Research Article

A New Characterization of Generalized Weighted Composition Operators from the Bloch Space into the Zygmund Space

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Received 10 April 2013; Accepted 16 June 2013

Academic Editor: Jiecheng Chen

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A new criterion for the boundedness and the compactness of the generalized weighted composition operators from the Bloch space into the Zygmund space is given in this paper.

1. Introduction

Let $H(\mathbb{D})$ be the space of analytic functions on the unit disk $\mathbb{D} = \{z: |z| < 1\}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathcal{B}$, if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty. \quad (1)$$

Let $\mathcal{B}_0$ be the little Bloch space, which consists of those $f \in \mathcal{B}$ such that $(1 - |z|^2)|f'(z)| \rightarrow 0$, as $|z| \rightarrow 1$. Denote by $\mathcal{Z}$ the set of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty, \quad (2)$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. In fact, $f \in \mathcal{Z}$ if and only if $\sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty$ by Theorem 5.3 of [1] and the Closed Graph Theorem. The class $\mathcal{Z}$ with the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| \quad (3)$$

is a Banach space. We call $\mathcal{Z}$ the Zygmund space. For more information on the Zygmund space, see, for example, [1, 2].

Let $S(\mathbb{D})$ denote the set of analytic self-map of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_{φ} , which is defined by $C_{\varphi}(f)(z) = f(\varphi(z))$ for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

We refer the readers to the book [3] for the theory of the composition operator on various function spaces.

Let $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_{φ} , is defined by

$$(uC_{\varphi})(f)(z) = u(z) f(\varphi(z)), \quad z \in \mathbb{D}, f \in H(\mathbb{D}). \quad (4)$$

Let n be a nonnegative integer. The generalized weighted composition operator, denoted by $D_{\varphi,u}^n$, is defined as follows (see [4–7]):

$$(D_{\varphi,u}^n f)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \quad z \in \mathbb{D}, f \in H(\mathbb{D}). \quad (5)$$

When $n = 0$, then $D_{\varphi,u}^n$ becomes the weighted composition operator. If $n = 0$ and $u(z) = 1$, then $D_{\varphi,u}^n = C_{\varphi}$. If $n = 1$ and $u(z) = \varphi'(z)$, then $D_{\varphi,u}^n = DC_{\varphi}$. If $n = 1$ and $u(z) = 1$, then $D_{\varphi,u}^n = C_{\varphi}D$. The operators DC_{φ} and $C_{\varphi}D$ were studied, for example, in [8–15].

Composition operators, weighted composition operators, and related operators on the Bloch space were studied in [10, 12–22], while composition operators and weighted composition operators between the Zygmund space and some other spaces were studied in [2, 12, 23–28].

Recently, many researchers studied the generalized weighted composition operator $D_{\varphi,u}^n$ on various spaces; see,

for example, in [4–7, 25, 26, 29–32]. In [26], Stević has studied the operator $D_{\varphi,u}^n$ from the Bloch space to the n th weighted-type space, which includes the Zygmund space. Among other things, he obtained the following result (see [26]).

Theorem A. *Let $u \in H(\mathbb{D})$, φ , an analytic self-map of $\mathbb{D}$ and n a positive integer. Then the following assertions hold.*

- (a) *The operator $D_{\varphi,u}^n : \mathcal{B}(\text{or } \mathcal{B}_0) \rightarrow \mathcal{X}$ is bounded if and only if*

$$\begin{aligned} M_1 &:= \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2) |u''(z)|}{(1 - |\varphi(z)|^2)^n} < \infty, \\ M_2 &:= \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2) |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{n+1}} < \infty, \\ M_3 &:= \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2) |u(z)| |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{n+2}} < \infty. \end{aligned} \quad (6)$$

- (b) *Suppose that the operator $D_{\varphi,u}^n : \mathcal{B}(\text{or } \mathcal{B}_0) \rightarrow \mathcal{X}$ is bounded then $D_{\varphi,u}^n : \mathcal{B}(\text{or } \mathcal{B}_0) \rightarrow \mathcal{X}$ is compact if and only if*

$$\begin{aligned} \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2) |u''(z)|}{(1 - |\varphi(z)|^2)^n} &= 0, \\ \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2) |u(z)| |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{n+2}} &= 0, \\ \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2) |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{n+1}} &= 0. \end{aligned} \quad (7)$$

Here we give a new criterion for the boundedness or compactness of the operator $D_{\varphi,u}^n$; namely, we use three families of functions to characterize the operator $D_{\varphi,u}^n : \mathcal{B} \rightarrow \mathcal{X}$.

Throughout the paper, C denotes a positive constant which may differ from one occurrence to the other. The notation $A \asymp B$ means that there exists a positive constant C such that $B/C \leq A \leq CB$.

2. Main Results

In this section we give our main results and proofs. For $a \in \mathbb{D}$, set

$$f_{a,j}(z) = \left(\frac{1 - |a|^2}{1 - \bar{a}z} \right)^j, \quad j = 1, 2, 3. \quad (8)$$

Next, we will use these three families of functions to characterize generalized weighted composition operators $D_{\varphi,u}^n : \mathcal{B} \rightarrow \mathcal{X}$.

Theorem 1. *Let $u \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, and n a positive integer. Then the following conditions are equivalent:*

- (a) *the operator $D_{\varphi,u}^n : \mathcal{B} \rightarrow \mathcal{X}$ is bounded;*
 (b) *the operator $D_{\varphi,u}^n : \mathcal{B}_0 \rightarrow \mathcal{X}$ is bounded;*
 (c) $u \in \mathcal{X}$, $N_2 := \sup_{z \in \mathbb{D}} (1 - |z|^2) |u(z)| |\varphi'(z)|^2 < \infty$,

$$N_3 := \sup_{z \in \mathbb{D}} (1 - |z|^2) |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty,$$

$$A := \max \left\{ \sup_{w \in \mathbb{D}} \|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{X}}, \sup_{w \in \mathbb{D}} \|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{X}}, \sup_{w \in \mathbb{D}} \|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{X}} \right\} < \infty. \quad (9)$$

Proof. (a) $\Rightarrow$ (b). This implication is obvious.

(b) $\Rightarrow$ (c). Assume that $D_{\varphi,u}^n : \mathcal{B}_0 \rightarrow \mathcal{X}$ is bounded. Taking the functions z^n , z^{n+1} , and z^{n+2} and using the boundedness of $D_{\varphi,u}^n$, we see that $u \in \mathcal{X}$, and N_2 and N_3 are finite. For each $a \in \mathbb{D}$, it is easy to check that $f_{a,j} \in \mathcal{B}_0$, $j = 1, 2, 3$. Moreover $\|f_{a,j}\|_{\mathcal{B}_0}$, $j = 1, 2, 3$, are bounded by constants independent of a . By the boundedness of $D_{\varphi,u}^n : \mathcal{B}_0 \rightarrow \mathcal{X}$, we get

$$\begin{aligned} \sup_{a \in \mathbb{D}} \|D_{\varphi,u}^n f_{\varphi(a),j}\|_{\mathcal{X}} &\leq \|D_{\varphi,u}^n\| \sup_{a \in \mathbb{D}} \|f_{\varphi(a),j}\|_{\mathcal{B}_0} \\ &\leq C \|D_{\varphi,u}^n\| < \infty, \quad j = 1, 2, 3, \end{aligned} \quad (10)$$

as desired.

(c) $\Rightarrow$ (a). Suppose that $u \in \mathcal{X}$, and N_2 and N_3 and A are finite. To prove this implication, we only need to show that these conditions imply (6). A calculation shows that

$$\begin{aligned} f_{a,1}^{(n)}(a) &= \frac{n! \bar{a}^n}{(1 - |a|^2)^n}, & f_{a,2}^{(n)}(a) &= \frac{(n+1)! \bar{a}^n}{(1 - |a|^2)^n}, \\ f_{a,3}^{(n)}(a) &= \frac{(n+2)! \bar{a}^n}{2(1 - |a|^2)^n}. \end{aligned} \quad (11)$$

For the simplicity, we denote $2u'(z)\varphi'(z) + u(z)\varphi''(z)$ by $v(z)$. From (11), for $w \in \mathbb{D}$, we have

$$\begin{aligned} &(D_{\varphi,u}^n f_{\varphi(w),1})''(w) \\ &= \frac{n! u''(w) \overline{\varphi(w)}^n}{(1 - |\varphi(w)|^2)^n} + \frac{(n+1)! v(w) \overline{\varphi(w)}^{n+1}}{(1 - |\varphi(w)|^2)^{1+n}} \\ &\quad + \frac{(n+2)! u(w) (\varphi'(w))^2 \overline{\varphi(w)}^{n+2}}{(1 - |\varphi(w)|^2)^{2+n}}, \end{aligned} \quad (12)$$

$$\begin{aligned}
& (D_{\varphi,u}^n f_{\varphi(w),2})''(w) \\
&= \frac{(n+1)!u''(w)\overline{\varphi(w)}^n}{(1-|\varphi(w)|^2)^n} + \frac{(n+2)!v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{1+n}} \\
&+ \frac{(n+3)!u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{2+n}},
\end{aligned} \quad (13)$$

$$\begin{aligned}
& (D_{\varphi,u}^n f_{\varphi(w),3})''(w) \\
&= \frac{(n+2)!u''(w)\overline{\varphi(w)}^n}{2(1-|\varphi(w)|^2)^n} + \frac{(n+3)!v(w)\overline{\varphi(w)}^{n+1}}{2(1-|\varphi(w)|^2)^{1+n}} \\
&+ \frac{(n+4)!u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{2(1-|\varphi(w)|^2)^{2+n}}.
\end{aligned} \quad (14)$$

Multiplying (12) by $-(n+1)$ and then adding (13), we get

$$\begin{aligned}
& -(n+1)(D_{\varphi,u}^n f_{\varphi(w),1})''(w) + (D_{\varphi,u}^n f_{\varphi(w),2})''(w) \\
&= \frac{(n+1)!v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{1+n}} \\
&+ \frac{2(n+2)!u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{2+n}}.
\end{aligned} \quad (15)$$

Multiplying (12) by $-(n+1)(n+2)$ and (14) by 2, respectively, we obtain

$$\begin{aligned}
& -(n+1)(n+2)(D_{\varphi,u}^n f_{\varphi(w),1})''(w) + 2(D_{\varphi,u}^n f_{\varphi(w),3})''(w) \\
&= \frac{2(n+2)!v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{1+n}} \\
&+ \frac{(4n+10)(n+2)!u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{2+n}}.
\end{aligned} \quad (16)$$

Multiplying (15) by $2(n+2)$, we get

$$\begin{aligned}
& -2(n+1)(n+2)(D_{\varphi,u}^n f_{\varphi(w),1})''(w) \\
&+ 2(n+2)(D_{\varphi,u}^n f_{\varphi(w),2})''(w) \\
&= \frac{2(n+2)!v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{1+n}} \\
&+ \frac{(4n+8)(n+2)!u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{2+n}}.
\end{aligned} \quad (17)$$

Subtracting (17) from (16), we obtain

$$\begin{aligned}
& \frac{2(n+2)!u(w)(\varphi'(w))^2\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{2+n}} \\
&= (n+1)(n+2)(D_{\varphi,u}^n f_{\varphi(w),1})''(w) \\
&- 2(n+2)(D_{\varphi,u}^n f_{\varphi(w),2})''(w) + 2(D_{\varphi,u}^n f_{\varphi(w),3})''(w),
\end{aligned} \quad (18)$$

which implies that

$$\begin{aligned}
& \frac{(1-|w|^2)|u(w)(\varphi'(w))^2||\varphi(w)|^{n+2}}{(1-|\varphi(w)|^2)^{2+n}} \\
&\leq \frac{1}{2n!}(1-|w|^2)|(D_{\varphi,u}^n f_{\varphi(w),1})''(w)|
\end{aligned} \quad (19)$$

$$\begin{aligned}
& + \frac{1}{(n+1)!}(1-|w|^2)|(D_{\varphi,u}^n f_{\varphi(w),2})''(w)| \\
&+ \frac{1}{(n+2)!}(1-|w|^2)|(D_{\varphi,u}^n f_{\varphi(w),3})''(w)| \\
&\leq \frac{1}{2n!}\|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{Z}} \\
&+ \frac{1}{(n+1)!}\|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{Z}} + \frac{1}{(n+2)!}\|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{Z}}
\end{aligned} \quad (20)$$

$$\leq \frac{A}{2n!} + \frac{A}{(n+1)!} + \frac{A}{(n+2)!}. \quad (21)$$

From (16) and (18), we obtain

$$\begin{aligned}
& \frac{(n+2)!v(w)\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{1+n}} \\
&= -(n+1)(n+2)(n+3)(D_{\varphi,u}^n f_{\varphi(w),1})''(w) \\
&+ (n+2)(2n+5)(D_{\varphi,u}^n f_{\varphi(w),2})''(w) \\
&- 2(n+2)(D_{\varphi,u}^n f_{\varphi(w),3})''(w),
\end{aligned} \quad (22)$$

which implies that

$$\begin{aligned}
& \frac{(1-|w|^2)|v(w)||\varphi(w)|^{n+1}}{(1-|\varphi(w)|^2)^{1+n}} \\
&\leq \frac{(n+3)}{n!}(1-|w|^2)|(D_{\varphi,u}^n f_{\varphi(w),1})''(w)| \\
&+ \frac{(2n+5)}{(n+1)!}(1-|w|^2)|(D_{\varphi,u}^n f_{\varphi(w),2})''(w)| \\
&+ \frac{2}{(n+1)!}(1-|w|^2)|(D_{\varphi,u}^n f_{\varphi(w),3})''(w)|
\end{aligned} \quad (23)$$

$$\begin{aligned} &\leq \frac{(n+3)}{n!} \|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{X}} + \frac{(2n+5)}{(n+1)!} \|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{X}} \\ &\quad + \frac{2}{(n+1)!} \|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{X}} \end{aligned} \quad (24)$$

$$\leq \frac{(n+3)}{n!} A + \frac{(2n+5)}{(n+1)!} A + \frac{2}{(n+1)!} A. \quad (25)$$

By (12), (18), and (22), we have

$$\begin{aligned} &\frac{n!u''(w)\overline{\varphi(w)}^n}{(1-|\varphi(w)|^2)^n} \\ &= \frac{2+(n+1)(n+4)}{2} (D_{\varphi,u}^n f_{\varphi(w),1})''(w) \\ &\quad - (n+3) (D_{\varphi,u}^n f_{\varphi(w),2})''(w) + (D_{\varphi,u}^n f_{\varphi(w),3})''(w), \end{aligned} \quad (26)$$

which implies that

$$\begin{aligned} &\frac{(1-|w|^2)|u''(w)| |\varphi(w)|^n}{(1-|\varphi(w)|^2)^n} \\ &\leq \frac{2+(n+1)(n+4)}{2n!} (1-|w|^2) |(D_{\varphi,u}^n f_{\varphi(w),1})''(w)| \\ &\quad + \frac{(n+3)}{n!} (1-|w|^2) |(D_{\varphi,u}^n f_{\varphi(w),2})''(w)| \\ &\quad + \frac{1}{n!} (1-|w|^2) |(D_{\varphi,u}^n f_{\varphi(w),3})''(w)| \end{aligned} \quad (27)$$

$$\begin{aligned} &\leq \frac{2+(n+1)(n+4)}{2n!} \|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{X}} \\ &\quad + \frac{(n+3)}{n!} \|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{X}} + \frac{1}{n!} \|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{X}} \end{aligned} \quad (28)$$

$$\leq \frac{2+(n+1)(n+4)}{2n!} A + \frac{(n+3)}{n!} A + \frac{1}{n!} A. \quad (29)$$

Fix $r \in (0, 1)$. If $|\varphi(w)| > r$, then from (21) we obtain

$$\begin{aligned} &\frac{(1-|w|^2)|u(w)| |\varphi'(w)|^2}{(1-|\varphi(w)|^2)^{2+n}} \\ &\leq \frac{A}{2n!r^{n+2}} + \frac{A}{(n+1)!r^{n+2}} + \frac{A}{(n+2)!r^{n+2}} < \infty. \end{aligned} \quad (30)$$

On the other hand, if $|\varphi(w)| \leq r$, we get

$$\begin{aligned} &\frac{(1-|w|^2)|u(w)| |\varphi'(w)|^2}{(1-|\varphi(w)|^2)^{2+n}} \\ &\leq \frac{1}{(1-r^2)^{2+n}} \sup_{z \in \mathbb{D}} (1-|z|^2) |u(z)| |\varphi'(z)|^2 < \infty. \end{aligned} \quad (31)$$

From (30) and (31) we see that M_3 is finite. Similarly, from (25) and (29) we can obtain that M_1 and M_2 are finite as well. The proof of this theorem is finished. $\square$

To get the characterization of the compactness of $D_{\varphi,u}^n : \mathcal{B}(\text{or } \mathcal{B}_0) \rightarrow \mathcal{X}$, we need the following criterion, which follows from standard arguments similar to those outlined in Proposition 3.11 of [3].

Lemma 2. Let $u \in H(\mathbb{D})$, φ , an analytic self-map of $\mathbb{D}$ and n be a positive integer. The operator $D_{\varphi,u}^n : \mathcal{B}(\text{or } \mathcal{B}_0) \rightarrow \mathcal{X}$ is compact if and only if $D_{\varphi,u}^n : \mathcal{B}(\text{or } \mathcal{B}_0) \rightarrow \mathcal{X}$ is bounded, and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{B}(\text{or } \mathcal{B}_0)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, one has $\|D_{\varphi,u}^n f_k\|_{\mathcal{X}} \rightarrow 0$ as $k \rightarrow \infty$.

Theorem 3. Let $u \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, and n a positive integer. Suppose that the operator $D_{\varphi,u}^n : \mathcal{B} \rightarrow \mathcal{X}$ is bounded; then the following conditions are equivalent:

- (a) the operator $D_{\varphi,u}^n : \mathcal{B} \rightarrow \mathcal{X}$ is compact;
- (b) the operator $D_{\varphi,u}^n : \mathcal{B}_0 \rightarrow \mathcal{X}$ is compact;
- (c) $\lim_{|\varphi(w)| \rightarrow 1} \|D_{\varphi,u}^n f_{\varphi(w),j}\|_{\mathcal{X}} = 0$, $j = 1, 2, 3$.

Proof. (a) $\Rightarrow$ (b). This implication is clear.

(b) $\Rightarrow$ (c). Assume that $D_{\varphi,u}^n : \mathcal{B}_0 \rightarrow \mathcal{X}$ is compact. Let $\{w_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k \rightarrow \infty} |\varphi(w_k)| = 1$ (if such a sequence does not exist, then the limits in (c) automatically hold). Since the sequences $\{f_{\varphi(w_k),j}\}$, $j = 1, 2, 3$, are bounded in $\mathcal{B}_0$ and converge to 0 uniformly on compact subsets of $\mathbb{D}$, by Lemma 2, we get $\|D_{\varphi,u}^n f_{\varphi(w_k),j}\|_{\mathcal{X}} \rightarrow 0$, as $k \rightarrow \infty$, which means that (c) holds.

(c) $\Rightarrow$ (a). Suppose that the limits in (c) are 0. To prove this implication, we only need to show that (7) hold. Using the inequality (28), we get

$$\begin{aligned} &\frac{(1-|w|^2)|u''(w)|}{(1-|\varphi(w)|^2)^n} \\ &\leq \frac{[2+(n+1)(n+4)] \|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{X}}}{2n!|\varphi(w)|^n} \\ &\quad + \frac{(n+3) \|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{X}}}{n!|\varphi(w)|^n} + \frac{\|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{X}}}{n!|\varphi(w)|^n} \rightarrow 0, \end{aligned} \quad (32)$$

as $|\varphi(w)| \rightarrow 1$.

Using inequality (24), we get

$$\begin{aligned} &\frac{(1-|w|^2)|2u'(w)\varphi'(w) + u(w)\varphi''(w)|}{(1-|\varphi(w)|^2)^{1+n}} \\ &\leq \frac{(n+3) \|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{X}}}{n!|\varphi(w)|^{n+1}} + \frac{(2n+5) \|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{X}}}{(n+1)!|\varphi(w)|^{n+1}} \\ &\quad + \frac{2 \|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{X}}}{(n+1)!|\varphi(w)|^{n+1}} \rightarrow 0, \end{aligned} \quad (33)$$

as $|\varphi(w)| \rightarrow 1$.

Using inequality (20), we get

$$\begin{aligned} & \frac{(1 - |w|^2) |u(w) (\varphi'(w))^2|}{(1 - |\varphi(w)|^2)^{2+n}} \\ & \leq \frac{\|D_{\varphi,u}^n f_{\varphi(w),1}\|_{\mathcal{F}}}{2n! |\varphi(w)|^{n+2}} + \frac{\|D_{\varphi,u}^n f_{\varphi(w),2}\|_{\mathcal{F}}}{(n+1)! |\varphi(w)|^{n+2}} \quad (34) \\ & \quad + \frac{\|D_{\varphi,u}^n f_{\varphi(w),3}\|_{\mathcal{F}}}{(n+2)! |\varphi(w)|^{n+2}} \rightarrow 0, \end{aligned}$$

as $|\varphi(w)| \rightarrow 1$. The desired result follows. The proof of this theorem is complete. $\square$

Acknowledgment

The first author is supported by National Natural Science Foundation of China (no. 11126284).

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