscr{B}$ then (H_ϕ).

The function $f(t,\psi)(x)$ is Carathéodory, and

$$\left|f(t,\psi_1)(x) - f(t,\psi_2)(x)\right| \le e^{-t}\left|\psi_1(t,x) - \psi_2(t,x)\right|. \tag{43}$$

Thus, $k(t) = e^{-t}$; moreover, we have

$$k^* = \sup\left\{\int_0^t e^{-s} ds, \ t \in [0,+\infty)\right\} = 1, \qquad f_0 \equiv 0. \tag{44}$$

Then problem (1) in an abstract formulation of the problem (37) and conditions (H_1)–(H_4), (H_ϕ) are satisfied. Theorem 8 implies that the problem (37) has at least one mild solution on BC.

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