Research Article

On Asymptotic Behavior and Blow-Up of Solutions for a Nonlinear Viscoelastic Petrovsky Equation with Positive Initial Energy

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This paper deals with the initial boundary value problem for the nonlinear viscoelastic Petrovsky equation $u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(x, \tau) d\tau - \Delta u_t - \Delta u_{tt} + |u_t|^{m-1} u_t = |u|^{p-1} u$. Under certain conditions on g and the assumption that $m < p$, we establish some asymptotic behavior and blow-up results for solutions with positive initial energy.

1. Introduction

In this work, we investigate the following nonlinear viscoelastic Petrovsky problem:

$$\begin{aligned} & u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(x, \tau) d\tau \\ & - \Delta u_t - \Delta u_{tt} + |u_t|^{m-1} u_t \\ & = |u|^{p-1} u, \quad x \in \Omega, \quad t \geq 0, \\ & u(x, t) = \frac{\partial u(x, t)}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ & u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \end{aligned} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial\Omega$, $m, p > 1$; ν is the unit outer normal on $\partial\Omega$; and g is a nonnegative memory term.

In the absence of viscoelastic term (i.e., $g = 0$), Guesmia [1] considered the following equation:

$$u_{tt} + \Delta^2 u + q(x) u(x, t) + g(u_t) = 0, \quad x \in \Omega, \quad t > 0, \quad (2)$$

where $q : \Omega \rightarrow \mathbb{R}^+$ is a bounded function and g is continuous and increasing function satisfying $g(0) = 0$.

Under suitable growth conditions on g , the author established global existence, uniqueness, and decay results by using the semigroup method. Messaoudi [2] investigated a nonlinearly damped semilinear Petrovsky equation

$$u_{tt} + \Delta^2 u + a|u_t|^{m-2} u_t = b|u|^{p-2} u, \quad x \in \Omega, \quad t > 0, \quad (3)$$

where $a, b > 0$ and $p, m > 2$ and proved that the solution is global when $m \geq p$ while the solution blows up in finite time with negative initial energy when $m < p$. Later, this blow-up result was improved by Chen and Zhou [3] with positive initial energy. For a related study, we may see the work of Wu and Tsai [4]. In [5], Amroun and Benaissa studied (3) by generalizing the damping term into the form of $g(u_t)$ and obtained the global existence of the solutions by means of the stable set method combined with the Faedo-Galerkin procedure. Very recently, in the presence of the strong damping, Li et al. [6] considered the following Petrovsky equation:

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m-1} u_t = |u|^{p-1} u, \quad x \in \Omega, \quad t \geq 0. \quad (4)$$

Without any interaction between m and p , the authors obtained the global existence and uniform decay of solutions when the initial data are in some stable set. And a blow-up result was established when $m < p$ and the initial energy is less than the potential well depth.

In the presence of the viscoelastic term (i.e., $g \neq 0$), Muñoz Rivera et al. [7] studied the following equation:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(x, \tau) d\tau - \gamma \Delta u_{tt} = 0, \quad x \in \Omega, \quad t \geq 0. \quad (5)$$

They proved that the memory effect produces strong dissipation capable of making uniform rate of decay for the energy. Later, in the presence of strong damping term, M. M. Cavalcanti et al. [8] considered

$$|u_t|^p u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(x, \tau) d\tau - \gamma \Delta u_t - \Delta u_{tt} = 0, \quad x \in \Omega, \quad t \geq 0, \quad (6)$$

and obtained a global existence for $\gamma \geq 0$ and uniform exponential decay for $\gamma > 0$. This work was extended by Messaoudi and Tatar [9] to a situation where a nonlinear source term is competing with the damping induced by $-\gamma \Delta u_t$ and the integral term. Then in the case of $\gamma = 0$, the same authors [10] showed that the damping induced by the viscoelastic term is enough to ensure global existence and uniform decay of solutions provided that the initial data are in some stable set by introducing a new functional and using the potential well method. Recently, Wu [11] improved [10] by considering the nonlinear equation:

$$|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau) \Delta u(x, \tau) d\tau + |u_t|^m u_t = |u|^p u, \quad (7)$$

and a general decay result was obtained. In the presence of strong damping term Δu_t and dispersive term Δu_{tt} , Xu et al. [12] considered the initial boundary value problem for the following viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(x, \tau) d\tau - \Delta u_t - \Delta u_{tt} + u_t = |u|^{p-1} u. \quad (8)$$

By introducing a family of potential wells, the authors not only obtained the invariant sets, but also proved the existence and nonexistence of global weak solution under some conditions with low initial energy. Furthermore, they established a blow-up result for certain solutions with arbitrary positive initial energy (high energy case). Very recently, Tahamtani and Peyravi [13] considered problem (1) and obtained the exponential decay of the energy under some assumptions on g without any interaction between source term and damping term. Under an appropriate restriction on g , they also proved that the L^{p+1} norm of any solution grows as an exponential function if $m < p$ and the initial energy is negative. For other related works, we refer the readers to [14–24] and the references therein.

Motivated by the above works, in this paper, we intend to consider problem (1) and establish some asymptotic behavior

and blow-up results for solutions with positive initial energy. For our purpose, we use the functional $L(t) = \|u\|_2^2 + \|\nabla u\|_2^2$ and give a modified manner to estimate the term $|\int_\Omega u|u_t|^{m-1} u_t dx|$ so that the appearance of the form like $\gamma = RH^{-\sigma}(t)$ (for constants γ , R , and σ) which has been used in many earlier works (e.g., in [3, 19, 25]) can be avoided.

The paper is organized as follows. In Section 2 we present some assumptions and known results and state the main results. Section 3 is devoted to proof the the blow-up result—Theorem 4.

2. Preliminaries and Main Results

In this section, we first present some assumptions and known results which will be used throughout this work.

For the relaxation function g , we give the following assumptions:

(G1) $g(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s) ds = l > 0, \quad (9)$$

(G2) $g'(t) \leq 0$, for all $t \geq 0$.

Lemma 1 (Sobolev-Poincaré inequality). *Let q be a number with $2 \leq q < \infty$ ($n = 1, 2, 3, 4$) or $2 \leq q \leq 2n/(n-4)$ ($n \geq 5$); then for $u \in H_0^2(\Omega)$ there exists a positive $C_* = C(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\Delta u\|_2. \quad (10)$$

We assume that m, p satisfy

$$\begin{aligned} 1 < p < \infty \quad (n = 1, 2, 3, 4) \quad \text{or} \quad 1 < p \leq \frac{n}{n-4} \quad (n \geq 5), \\ 1 < m < \infty \quad (n = 1, 2, 3, 4) \quad \text{or} \quad 1 < m \leq \frac{n+4}{n-4} \quad (n \geq 5). \end{aligned} \quad (11)$$

We state a local existence theorem that can be established by adopting the arguments of [2, 5, 22].

Theorem 2. *Assume that (11), (G1), and (G2) hold. Let $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ be given. Then there exists a unique weak solution $u(t)$ such that*

$$\begin{aligned} u &\in C([0, T]; H_0^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \\ u_t &\in L^2([0, T]; H_0^1(\Omega)) \cap L^{m+1}(\Omega \times (0, T)), \end{aligned} \quad (12)$$

for $T > 0$ small enough.

Next, we define the following functionals:

$$\begin{aligned}
 I(t) &:= I(u(t)) \\
 &= \left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u\|_2^2 + (g \circ \Delta u)(t) - \|u\|_{p+1}^{p+1}, \\
 J(t) &:= J(u(t)) \\
 &= \frac{1}{2} \left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u\|_2^2 \\
 &\quad + \frac{1}{2} (g \circ \Delta u)(t) - \frac{1}{p+1} \|u\|_{p+1}^{p+1}, \\
 E(t) &:= E(u(t)) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|\nabla u_t\|_2^2 + J(t),
 \end{aligned} \tag{13}$$

where

$$(g \circ v)(t) = \int_0^t \int_{\Omega} g(t-\tau) |v(t) - v(\tau)|^2 dx d\tau. \tag{14}$$

Remark 3. A multiplication of (1) by u_t and integration over Ω easily yield

$$\begin{aligned}
 E'(t) &= -\frac{1}{2} g(t) \|\Delta u\|_2^2 - \|\nabla u_t\|_2^2 + \frac{1}{2} (g' \circ \Delta u)(t) - \|u_t\|_{m+1}^{m+1} \\
 &\leq -\|\nabla u_t\|_2^2 - \|u_t\|_{m+1}^{m+1} \leq 0,
 \end{aligned} \tag{15}$$

since $g'(t) \leq 0$.

Our main results read as follows.

Theorem 4. Suppose that (11), (G1), and (G2) hold and $m < p$. Let u be the unique solution to problem (1) and denote $d_1 = ((p-1)/2(p+1))(l/C_*^2)^{(p+1)/(p-1)}$. Then, for any fixed $0 < \theta < 1$, $E(0) = \theta d_1 < d_1$, $I(0) < 0$, and g satisfies

$$\int_0^\infty g(\tau) d\tau < \frac{(p-1)(1-\theta)}{p-1 + (1/(p+1-\theta(p-1)))}, \tag{16}$$

either there exists some T^* such that the solution of problem (1) blows up in T^* in the sense of $\lim_{t \rightarrow T^*-} E(t) = -\infty$, or one has $\lim_{t \rightarrow +\infty} E(t) = -\infty$.

3. Proof of the Main Results

We denote

$$\begin{aligned}
 d(t) &= \inf_{u \in H_0^2(\Omega) \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u), \\
 \mathcal{N} &= \{u \in H_0^2(\Omega) \setminus \{0\} : I(u(t)) = 0\},
 \end{aligned} \tag{17}$$

then we can prove the following lemma.

Lemma 5. For $t \geq 0$, we have

$$0 < d_1 \leq d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u), \tag{18}$$

$$d(t) = \inf_{u \in \mathcal{N}} J(u). \tag{19}$$

Proof. Obviously,

$$d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u). \tag{20}$$

Straightforward computations yield

$$\begin{aligned}
 J(\lambda u) &= \frac{\lambda^2}{2} \left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right] \\
 &\quad - \frac{\lambda^{p+1}}{p+1} \|u(t)\|_{p+1}^{p+1},
 \end{aligned} \tag{21}$$

then

$$\begin{aligned}
 \frac{d}{d\lambda} J(\lambda u) &= \lambda \left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right] \\
 &\quad - \lambda^p \|u(t)\|_{p+1}^{p+1}.
 \end{aligned} \tag{22}$$

Let

$$\frac{d}{d\lambda} J(\lambda u) = 0, \tag{23}$$

which leads to

$$\begin{aligned}
 \overline{\lambda_1} &= 0, \\
 \overline{\lambda_2} &= \frac{\left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right]^{1/(p-1)}}{\left(\|u(t)\|_{p+1}^{p+1} \right)^{1/(p-1)}}.
 \end{aligned} \tag{24}$$

An elementary calculation shows

$$\frac{d^2}{d\lambda^2} J(\overline{\lambda_1} u) > 0, \quad \frac{d^2}{d\lambda^2} J(\overline{\lambda_2} u) < 0. \tag{25}$$

Using (G1) and Lemma 1 we arrive at

$$\begin{aligned}
 \sup_{\lambda \geq 0} J(\lambda u) &= J(\overline{\lambda_2} u) \\
 &= \frac{p-1}{2(p+1)} \\
 &\quad \times \left[\frac{\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)}{\left(\|u(t)\|_{p+1}^{p+1} \right)^{2/(p+1)}} \right]^{(p+1)/(p-1)} \\
 &\geq \frac{p-1}{2(p+1)} \left[\frac{l \|\Delta u(t)\|_2^2}{\left(\|u(t)\|_{p+1}^{p+1} \right)^{2/(p+1)}} \right]^{(p+1)/(p-1)} \\
 &\geq \frac{p-1}{2(p+1)} \left(\frac{l}{C_*^2} \right)^{(p+1)/(p-1)} \\
 &= d_1 > 0,
 \end{aligned} \tag{26}$$

which implies that $d(t) \geq d_1$.

To get (19), straightforward computations lead to

$$\begin{aligned}
& \left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta(\bar{\lambda}_2 u)\|_2^2 \\
& + (g \circ \Delta(\bar{\lambda}_2 u))(t) - \|\bar{\lambda}_2 u\|_{p+1}^{p+1} \\
& = \left(1 - \int_0^t g(\tau) d\tau\right) \\
& \quad \times \frac{\left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)\right]^{2/(p-1)}}{\left(\|u(t)\|_{p+1}^{p+1}\right)^{2/(p-1)}} \\
& \quad \times \|\Delta u(t)\|_2^2 \\
& \quad + \frac{\left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)\right]^{2/(p-1)}}{\left(\|u(t)\|_{p+1}^{p+1}\right)^{2/(p-1)}} \\
& \quad \times (g \circ \Delta u)(t) \\
& \quad - \frac{\left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)\right]^{(p+1)/(p-1)}}{\left(\|u(t)\|_{p+1}^{p+1}\right)^{(p+1)/(p-1)}} \\
& \quad \times \|u(t)\|_{p+1}^{p+1} \\
& = \frac{\left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)\right]^{(p+1)/(p-1)}}{\|u(t)\|_{p+1}^{2(p+1)/(p-1)}} \\
& \quad \times \left\{ \frac{\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2}{\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)} \right. \\
& \quad + \frac{(g \circ \Delta u)(t)}{\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)} \\
& \quad \left. - \frac{\|u(t)\|_{p+1}^{p+1}}{\|u(t)\|_{p+1}^{p+1}} \right\} = 0,
\end{aligned} \tag{27}$$

which implies that $\bar{\lambda}_2 u \in \mathcal{N}$. Also, for any $u \in \mathcal{N}$, we note that

$$\begin{aligned}
& \bar{\lambda}_2(u) \\
& = \frac{\left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)\right]^{1/(p-1)}}{\left(\|u(t)\|_{p+1}^{p+1}\right)^{1/(p-1)}} = 1.
\end{aligned} \tag{28}$$

Therefore we have $\bar{\lambda}_2(u)u = u$ for all $u \in \mathcal{N}$. Hence, we complete the proof. $\square$

Lemma 6. Suppose that (11), (G1), and (G2) hold; $E(0) < d_1$; and $I(0) < 0$; then we have $I(t) < 0$, for all $t \in [0, T)$ and

$$\begin{aligned}
d_1 & < \frac{p-1}{2(p+1)} \left[\left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right] \\
& < \frac{p-1}{2(p+1)} \|u\|_{p+1}^{p+1}.
\end{aligned} \tag{29}$$

Proof. Using (15), we have $E(t) < d_1$, for all $t \in [0, T)$. To show that $I(t) < 0$ on $[0, T)$, we proceed by contradiction. Assume there exists $t_0 \in [0, T)$ such that $I(t_0) \geq 0$. Since $I(0) < 0$, it follows that there exists some $\tilde{t} \in (0, t_0]$ such that $I(u(\tilde{t})) = 0$. Now, we define

$$\begin{aligned}
t^* & = \inf \left\{ \tilde{t} \in (0, t_0] : \left(1 - \int_0^{\tilde{t}} g(\tau) d\tau\right) \right. \\
& \quad \times \|\Delta u(\tilde{t})\|_2^2 + (g \circ \Delta u)(\tilde{t}) = \|u(\tilde{t})\|_{p+1}^{p+1} \left. \right\}.
\end{aligned} \tag{30}$$

Then, we have $I(u(t^*)) = 0$ and

$$\begin{aligned}
& \left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \\
& < \|u\|_{p+1}^{p+1}, \quad \forall 0 \leq t < t^*.
\end{aligned} \tag{31}$$

Suppose that $\|\Delta u(t^*)\|_2^2 = 0$, by the regularity of $u(t)$, we have

$$\lim_{t \rightarrow t^{*-}} \|\Delta u(t)\|_2^2 = 0. \tag{32}$$

On the other hand, applying Lemma 1 to (31), we obtain

$$\begin{aligned}
& \left(1 - \int_0^t g(\tau) d\tau\right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \\
& < \|u\|_{p+1}^{p+1} \leq C_*^{p+1} \|\Delta u\|_2^{p+1}, \quad 0 \leq t < t^*.
\end{aligned} \tag{33}$$

From the above inequality, we can easily get $\|\Delta u(t)\|_2^2 \neq 0$, for all $t \in [0, t^*)$, and

$$\frac{l}{C_*^{p+1}} < \|\Delta u\|_2^{p-1}, \tag{34}$$

which implies

$$\lim_{t \rightarrow t^{*-}} \|\Delta u(t)\|_2 > \left(\frac{l}{C_*^{p+1}} \right)^{1/(p-1)} \tag{35}$$

and this contradicts to (32).

Suppose that $\|\Delta u(t^*)\|_2^2 \neq 0$. Applying Lemma 5, we see that $d(t)$ is the infimum of $J(u(t))$ over all functions u in $\mathcal{N}$ and $J(u(t^*)) \geq d(t) \geq d_1$, which contradicts to $J(u(t^*)) \leq E(t^*) < d_1$. Thus, we conclude that $I(t) < 0$ for all $t \in [0, T)$.

To get (29), using Lemma 5, (31) and the conclusion that $I(t) < 0$, for all $t \in [0, T]$, we obtain

$$\begin{aligned}
 d_1 &\leq \frac{p-1}{2(p+1)} \\
 &\times \left[\left(\left(1 - \int_0^t g(\tau) d\tau \right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right) \right. \\
 &\times \left[\left(1 - \int_0^t g(\tau) d\tau \right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right]^{2/(p-1)} \\
 &\times \left((\|u\|_{p+1}^{p+1})^{2/(p-1)} \right)^{-1} \\
 &< \frac{p-1}{2(p+1)} \left[\left(1 - \int_0^t g(\tau) d\tau \right) \|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t) \right] \\
 &< \frac{p-1}{2(p+1)} \|u\|_{p+1}^{p+1}, \quad 0 \leq t < T,
 \end{aligned} \tag{36}$$

which completes the proof. $\square$

Proof of Theorem 4. Suppose on the contrary that there exists a positive constant C_0 such that $E(t) \geq -C_0$, for all $t \geq 0$. An integration of (15) over $[0, t]$ yields

$$\begin{aligned}
 d + C_0 &> E(0) - E(t) \\
 &\geq \int_0^t (\|\nabla u_t(\tau)\|_2^2 + \|u_t(\tau)\|_{m+1}^{m+1}) d\tau, \tag{37} \\
 &\forall t \geq 0.
 \end{aligned}$$

Set $L(t) = \|u\|_2^2 + \|\nabla u\|_2^2$; then we have

$$\begin{aligned}
 L'(t) &= 2 \int_{\Omega} uu_t dx + 2 \int_{\Omega} \nabla u \cdot \nabla u_t dx, \tag{38} \\
 L''(t) &= 2 \int_{\Omega} u_t^2 dx + 2 \int_{\Omega} uu_{tt} dx \\
 &\quad + 2 \int_{\Omega} \nabla u_t^2 dx + 2 \int_{\Omega} \nabla u \cdot \nabla u_{tt} dx \\
 &= 2 (\|u_t\|_2^2 + \|\nabla u_t\|_2^2 - \|\Delta u\|_2^2) \\
 &\quad + 2 \int_{\Omega} \Delta u \int_0^t g(t-\tau) \Delta u(x, \tau) d\tau dx \\
 &\quad - 2 \int_{\Omega} \nabla u \cdot \nabla u_t dx - 2 \int_{\Omega} |u_t|^{m-1} u_t dx + 2 \|u\|_{p+1}^{p+1}.
 \end{aligned} \tag{39}$$

We estimate (39) as follows. By using Cauchy-Schwarz inequality and Young's inequality, we get for any $\eta > 0$

$$\begin{aligned}
 &\int_{\Omega} \int_0^t g(t-\tau) \Delta u(t) \Delta u(\tau) d\tau dx \\
 &= \int_{\Omega} \int_0^t g(t-\tau) \Delta u(t) (\Delta u(\tau) - \Delta u(t)) d\tau dx \\
 &\quad + \left(\int_0^t g(\tau) d\tau \right) \|\Delta u(t)\|_2^2 \\
 &\geq - \int_0^t g(t-\tau) \|\Delta u(t)\|_2 \|\Delta u(\tau) - \Delta u(t)\|_2 d\tau \\
 &\quad + \left(\int_0^t g(\tau) d\tau \right) \|\Delta u(t)\|_2^2 \\
 &\geq -\eta (g \circ \Delta u)(t) + \left(1 - \frac{1}{4\eta} \right) \left(\int_0^t g(\tau) d\tau \right) \|\Delta u(t)\|_2^2.
 \end{aligned} \tag{40}$$

Using Young's inequality, we get for any $\varepsilon_1 > 0$

$$\left| 2 \int_{\Omega} \nabla u \cdot \nabla u_t dx \right| \leq \varepsilon_1 \|\nabla u\|_2^2 + C(\varepsilon_1) \|\nabla u_t\|_2^2. \tag{41}$$

By Hölder and Young inequalities, we arrive at

$$\begin{aligned}
 &\left| 2 \int_{\Omega} uu_t |u_t|^{m-1} dx \right| \\
 &\leq 2 \|u\|_{m+1} \|u_t\|_{m+1}^m \\
 &\leq 2 |\Omega|^{(p-m)/((p+1)(m+1))} \|u\|_{p+1} \|u_t\|_{m+1}^m \\
 &\leq 2 |\Omega|^{(p-m)/((p+1)(m+1))} \|u\|_{p+1}^{1-(p+1)/(m+1)} \\
 &\quad \times \|u\|_{p+1}^{(p+1)/(m+1)} \|u_t\|_{m+1}^m \\
 &< 2 |\Omega|^{(p-m)/((p+1)(m+1))} \left(\frac{2(p+1)d_1}{p-1} \right)^{(1/(p+1))-(1/(m+1))} \\
 &\quad \times \|u\|_{p+1}^{(p+1)/(m+1)} \|u_t\|_{m+1}^m \\
 &\leq C_d (\varepsilon_2 \|u\|_{p+1}^{p+1} + C(\varepsilon_2) \|u_t\|_{m+1}^{m+1}),
 \end{aligned} \tag{42}$$

where $C_d = \frac{2|\Omega|^{(p-m)/((p+1)(m+1))}(2(p+1)d_1/(p-1))^{1/(p+1)-(1/(m+1))}}$. Combining (40)–(42), we obtain

$$\begin{aligned}
& L''(t) + C(\varepsilon_1) \|\nabla u_t\|_2^2 + C_d C(\varepsilon_2) \|u_t\|_{m+1}^{m+1} \\
& > 2(\|u_t\|_2^2 + \|\nabla u_t\|_2^2) \\
& \quad - 2 \left[1 - \left(1 - \frac{1}{4\eta}\right) \left(\int_0^t g(\tau) d\tau \right) \right] \|\Delta u(t)\|_2^2 \\
& \quad - 2\eta (g \circ \Delta u)(t) + (2 - C_d \varepsilon_2) \|u\|_{p+1}^{p+1} - \varepsilon_1 \|\nabla u\|_2^2 \\
& \geq 2(\|u_t\|_2^2 + \|\nabla u_t\|_2^2) \\
& \quad - 2 \left[1 + \frac{\varepsilon_1 B}{2} - \left(1 - \frac{1}{4\eta}\right) \left(\int_0^t g(\tau) d\tau \right) \right] \|\Delta u(t)\|_2^2 \\
& \quad - 2\eta (g \circ \Delta u)(t) + (2 - C_d \varepsilon_2) \|u\|_{p+1}^{p+1},
\end{aligned} \tag{43}$$

where B is the Poincaré constant. Since

$$\begin{aligned}
& -2 \left[1 + \frac{\varepsilon_1 B}{2} - \left(1 - \frac{1}{4\eta}\right) \left(\int_0^t g(\tau) d\tau \right) \right] \|\Delta u(t)\|_2^2 \\
& \quad - 2\eta (g \circ \Delta u)(t) + (2 - C_d \varepsilon_2) \|u\|_{p+1}^{p+1} \\
& \geq -2 \left[1 + \frac{\varepsilon_1 B}{2} - \left(1 - \frac{1}{4\eta}\right) \left(\int_0^t g(\tau) d\tau \right) \right] \|\Delta u(t)\|_2^2 \\
& \quad - 2\eta (g \circ \Delta u)(t) + (2 - C_d \varepsilon_2) \|u\|_{p+1}^{p+1} \\
& \quad + \sigma (E(t) - E(0)) \\
& \geq \left\{ \frac{\sigma}{2} - 2 - \varepsilon_1 B - \left[\frac{\sigma}{2} - 2 \left(1 - \frac{1}{4\eta}\right) \right] \int_0^t g(\tau) d\tau \right\} \\
& \quad \times \|\Delta u(t)\|_2^2 \\
& \quad + \frac{\sigma}{2} (\|u_t\|_2^2 + \|\nabla u_t\|_2^2) \\
& \quad + \left(\frac{\sigma}{2} - 2\eta \right) (g \circ \Delta u)(t) \\
& \quad + \left(2 - C_d \varepsilon_2 - \frac{\sigma}{p+1} \right) \|u\|_{p+1}^{p+1} - \sigma E(0), \quad \forall t \geq 0,
\end{aligned} \tag{44}$$

where $\sigma > 0$ will be chosen later. Then, we have

$$\begin{aligned}
& L''(t) + C(\varepsilon_1) \|\nabla u_t\|_2^2 + C_d C(\varepsilon_2) \|u_t\|_{m+1}^{m+1} \\
& > \left(2 + \frac{\sigma}{2} \right) (\|u_t\|_2^2 + \|\nabla u_t\|_2^2) \\
& \quad + \left\{ \frac{\sigma}{2} - 2 - \varepsilon_1 B - \left[\frac{\sigma}{2} - 2 \left(1 - \frac{1}{4\eta}\right) \right] \int_0^t g(\tau) d\tau \right\} \\
& \quad \times \|\Delta u(t)\|_2^2
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\sigma}{2} - 2\eta \right) (g \circ \Delta u)(t) \\
& + \left(2 - C_d \varepsilon_2 - \frac{\sigma}{p+1} \right) \|u\|_{p+1}^{p+1} - \sigma E(0), \\
& \quad \forall t \geq 0.
\end{aligned} \tag{45}$$

Choosing $\eta < (\sigma/4) - (\theta(p-1)/2)$ and ε_1 small enough and using (16), we can get a certain constant $\sigma \in (2\theta(p-1) + 4, 2(p+1))$, such that

$$\begin{aligned}
& \left\{ \frac{\sigma}{2} - 2 - \varepsilon_1 B - \left[\frac{\sigma}{2} - 2 \left(1 - \frac{1}{4\eta}\right) \right] \int_0^t g(\tau) d\tau \right\} > \theta(p-1), \\
& \quad \frac{\sigma}{2} - 2\eta > \theta(p-1).
\end{aligned} \tag{46}$$

Therefore, it follows from (G1) and Lemma 6 that

$$\begin{aligned}
& \left\{ \frac{\sigma}{2} - 2 - \varepsilon_1 B - \left[\frac{\sigma}{2} - 2 \left(1 - \frac{1}{4\eta}\right) \right] \int_0^t g(\tau) d\tau \right\} \|\Delta u(t)\|_2^2 \\
& \quad + \left(\frac{\sigma}{2} - 2\eta \right) (g \circ \Delta u)(t) - \sigma E(0) \\
& > \theta(p-1) (\|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)) \\
& \quad - \sigma E(0) > 2\theta(p+1) d_1 - \sigma E(0) \\
& \geq 2(p+1) E(0) - \sigma E(0) > 0,
\end{aligned} \tag{47}$$

since the choice of σ . Finally, we take ε_2 small enough so that $2 - C_d \varepsilon_2 - (\sigma/(p+1)) := C_1 > 0$ and use Lemma 6 to get

$$\begin{aligned}
& L''(t) + C(\varepsilon_1) \|\nabla u_t\|_2^2 + C_d C(\varepsilon_2) \|u_t\|_{m+1}^{m+1} \\
& > C_1 \|u\|_{p+1}^{p+1} > \frac{2C_1(p+1)}{p-1} d_1.
\end{aligned} \tag{48}$$

Integrate twice the above inequality over $[0, t]$ and use (37); we have

$$\begin{aligned}
& L(t) > \frac{C_1(p+1)d_1}{p-1} t^2 \\
& \quad + [L'(0) - (C(\varepsilon_1) + C(\varepsilon_2)(C_0 + d_1))] t + L(0),
\end{aligned} \tag{49}$$

which implies that $L(t)$ grows more quickly than the linear growth for $t \geq 0$.

On the other hand, by (37) and Hölder's inequality, we have

$$\begin{aligned}
& \|u(t)\|_2 \leq \|u_0\|_2 + \int_0^t \|u_t(\tau)\|_2 d\tau \\
& \leq \|u_0\|_2 + C(C_0 + d_1)^{1/(m+1)} t^{(m-1)/(m+1)}, \\
& \|\nabla u\|_2 \leq \|\nabla u_0\|_2 + \int_0^t \|\nabla u_t(\tau)\|_2 d\tau \leq \|\nabla u_0\|_2 + (C_0 + d_1)^{1/2}.
\end{aligned} \tag{50}$$

Thus,

$$\begin{aligned} L(t) &= \|u\|_2^2 + \|\nabla u\|_2^2 \\ &\leq C_2 t^{2(m-1)/(m+1)} + C_3 t + C_4, \quad \forall t \geq 0, \end{aligned} \quad (51)$$

where C_2 , C_3 , and C_4 are positive constants. Obviously, (51) contradicts (49). $\square$

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