Research Article

A Generalization on Some New Types of Hardy-Hilbert's Integral Inequalities

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Sulaiman presented, in 2008, new kinds of Hardy-Hilbert's integral inequality in which the weight function is homogeneous. In this paper, we present a generalization on the kinds of Hardy-Hilbert's integral inequality.

1. Introduction and Preliminaries

For any two nonnegative measurable functions f and g such that

$$0 < \int_0^\infty f^2(x) dx < \infty, \quad 0 < \int_0^\infty g^2(y) dy < \infty, \quad (1)$$

we have the Hilbert's integral inequality [1] that

$$\begin{aligned} & \iint_0^\infty \frac{f(x)g(y)}{x+y} dx dy \\ & < \pi \left(\int_0^\infty f^2(x) dx \int_0^\infty g^2(y) dy \right)^{1/2}. \end{aligned} \quad (2)$$

The constant π is the best possible. In 1925, Hardy [2] extended the Hilbert's integral inequality into the integral inequality as follows. If $p > 1$, $1/p + 1/q = 1$, and $f, g \geq 0$ such that

$$0 < \int_0^\infty f^p(x) dx < \infty, \quad 0 < \int_0^\infty g^q(y) dy < \infty, \quad (3)$$

then we have the Hardy-Hilbert's integral inequality that

$$\begin{aligned} & \iint_0^\infty \frac{f(x)g(y)}{x+y} dx dy \\ & < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(y) dy \right)^{1/q}. \end{aligned} \quad (4)$$

The constant $\pi/\sin(\pi/p)$ is the best possible. Both the two inequalities are important in mathematical analysis and its applications [3].

In 1938, Widder [4] studied on the Stieltjes Transform $S_f(y) = \int_0^\infty f(x)/(x+y) dx$.

Now, we recall the beta function B as follows:

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad \text{where } p, q > 0. \quad (5)$$

In 2001, Yang [5] extended the Hardy-Hilbert's integral inequality into the following integral inequality. If $p, q > 0$, $\lambda > 2 - \min\{p, q\}$, $1/p + 1/q = 1$, and $f, g \geq 0$ such that

$$\begin{aligned} 0 & < \int_0^\infty x^{1-\lambda} f^p(x) dx < \infty, \\ 0 & < \int_0^\infty y^{1-\lambda} g^q(y) dy < \infty, \end{aligned} \quad (6)$$

then we have

$$\begin{aligned} & \iint_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \\ & < k_\lambda(p) \left(\int_0^\infty x^{1-\lambda} f^p(x) dx \right)^{1/p} \left(\int_0^\infty y^{1-\lambda} g^q(y) dy \right)^{1/q}, \end{aligned} \quad (7)$$

where $k_\lambda(p) = B(1 + (\lambda - 2)/p, 1 + (\lambda - 2)/p)$. The constant $k_\lambda(p)$ is the best possible.

We also recall that a nonnegative function $f(x, y)$ which is said to be homogeneous function of degree λ if $f(tx, ty) = t^\lambda f(x, y)$ for all $t > 0$. And we say that $K(u, v)$ is increasing if $K(1, t)$ and $K(t, 1)$ are increasing functions.

In 2008, Sulaiman [6] gave new integral inequality similar to the Hardy-Hilbert's integral inequality. If $a, b > 0$, $p > 1$, $1/p + 1/q = 1$, $0 < \lambda \leq \min\{(1-b)p/q, (1-a)q/p\}$, $K(u, v)$ is a positive increasing homogeneous function of degree λ , and $f, g \geq 0$ and

$$F(x) = \int_0^x f(t) dt, \quad G(x) = \int_0^x g(t) dt \quad \forall x > 0, \quad (8)$$

then, for all $T > 0$, we have

$$\begin{aligned} & \iint_0^T \frac{F(u)G(v)}{K(u, v)} du dv \\ & \leq T^\alpha \sqrt[p]{pK_1} \sqrt[q]{qK_2} \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\ & \quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q}, \end{aligned} \quad (9)$$

where

$$K_1 = \int_0^1 \frac{t^{a-1}}{K(1, t)} dt, \quad K_2 = \int_0^1 \frac{t^{b-1}}{K(t, 1)} dt. \quad (10)$$

In this paper, we present a generalization of the integral inequality (9) and its applications. Next proposition will be used in the next section.

Proposition 1 (see [6]). *Let g be a positive increasing function, and $a, b > 0$. Then, for all $x \geq 1$, one has*

$$x^{-a} \int_0^x \frac{t^{a-1}}{g(t)} dt \leq \int_0^1 \frac{t^{a-1}}{g(t)} dt. \quad (11)$$

2. Main Results

Theorem 2. *Let $0 < a, b < 1 < p$, $1/p + 1/q = 1$, $0 < \lambda \leq \min\{(1-b)p/q, (1-a)q/p\}$, and let $K(u, v)$ be positive increasing homogeneous function of degree λ , and $f, g \geq 0$ and*

$$F(x) = \int_0^x f(t) dt, \quad G(x) = \int_0^x g(t) dt \quad \forall x > 0, \quad (12)$$

and let ψ be a function such that $\psi(x) \geq x$ for all $x > 0$.

Then, for all $T > 0$, one has

$$\begin{aligned} & \iint_0^T \frac{F(u)G(v)}{\psi(K(u, v))} du dv \\ & \leq T^{1-\lambda} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\ & \quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q}, \end{aligned} \quad (13)$$

where

$$K_1 = \int_0^1 \frac{t^{a-1}}{K(1, t)} dt, \quad K_2 = \int_0^1 \frac{t^{b-1}}{K(t, 1)} dt. \quad (14)$$

Proof. Let $T > 0$ and $M = \iint_0^T F(u)G(v)/\psi(K(u, v)) du dv$.

By the Hölder inequality, the assumption of ψ , and the Tonelli theorem, we have

$$\begin{aligned} M &= \iint_0^T \frac{F(u) v^{(a-1)/p}}{u^{(b-1)/q} \psi^{1/p}(K(u, v))} \\ & \quad \times \frac{G(v) u^{(b-1)/q}}{v^{(a-1)/p} \psi^{1/q}(K(u, v))} du dv \\ &\leq \left(\iint_0^T \frac{F^p(u) v^{a-1}}{u^{(b-1)p/q} \psi(K(u, v))} du dv \right)^{1/p} \\ & \quad \times \left(\iint_0^T \frac{G^q(v) u^{b-1}}{v^{(a-1)q/p} \psi(K(u, v))} du dv \right)^{1/q} \\ &= \left(\int_0^T F^p(u) \int_0^T \frac{u^{(1-b)p/q} v^{a-1}}{\psi(K(u, v))} dv du \right)^{1/p} \\ & \quad \times \left(\int_0^T G^q(v) \int_0^T \frac{u^{b-1} v^{(1-a)q/p}}{\psi(K(u, v))} du dv \right)^{1/q} \\ &\leq \left(\int_0^T F^p(u) \int_0^T \frac{u^{(1-b)p/q} v^{a-1}}{K(u, v)} dv du \right)^{1/p} \\ & \quad \times \left(\int_0^T G^q(v) \int_0^T \frac{u^{b-1} v^{(1-a)q/p}}{K(u, v)} du dv \right)^{1/q} \\ &= \left(\int_0^T F^p(u) \int_0^T \frac{u^{(1-b)(p/q)+a-1} (v/u)^{a-1}}{u^\lambda K(1, v/u)} dv du \right)^{1/p} \\ & \quad \times \left(\int_0^T G^q(v) \int_0^T \frac{v^{(1-a)(q/p)+b-1} (u/v)^{b-1}}{v^\lambda K(u/v, 1)} du dv \right)^{1/q}. \end{aligned} \quad (15)$$

Now, we put $t = v/u$ and $dt = dv/u$ for the first integral, and then we put $t = u/v$ and $dt = du/v$ for the second integral.

And, by Proposition 1, one has

$$\begin{aligned} M &\leq \left(\int_0^T u^{a+(1-b)(p/q)-\lambda} F^p(u) \int_0^{T/u} \frac{t^{a-1}}{K(1, t)} dt du \right)^{1/p} \\ & \quad \times \left(\int_0^T v^{b+(1-a)(q/p)-\lambda} G^q(v) \int_0^{T/u} \frac{t^{b-1}}{K(t, 1)} dt dv \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&= \left(T^{a+(1-b)(p/q)-\lambda} \int_0^T F^p(u) \left(\frac{u}{T} \right)^{a+(1-b)(p/q)-\lambda} \right. \\
&\quad \times \left. \int_0^{T/u} \frac{t^{a-1}}{K(1,t)} dt du \right)^{1/p} \\
&\quad \times \left(T^{b+(1-a)(q/p)-\lambda} \int_0^T G^q(v) \left(\frac{v}{T} \right)^{b+(1-a)(q/p)-\lambda} \right. \\
&\quad \times \left. \int_0^{T/v} \frac{t^{b-1}}{K(t,1)} dt dv \right)^{1/q} \\
&\leq \left(T^{a+(1-b)(p/q)-\lambda} \int_0^T F^p(u) \left(\frac{u}{T} \right)^a \right. \\
&\quad \times \left. \int_0^{T/u} \frac{t^{a-1}}{K(1,t)} dt du \right)^{1/p} \\
&\quad \times \left(T^{b+(1-a)(q/p)-\lambda} \int_0^T G^q(v) \left(\frac{v}{T} \right)^b \right. \\
&\quad \times \left. \int_0^{T/v} \frac{t^{b-1}}{K(t,1)} dt dv \right)^{1/q} \\
&= T^{1-\lambda} \left(\int_0^T F^p(u) \left(\frac{u}{T} \right)^a \int_0^{T/u} \frac{t^{a-1}}{K(1,t)} dt du \right)^{1/p} \\
&\quad \times \left(\int_0^T G^q(v) \left(\frac{v}{T} \right)^b \int_0^{T/v} \frac{t^{b-1}}{K(t,1)} dt dv \right)^{1/q} \\
&\leq T^{1-\lambda} \left(\int_0^T F^p(u) \int_0^1 \frac{t^{a-1}}{K(1,t)} dt du \right)^{1/p} \\
&\quad \times \left(\int_0^T G^q(v) \int_0^1 \frac{t^{b-1}}{K(t,1)} dt dv \right)^{1/q} \\
&= T^{1-\lambda} \left(K_1 \int_0^T F^p(u) du \right)^{1/p} \\
&\quad \times \left(K_2 \int_0^T G^q(v) dv \right)^{1/q}.
\end{aligned} \tag{16}$$

Then, by the assumption, one has

$$\begin{aligned}
M &\leq T^{1-\lambda} \left(K_1 \int_0^T \int_0^u (F^p(t))' dt du \right)^{1/p} \\
&\quad \times \left(K_2 \int_0^T \int_0^v (G^q(t))' dt dv \right)^{1/q} \\
&= T^{1-\lambda} \left(pK_1 \int_0^T \int_0^u F^{p-1}(t) f(t) dt du \right)^{1/p} \\
&\quad \times \left(qK_2 \int_0^T \int_0^v G^{q-1}(t) g(t) dt dv \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
&= T^{1-\lambda} \left(pK_1 \int_0^T F^{p-1}(t) f(t) \int_t^T du dt \right)^{1/p} \\
&\quad \times \left(qK_2 \int_0^T G^{q-1}(t) g(t) \int_t^T dv dt \right)^{1/q} \\
&= T^{1-\lambda} \left(pK_1 \int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\
&\quad \times \left(qK_2 \int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q} \\
&= T^{1-\lambda} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \\
&\quad \times \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\
&\quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q}.
\end{aligned} \tag{17}$$

This proof is completed. $\square$

3. Applications

Corollary 3. Let $0 < a, b < 1 < p, 1/p + 1/q = 1$ and $0 < \lambda \leq \min\{(1-b)p/q, (1-a)q/p\}$, and let $K(u, v)$ be a positive increasing homogeneous function of degree λ , and $f, g \geq 0$ and

$$F(x) = \int_0^x f(t) dt, \quad G(x) = \int_0^x g(t) dt \quad \forall x > 0. \tag{18}$$

Then, for all $T > 0$, one has

$$\begin{aligned}
(a) \quad &\iint_0^T \frac{F(u)G(v)}{K(u,v)} du dv \\
&\leq T^{1-\lambda} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \\
&\quad \times \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\
&\quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q},
\end{aligned} \tag{19}$$

$$\begin{aligned}
(b) \quad &\iint_0^T \frac{F(u)G(v)}{1+K(u,v)} du dv \\
&\leq T^{1-\lambda} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \\
&\quad \times \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\
&\quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q},
\end{aligned} \tag{20}$$

$$\begin{aligned}
(c) \quad & \iint_0^T \frac{F(u)G(v)}{(1+K(u,v))K(u,v)} du dv \\
& \leq T^{1-\lambda} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \\
& \quad \times \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\
& \quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q},
\end{aligned} \tag{21}$$

$$\begin{aligned}
(d) \quad & \iint_0^T \frac{F(u)G(v)}{e^{K(u,v)}} du dv \\
& \leq T^{1-\lambda} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{1/p} \\
& \quad \times \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{1/q},
\end{aligned} \tag{22}$$

where

$$K_1 = \int_0^1 \frac{t^{a-1}}{K(1,t)} dt, \quad K_2 = \int_0^1 \frac{t^{b-1}}{K(t,1)} dt. \tag{23}$$

Proof. (a) This follows from Theorem 2 where $\psi(x) = x$ for all x .

(b) This follows from Theorem 2 where $\psi(x) = 1 + x$ for all x .

(c) This follows from Theorem 2 where $\psi(x) = x + x^2$ for all x .

(d) This follows from Theorem 2 where $\psi(x) = e^x$ for all x . $\square$

4. Open Problem

In this section, we pose a question that is how to generalize the integral inequality (13) if ψ may not satisfy the property $\psi(x) \geq x$ for all $x > 0$.

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