Research Article

Fundamental Spectral Theory of Fractional Singular Sturm-Liouville Operator

Erdal Bas

Department of Mathematics, Faculty of Science, Firat University, 23119 Elazig, Turkey

Correspondence should be addressed to Erdal Bas; erdalmat@yahoo.com

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We give the theory of spectral properties for eigenvalues and eigenfunctions of Bessel type of fractional singular Sturm-Liouville problem. We show that the eigenvalues and eigenfunctions of the problem are real and orthogonal, respectively. Furthermore, we prove new approximations about the topic.

1. Introduction

Sturm-Liouville problem was first developed in a number of papers that were published by these authors in 1836 and 1837. Charles-François Sturm (1803–1855), Professor of Mechanics at the Sorbonne, had been interested, since about 1833, in the problem of heat flow in bars, so he was well aware of eigenvalue-type problems. He worked closely with his friend Joseph Liouville (1809–1882), Professor of Mathematics at the Collège de France, on the general properties of second-order differential equations. Liouville also made many contributions to the general field of analysis, see [1].

A Sturm-Liouville boundary value problem consists of a second order linear ordinary differential equation

$$-(py')' + qy = \lambda y, \quad (a, b) \quad (1)$$

and boundary conditions. Here (a, b) is a bounded or unbounded open interval of the real line R . The coefficients $p, q, w : (a, b) \rightarrow R$; $\lambda \in \mathbb{C}$, the complex field. Spectral analysis finds applications in many diverse fields. Mathematical techniques could be developed into a more suitable and significant course by presenting them within the more general Sturm-Liouville theory in L_2 . The Sturm-Liouville problems are important in many areas of science, engineering and mathematics. It is known that the spectral characteristics are spectra, spectral functions, scattering data, norming constants, etc. According to the theory a linear second-order differential operator which is self-adjoint has

an orthogonal sequence of eigenfunctions in L_2 . Spectral properties of Sturm-Liouville operators are often derived, directly or indirectly, as a consequence of an established link between large distance asymptotic behavior of solutions of the associated differential equation and spectral properties of the corresponding differential operator. Sturm-Liouville problems are divided into regular and singular types. Differential equations such as Bessel, hydrogen atom, Hermite, Jacobi, and Legendre equations can be transformed into Sturm-Liouville equations. There are many studies on these issues [2–7]. We also discuss the radial part of Schrödinger's equation for the Bessel equation.

Fractional calculus is “the theory of derivatives and integrals of any arbitrary real or complex order, which unify and generalize the notions of integer-order differentiation and n -fold integration” [6–13]. In recent years, the concept of fractional calculus, originated from Leibniz, has achieved increasing interest during the last two decades. In particular, the last decade has scientific papers concerning fractional quantum mechanics. It has been proved that many systems in different fields of science and engineering can be modeled more accurately using fractional derivatives [8–17]. Fractional calculus has increasing importance for the last years because fractional calculus has been applied to almost every field of science. They are viscoelasticity, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory. We note that

ordinary derivatives in a traditional Sturm-Liouville problem are replaced with fractional derivatives, and the resulting problems are solved using some numerical methods [18–23]. Furthermore, Klimek and Argawal [24] define a fractional Sturm-Liouville operator, introduce a regular fractional Sturm-Liouville problem, and investigate the properties of the eigenfunctions and the eigenvalues of the operator. In this paper, our purpose is to introduce singular fractional Sturm-Liouville problem having Bessel type and prove spectral properties of spectral data for the operator.

Let us give the boundary value problem for Bessel equation and necessary data as follows.

2. Preliminaries

Now, consider the following Bessel equation:

$$\frac{d^2 y}{dx^2} + \left(\lambda - \frac{\nu^2 - 1/4}{x^2} \right) y = 0, \quad (2)$$

where λ and ν are real numbers. The Bessel equation for having the analogous singularity is given in [5].

Definition 1 (see [10]). Let $0 < \alpha \leq 1$. The left-sided and right-sided Riemann-Liouville integrals of order α , respectively, are given by the formulas

$$\begin{aligned} (I_{a,+}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \quad x > a, \\ (I_{b,-}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds, \quad x < b, \end{aligned} \quad (3)$$

where Γ denotes the gamma function.

Definition 2 (see [10]). Let $0 < \alpha \leq 1$. The left-sided and right-sided Riemann-Liouville derivatives of order α , respectively, are defined as follows:

$$\begin{aligned} (D_{a,+}^\alpha f)(x) &= D(I_{a,+}^{1-\alpha} f)(x) \quad x > a, \\ (D_{b,-}^\alpha f)(x) &= -D(I_{b,-}^{1-\alpha} f)(x) \quad x < b. \end{aligned} \quad (4)$$

Analogous formulas yield the left-sided and right-sided Caputo derivatives of order α :

$$\begin{aligned} ({}^C D_{a,+}^\alpha f)(x) &= (I_{a,+}^{1-\alpha} Df)(x) \quad x > a, \quad 0 < \alpha \leq 1 \\ ({}^C D_{b,-}^\alpha f)(x) &= (I_{b,-}^{1-\alpha} (-D)f)(x) \quad x < b, \quad 0 < \alpha \leq 1. \end{aligned} \quad (5)$$

Definition 3 (see [14]). The general function ${}_p\Psi_q(z)$ is defined for $z \in \mathbb{C}$, $a_l, b_j \in \mathbb{C}$, and $\alpha_l, \beta_j \in \mathbb{R}$ ($l = 1, \dots, p; j = 1, \dots, q$) by the series

$${}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^p \Gamma(a_l + \alpha_l k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}. \quad (6)$$

This general Wright function was investigated by Fox who presented its asymptotic expansion for large values of the argument z under the condition

$$\sum_{j=1}^q \beta_j - \sum_{l=1}^p \alpha_l > 1. \quad (7)$$

If these conditions are satisfied, the series in (6) is convergent for any $z \in \mathbb{C}$.

Theorem 4 (see [14]). Let $a_l, b_j \in \mathbb{C}$, and $\alpha_l, \beta_j \in \mathbb{R}$ ($l = 1, \dots, p; j = 1, \dots, q$), and let

$$\begin{aligned} \Delta &= \sum_{j=1}^q \beta_j - \sum_{l=1}^p \alpha_l, \\ \delta &= \prod_{l=1}^p |\alpha_l|^{-\alpha_l} \prod_{j=1}^q |\beta_j|^{\beta_j}, \\ \mu &= \sum_{j=1}^q \beta_j - \sum_{l=1}^p \alpha_l + \frac{p-q}{2}. \end{aligned} \quad (8)$$

(i) If $\Delta > -1$, then the series in (6) is absolutely convergent for all $z \in \mathbb{C}$.

(ii) If $\Delta = -1$, then the series in (6) is absolutely convergent for $|z| < \delta$ and for $|z| = \delta$ and $\Re(\mu) > 1/2$.

Property 1. The fractional differential operators defined in (4)-(5) satisfy the following identities:

$$\begin{aligned} (i) \quad & \int_a^b f(x) D_{b,-}^\alpha g(x) dx \\ &= \int_a^b g(x) {}^C D_{a,+}^\alpha f(x) dx - f(x) I_{b,-}^{1-\alpha} g(x) \Big|_a^b, \end{aligned} \quad (9)$$

$$\begin{aligned} (ii) \quad & \int_a^b f(x) D_{b,-}^\alpha g(x) {}^C D_{a,+}^\alpha k(x) dx \\ &= \int_a^b g(x) {}^C D_{a,+}^\alpha f(x) {}^C D_{a,+}^\alpha k(x) dx \\ & \quad - f(x) I_{b,-}^{1-\alpha} g(x) {}^C D_{a,+}^\alpha k(x) \Big|_a^b, \end{aligned} \quad (10)$$

$$\begin{aligned} (iii) \quad & \int_a^b f(x) D_{a,+}^\alpha g(x) dx \\ &= \int_a^b g(x) {}^C D_{b,-}^\alpha f(x) dx + f(x) I_{a,+}^{1-\alpha} g(x) \Big|_a^b. \end{aligned} \quad (11)$$

Property 2 (see [24]). Assume that $\alpha \in (0, 1)$, $\beta > \alpha$, and $f \in C[a, b]$. Then the relations

$$\begin{aligned} D_{a,+}^{\alpha} I_{a,+}^{\alpha} f(x) &= f(x), \\ D_{b,-}^{\alpha} I_{b,-}^{\alpha} f(x) &= f(x), \\ D_{a,+}^{\alpha} I_{a,+}^{\beta} f(x) &= I_{a,+}^{\beta-\alpha} f(x), \\ D_{b,-}^{\alpha} I_{b,-}^{\beta} f(x) &= I_{b,-}^{\beta-\alpha} f(x), \\ {}^C D_{a,+}^{\alpha} I_{a,+}^{\alpha} f(x) &= f(x), \\ {}^C D_{b,-}^{\alpha} I_{b,-}^{\alpha} f(x) &= f(x), \end{aligned} \quad (12)$$

hold for any $x \in [a, b]$. Furthermore, the integral operators defined in (3) satisfy the following semigroup properties:

$$I_{a,+}^{\alpha} I_{a,+}^{\beta} = I_{a,+}^{\alpha+\beta}, \quad I_{b,-}^{\alpha} I_{b,-}^{\beta} = I_{b,-}^{\alpha+\beta}. \quad (13)$$

Now, let us take up a singular fractional boundary problem for Bessel operator and give some spectral results.

3. Main Results

3.1. A Singular Fractional Sturm-Liouville Problem for Bessel Operator. Fractional Sturm-Liouville problem for Bessel operator denotes the differential part containing the left- and right-sided derivatives. Let us use the form of the integration by parts formulas (10), (11) for this new approximation. Properties of eigenfunctions and eigenvalues in the theory of classical Sturm-Liouville problems are related to the integration by parts formula for the first-order derivatives. In the corresponding fractional version, we note that both left and right derivatives appear and the essential pairs are the left Riemann-Liouville derivative with the right Caputo derivative and the right Riemann-Liouville derivative with the left Caputo one. Spectral properties of Sturm-Liouville operators are often derived, directly or indirectly, as a consequence of an established link between large distance asymptotic behavior of solutions of the associated differential equation and spectral properties of the corresponding Bessel operator.

Definition 5. Let $\alpha \in (0, 1)$. Fractional Bessel operator is written as

$$\mathcal{L}_{\alpha[B]} = D_{1,-}^{\alpha} p(x) {}^C D_{0,+}^{\alpha} + \left(q(x) - \frac{\nu^2 - 1/4}{x^2} \right). \quad (14)$$

Considering the fractional Bessel equation

$$\mathcal{L}_{\alpha[B]} y_{\lambda}(x) + \lambda w_{\alpha}(x) y_{\lambda}(x) = 0, \quad (15)$$

where $p(x) \neq 0$, $w_{\alpha}(x) > 0$, for all $x \in (0, 1]$, $w_{\alpha}(x)$ is weight function, and p, q are real valued continuous functions in interval $(0, 1]$.

The boundary conditions for the operator $\mathcal{L}$ are the following:

$$\begin{aligned} y_{\lambda}(0) &= 0, \\ d_1 y(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^C D_{0,+}^{\alpha} y(1) &= 0, \end{aligned} \quad (16)$$

where $d_1^2 + d_2^2 \neq 0$. The fractional boundary-value problem (15)-(16) is fractional Sturm-Liouville problem for Bessel operator.

Theorem 6. Fractional Bessel operator is self-adjoint on $(0, 1]$.

Proof. Let us consider the following equation:

$$\begin{aligned} \langle \mathcal{L}_{\alpha[B]} \varphi, \phi \rangle &= \int_0^1 \mathcal{L}_{\alpha[B]} \varphi(x) \cdot \phi(x) dx \\ &= \int_0^1 \phi(x) \left[D_{1,-}^{\alpha} p(x) {}^C D_{0,+}^{\alpha} \varphi(x) \right. \\ &\quad \left. + \left(q(x) - \frac{\nu^2 - 1/4}{x^2} \right) \varphi(x) \right] dx \\ &= \int_0^1 \phi(x) D_{1,-}^{\alpha} p(x) {}^C D_{0,+}^{\alpha} \varphi(x) dx \\ &\quad + \int_0^1 \left(q(x) - \frac{\nu^2 - 1/4}{x^2} \right) \varphi(x) \phi(x) dx. \end{aligned} \quad (17)$$

By means of equality (10) and boundary conditions (16), we obtain the identity

$$\begin{aligned} \langle \mathcal{L}_{\alpha[B]} \varphi, \phi \rangle &= \int_0^1 p(x) {}^C D_{0,+}^{\alpha} \phi(x) {}^C D_{0,+}^{\alpha} \varphi(x) dx \\ &\quad - \phi(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^{\alpha} \varphi(x) \Big|_0^1 \\ &\quad + \int_0^1 \left(q(x) - \frac{\nu^2 - 1/4}{x^2} \right) \varphi(x) \phi(x) dx \\ &= \int_0^1 p(x) {}^C D_{0,+}^{\alpha} \phi(x) {}^C D_{0,+}^{\alpha} \varphi(x) dx \\ &\quad + \frac{d_1}{d_2} \varphi(1) \phi(1) \\ &\quad + \int_0^1 \left(q(x) - \frac{\nu^2 - 1/4}{x^2} \right) \varphi(x) \phi(x) dx. \end{aligned} \quad (18)$$

On the other hand, by performing similar operations, we find

$$\begin{aligned} \langle \varphi, \mathcal{L}_{\alpha[B]} \phi \rangle &= \int_0^1 p(x) {}^C D_{0,+}^\alpha \varphi(x) {}^C D_{0,+}^\alpha \phi(x) dx \\ &\quad + \frac{d_1}{d_2} \varphi(1) \phi(1) \\ &\quad + \int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) \phi(x) \varphi(x) dx. \end{aligned} \quad (19)$$

The right-hand sides of (18) and (19) are equal; hence, we may see that the left sides are equal; that is,

$$\langle \mathcal{L}_{\alpha[B]} \varphi, \phi \rangle = \langle \varphi, \mathcal{L}_{\alpha[B]} \phi \rangle. \quad (20)$$

□

Theorem 7. *The eigenvalues of fractional Bessel operator (15)-(16) are real.*

Proof. Let us observe that the following relation results from equality (10):

$$\begin{aligned} &\int_0^1 f(x) \mathcal{L}_{\alpha[B]} g(x) dx \\ &= \int_0^1 p(x) {}^C D_{0,+}^\alpha f(x) {}^C D_{0,+}^\alpha g(x) dx \\ &\quad - f(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha g(x) \Big|_0^1 \\ &\quad + \int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) g(x) f(x) dx. \end{aligned} \quad (21)$$

Suppose that λ is the eigenvalue for (15)-(16) corresponding to eigenfunction y ; the following equalities satisfy y and its complex conjugate $\bar{y}$:

$$\mathcal{L}_{\alpha[B]} y(x) + \lambda w_\alpha(x) y(x) = 0, \quad (22)$$

$$y(0) = 0, \quad (23)$$

$$d_1 y(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^C D_{0,+}^\alpha y(1) = 0,$$

$$\mathcal{L}_{\alpha[B]} \bar{y}(x) + \bar{\lambda} w_\alpha(x) \bar{y}(x) = 0, \quad (24)$$

$$\bar{y}(0) = 0, \quad (25)$$

$$d_1 \bar{y}(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^C D_{0,+}^\alpha \bar{y}(1) = 0,$$

where $d_1^2 + d_2^2 \neq 0$. We multiply (22) by function $\bar{y}$ and (24) by function y , respectively, and subtract

$$\begin{aligned} &(\lambda - \bar{\lambda}) w_\alpha(x) y(x) \bar{y}(x) \\ &= y(x) \mathcal{L}_{\alpha[B]} \bar{y}(x) - \bar{y}(x) \mathcal{L}_{\alpha[B]} y(x). \end{aligned} \quad (26)$$

Now, we integrate over interval $(0, 1]$, and applying relation (21), and we note that the right-hand side of the integrated equality contains only boundary terms:

$$\begin{aligned} &(\lambda - \bar{\lambda}) \int_0^1 w_\alpha(x) y(x) \bar{y}(x) dx \\ &= \int_0^1 y(x) \mathcal{L}_{\alpha[B]} \bar{y}(x) dx - \int_0^1 \bar{y}(x) \mathcal{L}_{\alpha[B]} y(x) dx \\ &= \int_0^1 y(x) \left[D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha \bar{y}(x) \right. \\ &\quad \left. + \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) \bar{y}(x) \right] dx \\ &\quad - \int_0^1 \bar{y}(x) \left[D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha y(x) \right. \\ &\quad \left. + \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) y(x) \right] dx \\ &= -y(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha \bar{y}(x) \Big|_1^1 \\ &\quad + y(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha \bar{y}(x) \Big|_0^0 \\ &\quad + \bar{y}(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y(x) \Big|_1^1 \\ &\quad - \bar{y}(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y(x) \Big|_0^0. \end{aligned} \quad (27)$$

By virtue of the boundary conditions (23), (25), we find

$$(\lambda - \bar{\lambda}) \int_0^1 w_\alpha(x) |y(x)|^2 dx = 0. \quad (28)$$

Because y is a nontrivial solution and $w_\alpha(x) > 0$, it is easily seen that $\lambda = \bar{\lambda}$. The eigenvalues are real. □

Theorem 8. *The eigenfunctions corresponding with distinct eigenvalues of fractional Bessel operator (15)-(16) are orthogonal weight function w_α on $(0, 1]$; that is,*

$$\int_0^1 w_\alpha(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0, \quad \lambda_1 \neq \lambda_2. \quad (29)$$

Proof. We have by assumptions fractional Sturm-Liouville operator for Bessel type fulfilled by two different eigenvalues (λ_1, λ_2) and the respective eigenfunctions $(y_{\lambda_1}, y_{\lambda_2})$:

$$\mathcal{L}_{\alpha[B]} y_{\lambda_1}(x) + \lambda_1 w_\alpha(x) y_{\lambda_1}(x) = 0, \quad (30)$$

$$y_{\lambda_1}(x) = 0, \quad (31)$$

$$d_1 y_{\lambda_1}(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^C D_{0,+}^\alpha y_{\lambda_1}(1) = 0,$$

$$\mathcal{L}_{\alpha[B]} y_{\lambda_2}(x) + \lambda_2 w_\alpha(x) y_{\lambda_2}(x) = 0, \quad (32)$$

$$y_{\lambda_2}(x) = 0, \quad (33)$$

$$d_1 y_{\lambda_2}(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^C D_{0,+}^\alpha y_{\lambda_2}(1) = 0.$$

We multiply (30) by function y_{λ_2} and (32) by function y_{λ_1} , respectively, and subtract.

$$(\lambda_1 - \lambda_2) w_\alpha(x) y_{\lambda_1} y_{\lambda_2} = y_{\lambda_1} \mathcal{L}_{\alpha[B]} y_{\lambda_2} - y_{\lambda_2} \mathcal{L}_{\alpha[B]} y_{\lambda_1}. \quad (34)$$

Integrating over interval $(0, 1]$ and applying relation (21) we note that the right-hand side of the integrated equality contains only boundary terms:

$$\begin{aligned} & (\lambda_1 - \lambda_2) \int_0^1 w_\alpha(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx \\ &= \int_0^1 y_{\lambda_1}(x) \mathcal{L}_{\alpha[B]} y_{\lambda_2}(x) dx \\ & \quad - \int_0^1 y_{\lambda_2}(x) \mathcal{L}_{\alpha[B]} y_{\lambda_1}(x) dx \\ &= \int_0^1 y_{\lambda_1}(x) \left[D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha y_{\lambda_2}(x) \right. \\ & \quad \left. + \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) y_{\lambda_2}(x) \right] dx \\ & \quad - \int_0^1 y_{\lambda_2}(x) \left[D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha y_{\lambda_1}(x) \right. \\ & \quad \left. + \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) y_{\lambda_1}(x) \right] dx \\ &= -y_{\lambda_1}(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_{\lambda_2}(x) \Big|_1 \\ & \quad + y_{\lambda_1}(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_{\lambda_2}(x) \Big|_0 \\ & \quad + y_{\lambda_2}(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_{\lambda_1}(x) \Big|_1 \\ & \quad - y_{\lambda_2}(x) I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_{\lambda_1}(x) \Big|_0. \end{aligned} \quad (35)$$

Using the boundary conditions (31), (33), we obtain that

$$(\lambda_1 - \lambda_2) \int_0^1 w_\alpha(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0, \quad (36)$$

where $\lambda_1 \neq \lambda_2$. Then, the eigenfunctions are orthogonal of this operator. $\square$

Remark 9. Let us now give certain auxiliary functions. Because we use the functions, the first of them is as follows:

$$\begin{aligned} & I_{0,+}^\alpha \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)} \\ &= (1-0)^{\alpha-1} (x-0)^\alpha {}_1\Psi_2 \left[\begin{matrix} (1, 1) \\ (\alpha, -1) \end{matrix} \middle| -\frac{x-0}{1-0} \right], \end{aligned} \quad (37)$$

where ${}_1\Psi_2$ is the Fox-Wright function [14]:

$$\begin{aligned} & {}_1\Psi_2 \left[\begin{matrix} (a_1, \alpha_1) \\ (b_1, \beta_1) \end{matrix} \middle| z \right] \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 k)}{\Gamma(b_1 + \beta_1 k) \Gamma(b_2 + \beta_2 k)} \frac{z^k}{k!}. \end{aligned} \quad (38)$$

The properties of the function are determined by the parameters

$$\begin{aligned} \Delta &= \beta_1 + \beta_2 - \alpha_1 = -1, \\ \delta &= |\alpha_1|^{-\alpha_1} |\beta_1|^{\beta_1} |\beta_2|^{\beta_2} = 1, \\ \mu &= b_1 + b_2 - \alpha_1 + \frac{1-2}{2} = 2\alpha - \frac{1}{2}. \end{aligned} \quad (39)$$

Considering Theorem 4, we note that this function is continuous in $(0, 1]$ when order $\alpha > 1/2$, that is, $\mu > 1/2$. For $0 < \alpha \leq 1/2$; it is discontinuous at end $x = 1$. The explicitly calculated function allows to estimate the second component of stationary function ϕ_0 of the differential part of Sturm-Liouville operator

$$D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha \phi_0(x) = 0 \quad (40)$$

which looks as follows:

$$\phi_0(x) = \xi_1 + \xi_2 I_{0,+}^\alpha \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha) p(x)} = \xi_1 + \xi_2 \psi(\alpha, 0, x). \quad (41)$$

The next function is the following integral:

$$\begin{aligned} \varphi(x) &= I_{0,+}^\alpha I_{1,-}^\alpha 1 = I_{0,+}^\alpha \frac{(1-x)^\alpha}{\Gamma(\alpha+1)} \\ &= (1-0)^\alpha (x-0)^\alpha \\ & \quad \times {}_1\Psi_2 \left[\begin{matrix} (1, 1) \\ (\alpha+1, -1) \end{matrix} \middle| -\frac{x-0}{1-0} \right]. \end{aligned} \quad (42)$$

Again, using Theorem 4 and calculating parameters according to (39),

$$\Delta = -1, \quad \delta = 1, \quad \mu = 2\alpha + \frac{1}{2}. \quad (43)$$

Finally,

$$\alpha > 0 \implies \mu > \frac{1}{2}, \quad (44)$$

and the obtained Fox-Wright function (42) is continuous in interval $(0, 1]$ for any positive order α .

Theorem 10. Let $\alpha > 1/2$, $x \in (0, 1]$ and define

$$Y_\lambda(y) = \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) y_\lambda(x) + \lambda w_\alpha y_\lambda(x), \quad (45)$$

$$\tilde{\Delta} = d_2 + d_1 \psi(\alpha, 0, 1).$$

Assume that $\tilde{\Delta} \neq 0$. Then, (15)-(16) are equivalent to the integral equation

$$\begin{aligned} y_\lambda(x) &= -I_{0,+}^\alpha \frac{1}{p(x)} I_{1,-}^\alpha Y_\lambda(y) + A(x) \left(I_{0,+}^\alpha \frac{1}{p(x)} I_{1,-}^\alpha Y_\lambda(y) \right) \Big|_{x=1}, \end{aligned} \quad (46)$$

where the coefficient $A(x)$ is

$$A(x) = \frac{d_1}{\tilde{\Delta}} \psi(\alpha, 0, x) \quad (47)$$

and functions ψ are defined in (41).

Proof. By means of composition rules, (15) can be rewritten as follows:

$$D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha \left[y_\lambda(x) + I_{0,+}^\alpha \frac{1}{p(x)} I_{1,-}^\alpha Y_\lambda(y) \right] = 0. \quad (48)$$

The last equality suggests that is a stationary function of fractional singular Sturm-Liouville problem for Bessel operator. $D_{1,-}^\alpha p(x) {}^C D_{0,+}^\alpha$ which according to (41) can be found as

$$\phi_0 = \xi_1 + \xi_2 I_{0,+}^\alpha \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha) p(x)} = \xi_1 + \xi_2 \psi(\alpha, 0, x). \quad (49)$$

Equation (15) in the form of

$$y_\lambda(x) + I_{0,+}^\alpha \frac{1}{p(x)} I_{1,-}^\alpha Y_\lambda(y) = \xi_1 + \xi_2 \psi(\alpha, 0, x) \quad (50)$$

proves we should connect coefficients ξ_j values d_j , $j = 1, 2$ determining the boundary conditions (16).

Let us note that the following formula results from composition rules (11) and (50):

$$I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_\lambda(x) = -I_{1,-}^1 Y_\lambda(y) + \xi_2. \quad (51)$$

For continuous function y_λ , we obtain the following values as the ends

$$\begin{aligned} I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_\lambda(x) \Big|_{x=0} &= - \int_0^\pi Y_\lambda(y) + \xi_2, \\ I_{1,-}^{1-\alpha} p(x) {}^C D_{0,+}^\alpha y_\lambda(x) \Big|_{x=1} &= \xi_2, \end{aligned} \quad (52)$$

respectively, for y_λ . Using (50), we find

$$\begin{aligned} y_\lambda(0) &= \phi_0(0) = \xi_1, \\ y_\lambda(1) &= \phi_0(1) - I_{0,+}^\alpha \frac{1}{p(x)} I_{1,-}^\alpha Y_\lambda(y) \Big|_{x=1} \\ &= \xi_1 + \xi_2 \psi(\alpha, 0, 1) - I_{0,+}^\alpha \frac{1}{p(x)} I_{1,-}^\alpha Y_\lambda(y) \Big|_{x=1}. \end{aligned} \quad (53)$$

The following set of linear equations for coefficients ξ_j results from (52)-(54)

$$\begin{aligned} \xi_1 &= 0, \\ d_1 \xi_1 + \xi_2 (d_2 + d_1 \psi(\alpha, 0, 1)) &= d_1 F, \end{aligned} \quad (54)$$

where $F = I_{0,+}^\alpha (1/p(x)) I_{1,-}^\alpha Y_\lambda(y) \Big|_{x=1}$.

Since $\tilde{\Delta} \neq 0$, the solution for coefficients ξ_j ($j = 1, 2$) is unique:

$$\begin{aligned} \xi_1 &= 0, \\ \xi_2 &= \frac{d_1 F}{\tilde{\Delta}}, \end{aligned} \quad (55)$$

Substituting the previous solution into (50) we recover the equivalent integral equation (46).

Furthermore, we give notation such as

$$m_p = \min_{x \in [0,1]} |p(x)|, \quad (56)$$

$$A = \|A(x)\|, \quad M_\varphi = \|\varphi(x)\|.$$

The proof is completed. $\square$

4. Conclusion

In the paper, we have extended the scope of some spectral properties of singular fractional Sturm-Liouville problem. We pointed that its eigenvalues related to the Bessel operator with the certain boundary conditions are real and its eigenfunctions corresponding to distinct eigenvalues are orthogonal. Furthermore, we showed that fractional Bessel operator is self-adjoint. Spectral properties of Sturm-Liouville theory are applied to the fractional theory. Our results are important in point of the fractional Sturm-Liouville theory.

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