Research Article

Bounded Domains of Generalized Riesz Methods with the Hahn Property

Maria Zeltser

Department of Mathematics, Tallinn University, Narva Maantee 25, 10120 Tallinn, Estonia

Correspondence should be addressed to Maria Zeltser; mariaz@tlu.ee

Received 29 May 2013; Revised 28 August 2013; Accepted 4 September 2013

Academic Editor: Antonio S. Granero

Copyright © 2013 Maria Zeltser. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In 2002 Bennett et al. started the investigation to which extent sequence spaces are determined by the sequences of 0s and 1s that they contain. In this relation they defined three types of Hahn properties for sequence spaces: the Hahn property, separable Hahn property, and matrix Hahn property. In general all these three properties are pairwise distinct. If a sequence space $E$ is solid and $\{\{0,1\}^{N} \cap E\}^{\beta} = E^{\beta} = \ell_{1}$ then the two last properties coincide. We will show that even on these additional assumptions the separable Hahn property and the Hahn property still do not coincide. However if we assume $E$ to be the bounded summability domain of a regular Riesz matrix $R_{p}$ or a regular nonnegative Hausdorff matrix $H_{p}$, then this assumption alone guarantees that $E$ has the Hahn property. For any (infinite) matrix $A$ the Hahn property of its bounded summability domain is related to the strongly nonatomic property of the density $d_{A}$ defined by $A$. We will find a simple necessary and sufficient condition for the density $d_{A}$ defined by the generalized Riesz matrix $R_{p,m}$ to be strongly nonatomic. This condition appears also to be sufficient for the bounded summability domain of $R_{p,m}$ to have the Hahn property.

1. Preliminaries and Introduction

We start with some preliminaries. For other notations and preliminary results we refer the reader to [1–3].

Let $\chi$ denote the set of all sequences of 0s and 1s and let $\chi(E)$ denote the linear hull of $\chi \cap E$.

An FK-space is a sequence space endowed with a complete, metrizable, locally convex topology under which all coordinate mappings $x = (x_{j}) \rightarrow x_{k}$ ($k \in \mathbb{N}$) are continuous.

A sequence space $E$ is said to have the Hahn property, the separable Hahn property, and the matrix Hahn property, if $\chi(E) \subset F$ implies $E \subset F$ whenever $F$ is any FK-space, a separable FK-space, and a matrix domain $c_{0}$, respectively. Obviously, the Hahn property implies the separable Hahn property, and the latter implies the matrix Hahn property.

If $E$ has the matrix Hahn property then $\chi(E)^{\beta} = E^{\beta}$ (cf. Theorem 5.1 in [1]), but, in general, the inverse implication does not hold even for monotone sequence spaces (see Theorem 1.1 in [4]). Still if we ask $E$ to be a solid sequence space containing $\varphi$ (the set of all finite sequences) and satisfying $E^{\beta} = \ell_{1}$, then $\chi(E)^{\beta} = E^{\beta}$ implies the separable Hahn property of $E$ (see [2, Theorem 6]). This result suggests the following problem due to Boos and Leiger (cf. Problem 3 in [2]). Now, we formulate this question as a problem and give a negative answer to it.

Problem 1. Let $E$ be a solid sequence space containing $\varphi$ and satisfying $\chi(E)^{\beta} = E^{\beta} = \ell_{1}$. Then $E$ has the separable Hahn property. Does it have the Hahn property?

To answer this problem we consider some facts from the theory of double sequence spaces.

A double sequence space is a linear subspace of $\Omega$, the space of all real double sequences $x = (x_{k,l})$. In particular, the following sets are double sequence spaces:

\[ \mathcal{M}_{u} := \left\{ x \in \Omega \mid \|x\|_{\infty} := \sup_{k,l} |x_{k,l}| < \infty \right\}, \]
We denote by ξ₁ the set of double sequences of zeros and ones in C_b(Ω) such that,

\[ ξ₁ := \{0,1\}^{N×N} \cap C_b(Ω) \cap (M_u \cap C_b(Ω)) \]  

(2)

The space of all double sequences Ω can be identified with the space of all sequences using a suitable isomorphism T : Ω → ω (see (2) in [5] for a possible definition of T).

**Theorem 1.** There exists a solid sequence space E containing p and satisfying \( \chi(E) = E = \ell^1 \) which does not have the Hahn property.

**Proof.** Set \( E := T(M_u \cap C_b(Ω)) \). Then E is a solid sequence space containing p and failing to have the Hahn property (see Example 3.7 in [5]). In view of Theorem 3.2 in [5] we have also \( \chi(E) = \ell^1 \) as well as \( E \). \( \square \)

Taking the explained situation concerning the sufficiency of \( \chi(E) = \ell^1 \) for the Hahn property of E into consideration, it is mathematically interesting to research for classes of sequence spaces E with the property that \( \chi(E) = \ell^1 \) implies the Hahn property of E (or another equivalent condition). In that sense we consider on the base of related results in [6, 7] the bounded domains of Riesz methods in Section 2 and, more general, of the generalized Riesz methods in Section 3.

2. Bounded Domains of Riesz Methods

Let p = \( (p_n) \) be a real sequence with

\[ P_1 > 0, \quad P_k \geq 0 \quad (k \in N), \]  

(3)

\[ P_n := \sum_{k=1}^{n} P_k \quad (n \in N). \]

The Riesz matrix \( R_p = (r_{nk}) \) (associated with p) is defined by

\[ r_{nk} := \begin{cases} \frac{P_k}{P_n} & \text{if } k \leq n, \\ 0 & \text{otherwise} \end{cases} \quad (n, k \in N). \]  

(4)

The summability method corresponding \( R_p \) is called Riesz method.

The Riesz matrix \( R_p \) is conservative, and it is either regular (being equivalent to \( p = \ell^1 \)) or coercive. If \( p = \ell^1 \), then \( \ell^1 \cap c_0 \) has the Hahn property since \( \ell^1 \cap c_0 = \ell^1 \). If \( p = \ell^1 \), then \( \ell^1 \cap c_0 \) has the Hahn property if and only if \( (p_n/P_n)^i \) in \( c_0 \) (cf. Corollary 3.9 in [6]).

In Section 1 we have seen that the relation \( \chi(E) = E = \ell^1 \) does not imply in general the Hahn property of E even for a solid space E. This result suggests the following problem due to Boos and Leiger (cf. Problem 3 in [7, Section 4]).

**Problem 2.** Does the relation \( \chi(\ell^1 \cap c_0) = \ell^1 \) imply the Hahn property of \( \ell^1 \cap c_0 \)?

Aiming to a positive result we first prove the following more general theorem.

**Theorem 2.** Let \( B = (b_{nk}) \) be an infinite matrix such that there exists a subset F of \( \chi \) with \( p \in \ell^1 \) and \( F^\beta = (F^\beta) \). Then \( \ell^1 \in \ell^1 \) (being equivalent to sup_n \( \sum b_{nk} \mid < \infty \)).

**Proof.** In view of \( F \subseteq \ell^1 \) we have \( \sup_n \mid \sum b_{nk} \mid < \infty \) for each \( x \in F \), so in particular \( (b_{nk}) \subseteq \ell^1 \) for each \( n \in N \).

Assume on contrary that there exists an \( x \in \ell^1 \cap \ell^1 \). Hence sup_n \( \sum b_{nk} \mid = \infty \) since

\[ \left| \sum_{k} b_{nk} \right| \leq \|x\| \sup_{n} \sum_{k} b_{nk} \quad (x \in \ell^1 \cap \ell^1). \]  

(5)

Aiming to a contradiction we will construct by induction two index sequences \( (k_i) \) and \( (n_i) \), and then with the help of them we will define \( \omega \in \ell^1 \cap \ell^1 \). First of all set \( k_1 := 0 \) and \( A_1 := 0 \). Now choose \( n_1 \in N \) such that

\[ \sum_{k} b_{nk_1} > 1^2 (A_1 + 1) + \frac{1}{2}. \]  

(6)

Then choose \( k_2 > k_1 \) such that

\[ \sum_{k=k_2+1}^{N} |b_{nk}| < \frac{1}{2}. \]  

(7)

and set

\[ B_1 := \sum_{k} |b_{nk}|. \]  

(8)

Now suppose that \( k_1, \ldots, k_i \) and \( n_1, \ldots, n_{i-1} \) are already chosen for \( i > 1 \). Since \( p \in \ell^1 \), then

\[ A_i := \sup_{n} \sum_{k=1}^{N} |b_{nk}| < \infty. \]  

(9)

We choose \( n_i > n_{i-1} \) such that

\[ \sum_{k} |b_{nk}| > 1^2 (A_i + 1) + \frac{1}{2^2}. \]  

(10)

and then we take \( k_{i+1} > k_i \) such that

\[ \sum_{k=k_{i+1}+1}^{N} |b_{nk}| < \frac{1}{2^i}. \]  

(11)

So

\[ B_i := \sum_{k=k_{i+1}+1}^{N} |b_{nk}| > 1^2 (A_i + 1), \quad A_i/B_i < \frac{1}{2^i}. \]  

(12)
Now we define a $w = (w_k)$ by
\[ w_k := \frac{b_{nk}}{B_k} \quad \text{for } k_i + 1 \leq k \leq k_{i+1}, \ i \in \mathbb{N} \tag{13} \]
and note that $w \notin \ell^1$ because
\[ \sum_k |w_k| = \sum_i \frac{1}{B_{i+1}} \sum_{k=k_i i+1}^{k_i i} |b_{nk}| = \sum_i \frac{1}{i} \to \infty. \tag{14} \]

We are going to verify $w \in E^\beta$ in contradiction to $F^\beta = \ell^1$. For that end let $u \in F$ and $s, t \in \mathbb{N}$ with $t > s$ be fixed. We choose $i_1, i_2 \in \mathbb{N}$ such that $k_{i_1 - 1} < s \leq k_{i_1}$ and $k_{i_2} < t \leq k_{i_2 + 1}$. If $k_{i_1 - 1} = k_{i_2}$ we have
\[ \left| \sum_{k=s}^{t} w_k u_k \right| \leq \frac{1}{B_{i+1}} \left( \sum_{k=k_{i_1 i+1}}^{k_{i_1 i}} |b_{nk}| + \sum_{k=k_{i_2 i+1}}^{s} |b_{nk}| \right) \leq \frac{1}{B_{i+1}} \left( M + A_1 + \frac{1}{2} \right) \to 0, \quad \text{as } s, t \to \infty. \tag{16} \]

Since $u \in F \subset \ell^{\infty}_B$, we have $\sup_{n} |Bu_n| := M < \infty$; hence,
\[ \left| \sum_{k=k_{i_1 i+1}}^{k_{i_1 i}} \frac{b_{nk} u_k}{B_k} \right| \leq \frac{1}{B_{i+1}} \left( \sum_{k=k_{i_1 i+1}}^{k_{i_1 i}} |b_{nk}| \right) \leq \frac{1}{B_{i+1}} \left( M + A_1 + \frac{1}{2} \right) \to 0. \tag{17} \]

The following two corollaries are basic for the answer to Problem 2.

**Corollary 3.** Let $B$ be a matrix satisfying $\mathcal{T} \subset c_0$, then $c_0 \subset c_p$. Moreover, if $\mathcal{T} \cup \{e\} \subset c_0$; then $B$ is conservative and $B \in K_G$ (i.e., each matrix $A$ with $\chi(c_0 \cap c_p) \subset c_A$ is conservative).

**Proof.** Taking $F = \mathcal{T}$ in Theorem 2 in view of $\mathcal{T}^\beta = \ell^1$ we have $\ell^\infty \subset \ell^\infty_B$, so $\sup_n \sum_k |b_{nk}| < \infty$. Since $\ell^\infty \in \langle \mathcal{T} \rangle \subset c_0$, it follows that $c_0 \subset c_p$.

The second part of the corollary follows immediately from the first one.

**Corollary 4.** Let $B_n$ be an infinite matrix, and let $F$ be a subset of $\mathcal{X}$ with $\varphi \in \langle F \rangle \subset c_0$ and $F^\beta = \ell^1$. Let $E$ be a subset of $\ell^\infty_B$ such that $E^\beta = \ell^\infty$ and $F \cap E \subset c_0$. Furthermore, let $(E, \tau_E)$ be an $F$-space with a dense subset $F \cap E$. Hence $E \subset c_0$.

**Proof.** We consider the functionals $f_n : E \to \mathbb{R}$ with $f_n(x) = \sum_k b_{nk} x_k$ ($n \in \mathbb{N}$). By Theorem 2 the sequence $(f_n)$ is pointwise bounded on $E \subset \ell^\infty_B$, and by our assumptions it converges pointwise on the dense subset $F \cap E$ of $E$. Hence by generalized version of Banach-Steinhaus theorem (cf. Theorem 6.8.6 in [3]) the sequence $(f_n)$ converges pointwise on $E$. So $E \subset c_0$.

**Corollary 5.** Let $A$ be a conservative matrix. Then $A \in K_G$ if and only if $\chi(c_A \cap \ell^\infty)^\beta = \ell^1$.

**Proof.** Suppose $A \in K_G$ and let $u \in \chi(c_A \cap \ell^\infty)^\beta$. We set $b_{nk} := u_k$ ($k, n \in \mathbb{N}$). Then $\chi(c_A \cap \ell^\infty) \subset c_0$; hence, $B$ is conservative. Therefore $u \in \ell^\infty = \ell^1$. Hence $\chi(c_A \cap \ell^\infty)^\beta \subset \ell^1$. On the other hand in view of conservativity of $A$ we have $\chi(c_A \cap \ell^\infty)^\beta \subset (c_A \cap \ell^\infty)^\beta = \ell^1$.

Now let $B$ be a matrix satisfying $\chi \cap c_A \subset c_B$. Take $E = c_0$ and $F = \chi(c_A \cap \ell^\infty)$ in Corollary 4. Then since $\chi(c_A \cap \ell^\infty) \cap c_0 = \varphi$ is dense in $(c_0, \|\cdot\|_{\ell^\infty})$, we have $c_0 \subset c_p$. Moreover, since $e \in \chi \cap c_A \subset c_B$, it follows that $c \subset c_p$. Hence $A \in K_G$.

Now we give a positive answer to Problem 2.
Proposition 6. Let $R_p$ be regular. Then $\ell^\infty \cap cR_p$ has the Hahn property if and only if $\chi(\ell^\infty \cap cR_p)^\beta = \ell^1$.

Proof. In view of Corollary 5 we have $\chi(\ell^\infty \cap cR_p)^\beta = \ell^1$ if and only if $R_p \in KG$, but the last condition is equivalent to the Hahn property of $\ell^\infty \cap cR_p$ (cf. Corollary 3.9 in [6]). □

In the same way as for Riesz matrices we obtain a similar result for nonnegative Hausdorff matrices, where we make use that a nonnegative regular Hausdorff matrix $H_p$ is KG if and only if $\ell^\infty \cap cH_p$ has the Hahn property (Theorem 3.2.1 in [7]).

Proposition 7. Let $H_p$ be a nonnegative regular Hausdorff matrix. Then $\ell^\infty \cap cH_p$ has the Hahn property if and only if $\chi(\ell^\infty \cap cH_p)^\beta = \ell^1$.

3. Bounded Domains of Generalized Riesz Methods

In the previous section we demonstrated that for $A = R_p$ the relation $\chi(\ell^\infty \cap cR_p)^\beta = \ell^1$ is equivalent to the Hahn property of $\ell^\infty \cap cR_p$. Now we consider two more conditions being equivalent to these properties for $A = R_p$ (cf. [6, Corollary 3.9 (ii)], [7, Theorem 3.1.1]):

(a) $A = (a_{nk})$ has spreading rows; that is, $(sup_{k \in N} a_{nk}) \in c_0$;

(b) there exists a sequence

\[(\mathcal{N}_t) = \left\{ \left( N_v \mid v \in N_{t,i} \right) \right\}, \quad v_i \in N \quad (t \in N) \quad (21)\]

of partitions of $N$ (called an admissible partition sequence) such that

\[\lim sup_{t \to \infty} \sup_{n \to \infty} \sup_{1 \leq k \leq n} A_{nkt} = 0 \quad \text{where} \quad A_{nkt} := \sum_{k \in N_{t,i}} a_{nk}. \quad (22)\]

Recall that $\mathcal{N}_t = \{ N_v \mid v \in N_{t,i} \}$ is called a partition of $N$ if $N = \bigcup_{t=1}^{\infty} N_v$ and $N_s \cap N_{s+1} = \emptyset$ if $v \neq \mu$.

So in fact conditions (a) and (b) are equivalent for $A = R_p$. Note that for any nonnegative and regular for null sequences matrix $A$ condition (b) is equivalent to the following condition (see [7] for the corresponding definitions).

(b’) Density $d_A$ defined by the matrix $A$ is strongly nonatomic on the power set $P(N)$ of $N$.

Now we replace a Riesz matrix $R_p$ with a more general matrix obtained as a row submatrix of $R_p$. More precisely, we consider the matrix $R_{p,m} = (a_{nk})$, called generalized Riesz matrix, defined by

\[a_{nk} := \begin{cases} \frac{p_k}{P_{m_n}} & \text{if } 1 \leq k \leq m_n, \\ 0 & \text{otherwise}, \end{cases} \quad (n,k \in N), \quad (23)\]

where $(m_n)$ is any fixed index sequence and $p = (p_i)$ is a sequence of positive reals with $p \notin \ell^1$.

Boos and Leiger showed (cf. Theorem 2.1 in [8]) that in the case of some sufficient conditions on terms of $R_{p,m}$ the properties (a) and (b) are equivalent for $A = R_{p,m}$. These conditions covered all possible cases except

\[\lim sup_n \frac{P_{m_n}}{P_{m_{n+1}}} = 1, \quad \lim inf_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0. \quad (24)\]

Note that even in this case the implication (b)⇒(a) holds (cf. Theorem 2.1 in [8]). So only the implication (a)⇒(b) in case (24) was under the question.

In relation with this in [8], Problem 2.3 Boos and Leiger posed the following.

Problem 3. Complete the distinction of cases in [8, Theorem 2.1]. May be the relation $\mathcal{F} \subset c_{0R_{p,m}}$ hold?

First we answer the first question in Problem 3 and then demonstrate that the relation $\mathcal{F} \subset c_{0R_{p,m}}$ does not hold in general for generalized Riesz matrices $R_{p,m}$ satisfying (24).

For our first aim we apply—in the same way as Boos and Leiger in [8]—the following lemma due to Kuttner and Parameswaran (cf. [9, Lemma 2.1]).

Lemma 8. Let $\{z_1, z_2, \ldots, z_s\}$ for any $n \in N$ be a set of nonnegative real numbers. Let $Z_n := \sum_{k=1}^{s} z_k$ and suppose that $B_n > 0$ with $z_k \leq B_n$ $(k \in N_{t,i})$, and let $t \in N$ be arbitrarily given. Then one can divide the set $N_{t,i}$ into $t$ (pairwise disjoint) subsets $N_1, N_2, \ldots, N_t$ (some of them may be empty), such that

\[\sum_{k \in N_{t,i}} z_k \leq \frac{1}{t} Z_n + B_n \quad (s = 1, 2, \ldots, t). \quad (25)\]

Proposition 9. Suppose that $R_{p,m} = (a_{nk})$ has spreading rows; that is,

\[\lim_n \frac{1}{P_{m_n}} \sup_{1 \leq k \leq m_n} p_k = 0, \quad (26)\]

\[\lim inf_n \frac{P_{m_n}}{P_{m_{n+1}}} = 0; \quad (27)\]

then there exists an admissible partition sequence of $N$ satisfying (22).

Proof. We set $A_n := P_{m_n}$. For a given $t \in N$ in view of (27) we can choose the minimal index $n_t$ such that $A_{n_t}/A_1 > (t+1)/t$. Afterwards (27) we choose the minimal index $n_{t+1} > n_t$ such that $A_{n}/A_{n-1} > (t+1)/t$. Continuing in the same way we choose the minimal index $n_t > n_{t-1}$ such that $A_{n}/A_{n-1} > (t+1)/t$. In view of (26) it follows that $m_{n_{t+1}} - m_{n_t} \to \infty$. Again by (26) we can choose $\alpha_i \in N, \alpha_i \geq n_t$ such that

\[\frac{p_k}{A_n} < \frac{1}{t^2} \quad \text{for each } k \in N_{m_n}, \quad n \geq \alpha_i; \quad (28)\]
also we can find \( \beta_t \in \mathbb{N}, \beta_t > \alpha_t \) such that
\[
\frac{A_{\alpha_t}}{A_n} < \frac{1}{t^2} \quad \text{for } n \geq \beta_t. \tag{29}
\]

By Lemma 8 for any positive integer \( s \geq \alpha_t \), we can divide the set \( I_s := (m_{n}, m_{n+1}] \cap \mathbb{N} \) into \( t^2 \) disjoint subsets (some of them may be empty), say \( N_{a_t}(1), N_{a_t}(1), \ldots, N_{a_t}(t^2) \) such that (cf. (28))
\[
\sum_{k \in N_{a_t}(v)} p_k \leq \frac{1}{t^2} \sum_{k} p_k + \frac{1}{t^2} A_{n+1} \leq \frac{2}{t^2} A_{n+1}. \tag{30}
\]

Now, setting
\[
N_s := \bigcup_s N_{a_t}(v), \quad N_i := \{ N_{a_t} \mid v \in \mathbb{N}_s \}, \tag{31}
\]
we get an admissible partition sequence \( \mathcal{A} := (\mathcal{A}_v) \) of \( \mathbb{N} \). We verify that this partition satisfies (22). Given \( n \geq \beta_t \), we find \( r \in \mathbb{N} \) such that \( n_{r} \leq n < n_{r+1} \). By construction of \( (n_r) \) it follows that \( A_{n_r} \geq tA_{n}/(t+1) \). Therefore
\[
\frac{1}{A_{n+1}} \sum_{k \in N_{n_k} \cap [m_{n+1}, k \leq m_n]} p_k \leq \frac{A_{\alpha_t}}{A_n} + \frac{1}{A_{n+1}} \sum_{k \in N_{n_k} \cap [m_{n+1}, k \leq m_n]} \frac{A_n - A_{n_k}}{A_n}
\]
\[
\leq \frac{1}{t^2} + \frac{1}{t^2} \sum_{n_k=1}^{\infty} \frac{1}{n_k+1} \leq \frac{1}{t^2} + \frac{1}{t+1}
\]
\[
\leq \frac{1}{t^2} + \frac{2}{t^2} \sum_{n_k=1}^{\infty} \frac{1}{n_k+1} \leq \frac{1}{t+1}
\]
\[
= \frac{2t + 3}{t^2} + \frac{1}{t+1}. \tag{32}
\]

Therefore
\[
\limsup_{n \to \infty} \limsup_{v \in \mathbb{N}_s} \limsup_{k \in N_{a_t} \cap [m_{n+1}, k \leq m_n]} p_k = 0. \tag{33}
\]

Now combining the obtained result with Theorem 2.1 in [8] we get the following.

**Theorem 10.** Let \( p = (p_i) \) be a sequence of positive reals with \( p \not\in \ell 1 \). Then the following conditions are equivalent:

(i) \( R_{p,m} \) has spreading rows; that is, \( \lim_n (1/\sum_{k \leq m} p_k) = 0 \);

(ii) \( d_{R_{p,m}} \) is strongly nonatomic.

The following example demonstrates that the assumption of Problem 3 that \( \mathcal{F} \subset \mathcal{G}_{R_{p,m}} \) in the case of (24) is not true in general.

**Example II.** Suppose that \( m_{2n} \) and \( p_k \) for \( k \leq m_{2n} \) are defined. We set \( m_{2n+1} := m_{2n} + (n+2) \) and \( p_{k} := 2^{k} \) for \( k < m_{2n} \). Then
\[
\frac{p_{m_{2n+1}}}{p_{m_{2n}}} \geq \frac{m_{2n} + (n+1) [p_{m_{2n}}]}{m_{2n+1} + \sum 2^{-k} \geq 1. \tag{34}
\]

Consequently the matrix \( R_{p,m} \) satisfies condition (24). On the other hand for \( x \in \mathcal{G} \) having \( x_n = 1 \) when \( p_n = 1 \) and \( x_n = 0 \) otherwise \( n \in \mathbb{N} \) we have
\[
[ Ax]_{m_{2n+1}} \geq \frac{1}{m_{2n+1}} \sum_{k=1}^{m_{2n+1} - k(n+1)} p_{m_{2n} + k(n+1)} \geq 1. \tag{35}
\]

Hence \( x \notin \mathcal{G}_{R_{p,m}} \).

As well as for Riesz matrices in the case of a generalized Riesz matrix the assumption that the matrix has spreading rows implies that its bounded summability domain has the Hahn property. To prove it we will adjust the methods developed in [6, Theorem 3.8].

Let \( I \) be an at most countable set, and let \( N = \{ N_i \mid i \in I \} \) be a partition of \( \mathbb{N} \). Let \( (n_k) \) be the sequence of all elements of \( N_i \) arranged in the ascending order \( i \in I \). We introduce the notation
\[
bs(N) := \{ x \in \omega \mid \forall j \in I : \| x \|_{bs(N)} := \sup_j \| x_k \|_{bs} < \infty \}. \tag{36}
\]

For the proof of the main result we need two lemmas.

**Lemma 12 (cf. Lemma 3.6 in [6]).** Let \( n \in \mathbb{N} \) and \( p_1, \ldots, p_n \) be a (finite) sequence of numbers. Then there exists a partition \( M_1, \ldots, M_t \) of \( \{1, \ldots, n\} \) such that \( t \leq 2 \sqrt{n} \) and \( (p_i)_{i \in M_j} \) is monotone \( (j = 1, \ldots, t) \).

Lemma 13. Let $N = \{N_i | i \in I\}$ be a partition of $\mathbb{N}$. If
\[ \frac{1}{P_{m_{\alpha}}(1_m)} \sum_{n \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}} |p_{n,j} - p_{m_{k_i}}| + \sum_{j \in \mathbb{N}} p_{n,j} \right) \rightarrow 0 \quad (n \rightarrow \infty), \]
then $bs(N, I) \subset c_{0}R_{p,m} \subset c_{2}R_{p,m}$.

Proof. To get a proof of the statement just replace everywhere $n$ with $m_{\alpha}$ in the proof of Lemma 3.7 [6], the case of $R_{p}$.

In the next theorem we generalize Theorem 3.8 in [6] in the case of Riesz matrices $R_{p}$. The proof requires nontrivial refinements of the methods used in the corresponding part of the proof in [6].

Theorem 14. Let $(p_{n})$ be a positive sequence such that $P_{n} \rightarrow \infty$ and $R_{p,m} = (a_{nk})$ has spreading rows. Then $c_{0}R_{p,m} \cap l^{\infty}$ has the Hahn property.

Proof. By Lemma 13 it is sufficient to verify the existence of a partition $N = \{N_i | i \in I\}$ of $\mathbb{N}$ satisfying the condition in the lemma. Aiming to that we set $\overline{k}_{i+1} := k_{i} + 1$ and suppose that $k_{1}, \ldots, k_{i}, \overline{k}_{i+1}, \ldots, \overline{k}_{i}$ are already chosen. Since $P_{n} \rightarrow \infty$, we can choose
\[ \overline{k}_{i+1} := \min \left\{ k \in \mathbb{N} | \frac{P_{m_{k_i}}}{P_{m_{k_i}}(1_m)} \geq 2 \right\}. \]
If $P_{m_{k_{i+1}}} / P_{m_{k_i}} \leq 8$ or $\overline{k}_{i+1} = k_{i+1}$ we set $k_{i+1} := \overline{k}_{i+1}$. Otherwise we set $k_{i+1} := \overline{k}_{i+1} - 1$ and $k_{i+2} := \overline{k}_{i+2} := k_{i+1}$. In the latter case we have $P_{m_{k_{i+1}}} / P_{m_{k_i}} < 2$ and $k_{i+2} = k_{i+1} + 1$. Hence in both cases
\[ \frac{P_{m_{k_{i+1}}} / P_{m_{k_i}}}{m_{k_{i+1}} / m_{k_i}} \geq 2, \quad \frac{P_{m_{k_{i+1}}} / P_{m_{k_i}}}{m_{k_{i+1}} / m_{k_i}} \leq 8 \quad \text{or} \quad k_{i+1} = k_{i} + 1. \]

Since $R_{p,m} = (a_{nk})$ has spreading rows, then
\[ \varepsilon_{k} := \max \{ p_{j} | j \leq m_{k_{i}} \} \rightarrow 0. \]

Let $i_{0} \in \mathbb{N}$ be such that $\varepsilon_{i} < 1$ for $i \geq i_{0}$.

For every $i \geq i_{0}$ we choose the minimal integer $j_{i} \in \mathbb{N}$ such that $\frac{m_{k_{i+1}}}{m_{k_{i}}} < \varepsilon_{j_{i}}^{-1/2}$. Since $\varepsilon_{i} = \max \{ p_{k} | k \leq m_{k_{i}} / P_{m_{k_{i}}} \geq 1 / m_{k_{i}} \}$, then $m_{k_{i}} \geq \varepsilon_{i}^{-1} (i \geq i_{0})$. Hence $j_{i} > 2 (i \geq i_{0})$. Note that $m_{k_{i}}^1/2 < \varepsilon_{j_{i}}^{1/2} \leq m_{k_{i+1}} / m_{k_{i}} \leq \varepsilon_{j_{i}}^{1/2}$, so $j_{i} / 2 + 1 \geq j_{i} / 2 + 1 \geq j_{i}$, which is equivalent to $j_{i} \leq 2$ and contradicts $j_{i} > 2$.

Set
\[ \alpha_{i} := \frac{P_{m_{k_{i+1}}}}{m_{k_{i+1}}}, \quad \beta_{i} := \max \{ p_{k} | k \leq m_{k_{i+1}} \} \quad (i \in \mathbb{N}). \]

For every $i \in \mathbb{N}$ and $s = 2, \ldots, j_{i+1} - 2$ we use the notations
\[ M_{i} := \{ m_{k_{i}} < k \leq m_{k_{i+1}} | p_{k} \leq \frac{m_{k_{i+1}}^{1/j_{i+1}}}{m_{k_{i+1}}^{j_{i+1}}} \alpha_{i} \}, \]
\[ M_{is} := \{ m_{k_{i}} < k \leq m_{k_{i+1}} | m_{k_{i+1}}^{(s-1)/j_{i+1}} \alpha_{i} < p_{k} \leq m_{k_{i+1}}^{s/j_{i+1}} \alpha_{i} \}, \]
\[ M_{ij_{i+1}-1} := \{ m_{k_{i}} < k \leq m_{k_{i+1}} | m_{k_{i+1}}^{(s-2)/j_{i+1}} \alpha_{i} < p_{k} \}. \]

Set $v_{s} := |M_{is}|$ ($s = 1, \ldots, j_{i+1} - 1$). Note that
\[ v_{s} \leq \frac{P_{m_{k_{i+1}}}}{m_{k_{i+1}}^{(s-1)/j_{i+1}}} \alpha_{i} < 4 \varepsilon_{j_{i+1}}^{1/2} \beta_{i}. \]

By Lemma 12 for every $i \in \mathbb{N}$ and $s = 1, \ldots, j_{i+1} - 1$ we may find a partition $(S_{i1}, \ldots, S_{irs})$ of $M_{is}$ with $r_{is} \leq 2 \varepsilon_{j_{i+1}}^{1/2}$ such that $(p_{j})_{j \in S_{irs}}$ is monotone ($i = 1, \ldots, r_{is}$). Let $\varepsilon_{i} := |S_{irs}|$ and let $(\xi_{irs})_{j \in S_{irs}}$ be the finite sequence of all elements of $S_{irs}$ arranged in the ascending order ($i \in \mathbb{N}$, $s = 1, \ldots, j_{i+1} - 1, k = 1, \ldots, r_{is}$). Then
\[ \sum_{k=1}^{r_{is}} \left( p_{\xi_{irs}^{i+1}} + \sum_{j=1}^{s-1} |p_{\xi_{irs}^{i+1}} - p_{\xi_{irs}^{i+1}+s/2} \right) \leq 2 r_{is} \varepsilon_{j_{i+1}}^{1/2} \beta_{i} \leq 4 \varepsilon_{j_{i+1}}^{1/2} \beta_{i}. \]

\[ \alpha_{i} := \frac{P_{m_{k_{i+1}}}}{m_{k_{i+1}}} < m_{k_{i+1}}^{1/j_{i+1}} \alpha_{i} < m_{k_{i+1}}^{s/j_{i+1}} \alpha_{i} \leq m_{k_{i+1}}^{s/j_{i+1}} \alpha_{i} < m_{k_{i+1}}^{s/j_{i+1}} \alpha_{i} \leq 4 \varepsilon_{j_{i+1}}^{1/2} \beta_{i}. \]
To define \( N_i \) \((i \in \mathbb{N})\) we use the notation \( N_i := \{ k \in N \mid k \leq n \} \) for a subset \( N \subset \mathbb{N} \) and \( n \in \mathbb{N} \).

Set \( N_i \mid m_{k_0} := \{1, \ldots, m_{k_0}\} \) and \( N_i \mid m_{k_0} := \emptyset \) for \( j > 1 \). Let \( s_1^0, \ldots, s_{t_0}^0 \) be all indexes such that \( M_{i_0} \mid \neq \emptyset \) \((t = 1, \ldots, t_0)\). We set

\[
N_i \mid m_{k_0+1} := \begin{cases} N_i \mid m_{k_0} \cup S_{i_0}^0 & \text{for } j = \sum_{\tau = 1}^{t_0} r_{i_0}^0 + k \\
N_i \mid m_{k_0} & \text{for } j > \sum_{\tau = 1}^{t_0} r_{i_0}^0,
\end{cases}
\tag{47}
\]

Continuing inductively for \( i > i_0 \) let \( s_1^{i-i_0}, \ldots, s_{t_{i-i_0}}^{i-i_0} \) be all indexes such that \( M_{i_0} \mid \neq \emptyset \) \((t = 1, \ldots, t_{i-i_0})\). We set

\[
N_i \mid m_{k_0+1} := \begin{cases} N_i \mid m_{k_0} \cup S_{i_0}^{i-i_0} & \text{for } j = \sum_{\tau = 1}^{t_{i-i_0}} r_{i_0}^0 + k \\
N_i \mid m_{k_0} & \text{for } j > \sum_{\tau = 1}^{t_{i-i_0}} r_{i_0}^0.
\end{cases}
\tag{48}
\]

Let \( I \) be the set of all indexes \( i \in \mathbb{N} \) such that \( N_i \neq \emptyset \). If \( I \) is infinite, then by our construction every \( N_i \) is infinite. If \( I \) is finite, then without loss of generality we may assume that \( N_i \) is infinite \((i \in I)\).

Let \( n \in \mathbb{N} \) with \( k_j < n \leq k_{j+1} \) and \( i \geq i_0 \) be fixed. Then

\[
B_n := \frac{1}{p_{m_n}} \times \sum_{m_{k_0} \leq m \leq m_{k_0+1}} \left( \sum_{s_{i_0}^0} \left| \rho_{n,s_{i_0}^0} - \rho_{n,m_{k_0+1}} \right| + \sum_{s_{i_0}^0} p_{m_{k_0}} \right)
\leq \frac{1}{p_{m_n}} \times \left( 2 \sum_{k=1}^{m_{k_0}} p_k \right.
\]

\[
+ \sum_{s_{i_0}^0} p_{i_0} \sum_{\tau = 1}^{t_{i_0}} + \sum_{s_{i_0}^0} \left| n_{s_{i_0}^0} \right| p_{n_{s_{i_0}^0}} - p_{n_{s_{i_0}^0}} \right)
\leq \frac{1}{p_{m_n}} \left( 2p_{m_{k_0}} + \sum_{s_{i_0}^0} \left( 16e^{1/8} + 4e^{1/2} \right) p_{m_{k_0+1}} \right).
\tag{49}
\]

If \( k_{i+1} = k_i + 1 \) then \( n = k_{i+1} \), so \( p_n = p_{k_{i+1}} \). If \( k_{i+1} > k_i + 1 \) then \( p_n \geq p_{k_i} \geq 8^{-1} p_{k_i+1} \). Hence

\[
B_n \leq \frac{8}{p_{k_{i+1}}} \left( 2p_{m_{k_0}} + \sum_{\tau = 1}^{t_{i_0}} \left( 16e^{1/8} + 4e^{1/2} \right) p_{m_{k_0+1}} \right)
\leq \frac{16}{2^{(i-i_0)/2}} + \sum_{\tau = 1}^{t_{i_0}} \left( 4e^{1/8} + e^{1/2} \right).
\tag{50}
\]

Consider the matrix \( A = (a_{i,j}) \) with \( a_{i,j} := 2^{(r-i)/2} \) for \( r \leq i \) and \( a_{r,i} = 0 \) otherwise. Evidently, \( A \) is regular for null sequences, so it sums the null sequence \((4e^{1/8} + e^{1/2})\tau\) to zero. Therefore \( B_n \rightarrow 0 \) \((n \rightarrow \infty)\). So in view of Lemma 13 the inclusion \( bs(N) \subset c_0 \) holds.

Now having in mind Proposition 2.8 in [7] we can prove the following theorem.

**Theorem 15.** Let \( p = (p_i) \) be a sequence of positive reals with \( p \notin \ell^1 \). Then one considers the following conditions:

(a) \( \ell^\infty \cap c_{0,p} \) has the matrix Hahn property,
(b) \( R_{p,m} \) has spreading rows; that is, \( \lim_{n}(1/P_{m_n}) \sup_{1 \leq k \leq m_n} p_k = 0 \),
(c) \( d_{R_{p,m}} \) is strongly nonatomic,
(d) \( R_{p,m} \in KG \),
(e) \( \ell^\infty \cap c_{0,p} \) has the separable Hahn property,
(f) \( \ell^\infty \cap c_{0,p} \) has the Hahn property,
(g) \( \chi(c_{0,p} \cap \ell^\infty) = \ell^1 \).

Then the implications \( (c) \iff (b) \iff (f) \iff (e) \iff (a) \iff (d) \iff (g) \) hold.

**Proof.** The implication chain \( (f) \iff (e) \iff (a) \iff (d) \) is obvious. \( (c) \iff (b) \) is Theorem 10, \( (b) \iff (f) \) is Theorem 14, and \( (d) \iff (g) \) follows from Corollary 5.

The author did not succeed to show that the condition

\[
\lim_{n} \frac{1}{p_{m_n}} \sup_{1 \leq k \leq m_n} p_k = 0
\tag{51}
\]

is necessary for \( R_{p,m} \in KG \). The following example gives a generalized Riesz matrix \( R_{p,m} \) not having spreading rows and failing to have \( R_{p,m} \in KG \).

**Example 16.** Suppose \( p \notin \ell^1 \) and

\[
\lim_{n} \frac{1}{p_{m_n}} \sup_{1 \leq k \leq m_n} p_k > \frac{1}{2}
\tag{52}
\]

Then there exists \( \delta > 1/2 \) and \( n_0 \in \mathbb{N} \) such that \( (1/p_{m_n}) \sup_{1 \leq k \leq m_n} p_k \geq \delta \) for \( n \geq n_0 \).

Let \( k_n \leq m_n \) be an integer such that \( p_{k_n} = \sup_{1 \leq k \leq m_n} p_k \).

Then

\[
\frac{1}{p_{m_n}} \sum_{k=k_n}^{m_n} p_k \leq 1 - \delta < \frac{1}{2}
\tag{53}
\]
Since $P_n \to \infty$ we can choose an index sequence $(n_i)$ such that $(k_{n_i})$ is increasing. Now if $x \in \chi \cap c_{R_{p,m}}$ then there exists $i_0 = i_0(x)$ such that $x_{k_{n_i}} = 0$ for all $i \geq i_0$ or $x_{k_{n_i}} = 1$ for all $i \geq i_0$. We set $y_{k_{n_i}} := (-1)^i/i$ for $i \in \mathbb{N}$ and $y_{k} := 0$ for $k \notin \{k_{n_i} | i \in \mathbb{N}\}$. Then the series
\begin{equation}
\sum_{k=k_{n_0}}^{\infty} x_{k} y_{k} = \sum_{i=i_0}^{\infty} \frac{(-1)^i}{i} x_{k_{n_i}}
\end{equation}
converges, so $y \in \chi(c_{R_{p,m}} \cap \ell^{\infty})^\beta \setminus \ell^1$.

**Problem 4.** In Theorem 15 we have shown that the fact that $R_{p,m}$ has spreading rows implies $R_{p,m} \in KG$. Does the inverse implication hold?

**Acknowledgments**

The author is much obliged to Professor J. Boos (FernUniversität in Hagen) for advice and criticism with this paper. The author thanks the reviewer for his/her thorough review and highly appreciates the comments and suggestions, which significantly contributed to improving the quality of the publication. This research was supported by the Estonian Science Foundation (Grant no. 8627), European Regional Development Fund (Centre of Excellence "Mesosystems: Theory and Applications," TK114), and Estonian Ministry of Education and Research (Project SF0130010s12).

**References**


