Research Article

Operator Characterizations and Some Properties of g -Frames on Hilbert Spaces

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Received 30 May 2013; Accepted 22 September 2013

Academic Editor: Simone Secchi

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Given the g -orthonormal basis for Hilbert space H , we characterize the g -frames, normalized tight g -frames, and g -Riesz bases in terms of the g -preframe operators. Then we consider the transformations of g -frames, normalized tight g -frames, and g -Riesz bases, which are induced by operators and characterize them in terms of the operators. Finally, we discuss the sums and g -dual frames of g -frames by applying the results of characterizations.

1. Introduction

A sequence $(f_i)_{i \in I}$ of elements of a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \quad (1)$$

The numbers A, B are called the *lower* (resp., *upper*) frame bounds. The frame is a *tight frame* if $A = B$ and a *normalized tight frame* if $A = B = 1$.

The concept of frame first appeared in the late 40s and early 50s (see, [1–3]). The development and study of wavelet theory during the last decades also brought new ideas and attentions to frames because of their close connections. There are many related references on this topic, see; [4–8].

In [9], Sun raised the concept of g -frame as follows, which generalized the concept of frame extensively. A sequence $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is called a g -frame for H with respect to $\{H_i : i \in N\}$, which is a sequence of closed subspaces of a Hilbert space K , if there exist two positive constants A and B such that for any $f \in H$

$$A\|f\|^2 \leq \sum_{i \in N} \|\Lambda_i f\|^2 \leq B\|f\|^2. \quad (2)$$

We simply call $\{\Lambda_i : i \in N\}$ a g -frame for H whenever the space sequence $\{H_i : i \in N\}$ is clear. The tight g -frame

and normalized tight g -frame are defined similarly. We call $\{\Lambda_i : i \in N\}$ a g -frame sequence, if it is a g -frame for $\overline{\text{span}}\{\Lambda_i^*(H_i) : i \in N\}$. We call $\{\Lambda_i : i \in N\}$ a g -Bessel sequence, if only the right inequality is satisfied. Recently, g -frames in Hilbert spaces have been studied intensively; for more details see [10–14] and the references therein.

Constructing frames and g -frames is an interesting problem in frame theory and it is also useful in applications. In this respect, many mathematicians considered the algebraic operations among frames, which allows us to construct a large number of new frames from existing frames. For more details, see [15–18] and the references therein. In this paper, we will consider the algebraic operations among g -frames; in particular, we will consider the direct sums of g -frames and general sums of g -frames. We also will consider the transforms of g -frames and their dual g -frames.

2. g -Preframe Operators and Transformations of g -Frames

In this section, we introduce the g -preframe operators and use them to characterize g -frames. We also discuss the transformations of g -frames. Since any countable infinite sequence can be made as one indexed by natural number N , so all infinite sequences in this paper are assumed to be

indexed by N . For each sequence $\{H_i : i \in N\}$, we define the space $(\sum_{i \in N} \oplus H_i)_{l_2}$ by

$$\left(\sum_{i \in N} \oplus H_i \right)_{l_2} = \left\{ \{f_i\}_{i \in N} : f_i \in H_i, i \in N, \sum_{i \in N} \|f_i\|^2 < +\infty \right\}, \quad (3)$$

with the inner product defined by

$$\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in N} \langle f_i, g_i \rangle. \quad (4)$$

Definition 1 (see, [9]). Let $\Lambda_i \in B(H, H_i)$, $i \in N$.

- (i) If $\{f : \Lambda_i f = 0; i \in N\} = \{0\}$, then we say that $\{\Lambda_i : i \in N\}$ is g -complete.
- (ii) If $\{\Lambda_i : i \in N\}$ is g -complete and there are positive constants A and B such that for any finite subset $J \subset N$ and $g_j \in H_j$, $j \in J$,

$$A \sum_{j \in J} \|g_j\|^2 \leq \left\| \sum_{j \in J} \Lambda_j^* g_j \right\|^2 \leq B \sum_{j \in J} \|g_j\|^2, \quad (5)$$

then we say that $\{\Lambda_i : i \in N\}$ is a g -Riesz basis for H with respect to $\{H_i : i \in N\}$.

- (iii) We say $\{\Lambda_i : i \in N\}$ is a g -orthonormal basis for H with respect to $\{H_i : i \in N\}$ if it satisfies the following:

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad \forall i, j \in N, g_i \in H_i, g_j \in H_j,$$

$$\sum_{j \in N} \|\Lambda_j f\|^2 = \|f\|^2, \quad \forall f \in H. \quad (6)$$

Remark 2. It is obvious that any g -frame $\{\Lambda_i : i \in N\}$ is g -complete and any g -orthonormal basis is a normalized tight g -frame.

Lemma 3 (see [11]). $\{\Lambda_i \in B(H, H_i), i \in N\}$ is g -complete if and only if $\overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in N} = H$.

Lemma 4 (see, [11]). $\{\Lambda_i \in B(H, H_i), i \in N\}$ is a g -Bessel sequence for H with bound B , if and only if the operator

$$T : \left(\sum_{i \in N} \oplus H_i \right)_{l_2} \longrightarrow H \quad (7)$$

defined by $T(\{g_i\}) = \sum_{j \in N} \Lambda_j^*(g_j)$ is a well-defined bounded operator with $\|T\| \leq \sqrt{B}$.

Lemma 5. Let $\{\theta_i \in B(H, H_i) : i \in N\}$ be a g -orthonormal basis for H with respect to $\{H_i : i \in N\}$; then the sequence $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence with respect to $\{H_i : i \in N\}$ if and only if there is a unique bounded operator $V : H \rightarrow H$ such that $\Lambda_i = \theta_i V^*$, for all $i \in N$.

Proof. $\Rightarrow$: Since $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis for H , $\{\theta_i f : i \in N\} \in (\sum_{i \in N} \oplus H_i)_{l_2}$, for any $f \in H$. If $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence, then the operator

$$V : H \longrightarrow H, \quad Vf = \sum_{i \in N} \Lambda_i^* \theta_i f \quad (8)$$

is well-defined and bounded operator by Lemma 5. Also by the definition of g -orthonormal basis, it is easy to see that $\theta_i \theta_j^* g = \delta_{ij} g$. So

$$V \theta_j^* g = \sum_{i \in N} \Lambda_i^* \theta_i \theta_j^* g = \Lambda_j^* \theta_j \theta_j^* g = \Lambda_j^* g \quad (9)$$

for all $g \in H_j$, $j \in N$. Hence $V \theta_j^* = \Lambda_j^*$, which implies that $\theta_j V^* = \Lambda_j$, $j \in N$. Suppose $V_1, V_2 \in B(H)$ and $\theta_i V_1^* = \theta_i V_2^* = \Lambda_i$ for any $i \in N$. Then for any $f \in H$, $g_i \in H_i$, we have $\langle \theta_i V_1^* f, g_i \rangle = \langle \theta_i V_2^* f, g_i \rangle$, that is, $\langle V_1^* f, \theta_i^* g_i \rangle = \langle V_2^* f, \theta_i^* g_i \rangle$. Since $\overline{\text{span}}\{\theta_i^*(H_i)\}_{i \in N} = H$ by Lemma 3, $V_1^* f = V_2^* f$, which means that $V_1 = V_2$. Thus the operator V is unique.

$\Leftarrow$: Since $\Lambda_i = \theta_i V^*$, for all $i \in N$, for any $f \in H$, we have

$$\sum_{i \in N} \|\Lambda_i f\|^2 = \sum_{i \in N} \|\theta_i V^* f\|^2 = \|V^* f\|^2 \leq \|V^*\|^2 \|f\|^2. \quad (10)$$

So $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence with respect to $\{H_i : i \in N\}$. $\square$

Definition 6. Let $\{\theta_n : n \in N\}$ be a g -orthonormal basis for H , $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence for H with respect to $\{H_i : i \in N\}$, and the g -preframe operator associated with $\{\Lambda_i : i \in N\}$ is the operator $V \in B(H)$ such that $\theta_n V^* = \Lambda_n$ for any $n \in N$. The g -frame operator S associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is defined by $Sf = \sum_{i \in N} \Lambda_i^* \Lambda_i f$.

Remark 7. It is well known that if $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -frame, then S is a well-defined positive invertible operator on H and $\{\tilde{\Lambda}_i = \Lambda_i S^{-1} : i \in N\}$ is the canonical dual g -frame of $\{\Lambda_i : i \in N\}$; for details see [9]. In this case, for any $f \in H$, we have the Fourier series type decomposition: $f = \sum_{i \in N} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i \in N} \tilde{\Lambda}_i^* \Lambda_i f$. In general, there are many g -frames $\{\Gamma_i : i \in N\}$ for which the formula $f = \sum_{i \in N} \Lambda_i^* \Gamma_i f = \sum_{i \in N} \Gamma_i^* \Lambda_i^* f$ holds. All such g -frames are called alternate dual g -frames of $\{\Lambda_i : i \in N\}$.

Lemma 8. Suppose $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis for H , $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence for H , and V and S are the g -preframe operator and g -frame operator associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$, respectively. Then $S = VV^*$.

Proof. Since $\Lambda_i = \theta_i V^*$ for all $i \in N$, for any $f \in H$ we have

$$\begin{aligned} Sf &= \sum_{i \in N} \Lambda_i^* \Lambda_i f = \sum_{i \in N} V \theta_i^* \theta_i V^* f \\ &= V \sum_{i \in N} \theta_i^* \theta_i V^* f = VV^* f. \end{aligned} \quad (11)$$

So $S = VV^*$. $\square$

Lemma 9 (see, [11]). A sequence $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g -frame for H if and only if

$$S : f \longrightarrow \sum_{i \in N} \Lambda_i^* \Lambda_i f \quad (12)$$

is a well-defined and bounded mapping from H onto H .

Remark 10. Since S is adjoint, the above condition can be replaced with S is a well-defined and bounded invertible mapping.

Lemma 11 (see, [19]). Suppose that $T : K \rightarrow H$ is a bounded surjective operator. Then there exists a bounded operator (called the pseudoinverse of T) $T^\dagger : H \rightarrow K$ for which

$$TT^\dagger f = f, \quad \forall f \in H. \quad (13)$$

Theorem 12. Suppose $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis for H , $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence for H , and V and S are the g -preframe operator and g -frame operator associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$, respectively. Then

- (1) $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -frame if and only if V is onto.
- (2) $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a normalized tight g -frame if and only if V is a coisometry.
- (3) $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Riesz basis if and only if V is invertible.
- (4) $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis if and only if V is unitary.

Proof. (1) $\Rightarrow$: By Lemma 9, if $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -frame, then S is invertible, but $S = VV^*$ by Lemma 8, so V is onto. $\Leftarrow$: If V is onto, then $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Bessel sequence by Lemma 5. So we just need to show the existence of lower frame bound. Since V is onto, $VV^\dagger = I$ by Lemma 11. Hence $(V^\dagger)^* V^* = I$, so for any $f \in H$, $\|f\|^2 = \|(V^\dagger)^* V^* f\|^2 \leq \|(V^\dagger)^*\|^2 \|V^* f\|^2$. So we have

$$\begin{aligned} \sum_{i \in N} \|\Lambda_i f\|^2 &= \sum_{i \in N} \|\theta_i V^* f\|^2 \\ &= \|V^* f\|^2 \geq \frac{1}{\|(V^\dagger)^*\|^2} \|f\|^2. \end{aligned} \quad (14)$$

So $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -frame.

(2) Since $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a normalized tight g -frame if and only if $S = I$, but $S = VV^*$ by Lemma 8, so $S = I$ if and only if V is coisometry.

(3) If $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Riesz basis then V is invertible by [11, Corollary 2.18]. Conversely, suppose V is invertible. If for any $i \in N$, $\Lambda_i f = \theta_i V^* f = 0$, since $\{\theta_i : i \in N\}$ is g -complete, then $V^* f = 0$; hence $f = 0$. It follows that $\{\Lambda_i : i \in N\}$ is g -complete. Now for any finite set $J \subset N$, and $g_j \in H_j, j \in J$, we have

$$\left\| \sum_{j \in J} \Lambda_j^* g_j \right\|^2 = \left\| \sum_{j \in J} V \theta_j^* g_j \right\|^2 = \left\| V \sum_{j \in J} \theta_j^* g_j \right\|^2. \quad (15)$$

Since

$$\begin{aligned} \left\| \sum_{j \in J} \theta_j^* g_j \right\|^2 &= \left\| V V^{-1} \sum_{j \in J} \theta_j^* g_j \right\|^2 \leq \|V^{-1}\|^2 \left\| V \sum_{j \in J} \theta_j^* g_j \right\|^2, \\ \left\| V \sum_{j \in J} \theta_j^* g_j \right\|^2 &\leq \|V\|^2 \left\| \sum_{j \in J} \theta_j^* g_j \right\|^2, \quad \left\| \sum_{j \in J} \theta_j^* g_j \right\|^2 = \sum_{j \in J} \|g_j\|^2. \end{aligned} \quad (16)$$

So we have

$$\frac{1}{\|V^{-1}\|^2} \sum_{j \in J} \|g_j\|^2 \leq \left\| \sum_{j \in J} \Lambda_j^* g_j \right\|^2 \leq \|V\|^2 \sum_{j \in J} \|g_j\|^2. \quad (17)$$

Hence $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -Riesz basis.

(4) If $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis, then V is unitary by [11, Corollary 2.19]. Conversely, suppose that V is unitary. Then for $i, j \in N$, and $g_i \in H_i, g_j \in H_j$ we have

$$\begin{aligned} \langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle &= \langle V \theta_i^* g_i, V \theta_j^* g_j \rangle \\ &= \langle \theta_i^* g_i, \theta_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \end{aligned} \quad (18)$$

$$\sum_{i \in N} \|\Lambda_i^* f\|^2 = \sum_{i \in N} \|\theta_i V^* f\|^2 = \|V^* f\|^2 = \|f\|^2.$$

So $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis. $\square$

Lemma 13. (1) Suppose that $T, V \in B(H)$ and V is onto. Then TV is onto if and only if T is onto.

(2) Suppose that $T, V \in B(H)$ and V is a coisometry. Then TV is a coisometry if and only if T is a coisometry.

(3) Suppose that $T, V \in B(H)$ and V is invertible. Then TV is invertible if and only if T is invertible.

(4) Suppose that $T, V \in B(H)$ and V is a unitary. Then TV is unitary if and only if T is unitary.

Proof. (1) If TV is onto, then for any $y \in H$, there exists $x \in H$ such that $y = TVx = T(Vx)$, so T is onto. Conversely, if T is onto, then for any $y \in H$ there exists $x \in H$ such that $y = Tx$. Since V is onto, there exists $z \in H$ such that $x = Vz$. So $y = TVz$, which implies that TV is onto.

(2) If TV is a coisometry, then $(TV)(TV)^* = I$. Since V is a coisometry, $VV^* = I$. So $TT^* = T(VV^*)T^* = (TV)(TV)^* = I$, which implies that T is a coisometry. Conversely, if T is a coisometry, then $(TV)(TV)^* = TVV^*T^* = T(VV^*)T^* = TT^* = I$. Hence TV is a coisometry.

(3) If TV is invertible, then $T = (TV)V^{-1}$, so T is invertible. Conversely, if T is invertible, since V is invertible, then TV is invertible.

(4) If TV is unitary, then $T = TVV^* = (TV)V^*$, so T is unitary. Conversely, if T is unitary, since V is unitary, so TV is unitary. $\square$

By Theorem 12 and Lemma 13, it is easy to get the following results. We leave the details to the readers.

Theorem 14. Suppose that $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g -orthonormal basis for H with respect to $\{H_i : i \in N\}$, $T \in B(H)$. Define the transformation $\Phi_T : \{\Lambda_i \in B(H, H_i) : i \in N\} \rightarrow \{\Lambda_i T^* : i \in N\}$, then:

- (1) It transforms g -frames to g -frames if and only if T is onto.
- (2) It transforms normalized tight g -frames to normalized tight g -frames if and only if T is a coisometry.
- (3) It transforms g -Riesz bases to g -Riesz bases if and only if T is invertible.
- (4) It transforms g -orthonormal bases to g -orthonormal bases if and only if T is unitary.

Remark 15. The above results are established under the constraint that H has a g -orthonormal basis with respect to $\{H_i : i \in N\}$. But in general, the statement (1) in the above Theorem is still true.

Theorem 16. Given $T \in B(H)$. The transformation $\Phi_T : \{\Lambda_i \in B(H, H_i) : i \in N\} \rightarrow \{\Lambda_i T^* : i \in N\}$, transforms g -frames to g -frames if and only if T is onto.

Proof. $\Rightarrow$: If Φ_T transforms g -frames to g -frames, then $\{\Lambda_i T^* : i \in N\}$ is a g -frame whenever $\{\Lambda_i : i \in N\}$ is a g -frame. Suppose that A_1, B_1 and A_2, B_2 are the lower and upper frame bounds of $\{\Lambda_i T^* : i \in N\}$ and $\{\Lambda_i : i \in N\}$ respectively. Then for any $f \in H$, we have

$$A_1 \|f\|^2 \leq \sum_{i \in N} \|\Lambda_i T^* f\|^2 \leq B_1 \|f\|^2. \quad (19)$$

So if $T^* f = 0$, then $f = 0$, which implies that T^* is one to one. If $T^* f_n \rightarrow y (n \rightarrow \infty)$, then $\{f_n\}$ is convergent; that is, there exists $y_0 \in H$ such that $f_n \rightarrow y_0$. Hence $T^* f_n \rightarrow T^* y_0$, so $y = T^* y_0$, that is, $y \in \text{range}(T^*)$, which implies that T^* has closed range. So T has closed range. Since $\text{range}(T) = \text{range}(T) = (\ker T^*)^\perp = H$, T is onto.

$\Leftarrow$: If T is onto and $\{\Lambda_i : i \in N\}$ is a g -frame for H with frame bounds A and B , then for any $f \in H$, we have

$$\sum_{i \in N} \|\Lambda_i T^* f\|^2 \leq B \|T^* f\|^2 \leq B \|T^*\|^2 \|f\|^2 \quad (20)$$

and since $TT^\dagger = I$ by Lemma 11, $(T^\dagger)^* T^* = I$. So

$$\sum_{i \in N} \|\Lambda_i T^* f\|^2 \geq A \|T^* f\|^2 \geq \frac{A}{\|(T^\dagger)^*\|^2} \|f\|^2. \quad (21)$$

So $\{\Lambda_i T^* : i \in N\}$ is a g -frame for H , which implies that Φ_T transforms g -frames to g -frames. $\square$

3. Sums of g -Frames

In this section, we will consider the direct sums and usual sums of g -frames, which generate g -frames for direct sum of Hilbert spaces and sum of Hilbert spaces, respectively.

Definition 17. Suppose $T, L \in B(H, K)$, then $T \oplus L \in B(H \oplus H, K)$ is defined by $(T \oplus L)(x \oplus y) = Tx + Ly$.

Remark 18. Since for any scalars a and b , any $x, y \in H$, we have

$$\begin{aligned} (T \oplus L) [a(x_1 \oplus y_1) + b(x_2 \oplus y_2)] \\ &= (T \oplus L) [(ax_1 + bx_2) \oplus (ay_1 + by_2)] \\ &= T(ax_1 + bx_2) + L(ay_1 + by_2) \\ &= aTx_1 + bTx_2 + aLy_1 + bLy_2 \\ &= a(Tx_1 + Ly_1) + b(Tx_2 + Ly_2) \\ &= a(T \oplus L)(x_1 \oplus y_1) + b(T \oplus L)(x_2 \oplus y_2). \end{aligned} \quad (22)$$

So $T \oplus L$ is linear. Since for any $x, y \in H$, we have

$$\begin{aligned} \|(T \oplus L)(x \oplus y)\| &= \|Tx + Ly\| \leq \|Tx\| + \|Ly\| \\ &\leq \|T\| \|x\| + \|L\| \|y\| \leq M(\|x\| + \|y\|) \\ &\leq M\sqrt{2} \sqrt{\|x\|^2 + \|y\|^2} = \sqrt{2}M \|x \oplus y\|, \end{aligned} \quad (23)$$

where $M = \max\{\|T\|, \|L\|\}$. So $T \oplus L$ is bounded. It is easy to see that $(T_1 \oplus T_2)^* z = T_1^* z \oplus T_2^* z$ for any $z \in H_i$. In fact, for any $x, y \in H$, we have

$$\begin{aligned} \langle (T_1 \oplus T_2)(x \oplus y), z \rangle &= \langle (x \oplus y), (T_1 \oplus T_2)^* z \rangle, \\ \langle (T_1 \oplus T_2)(x \oplus y), z \rangle &= \langle T_1 x + T_2 y, z \rangle \\ &= \langle T_1 x, z \rangle + \langle T_2 y, z \rangle \\ &= \langle x, T_1^* z \rangle + \langle y, T_2^* z \rangle \\ &= \langle x \oplus y, T_1^* z \oplus T_2^* z \rangle. \end{aligned} \quad (24)$$

So $(T_1 \oplus T_2)^* z = T_1^* z \oplus T_2^* z$. The finite direct sum of operators have the same meaning.

Definition 19. Let $\{\Lambda_j : j \in N\}, \{\Gamma_j : j \in N\}$ be normalized tight g -frames for Hilbert space H with respect to $\{H_i : i \in N\}$, if $\{\Lambda_j \oplus \Gamma_j : j \in N\}$ is a normalized tight g -frame for $H \oplus H$ with respect to $\{H_i : i \in N\}$, then we call them *strongly disjoint*. More generally, if $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are g -frames with respect to $\{H_i : i \in N\}$ such that $\{\Lambda_j \oplus \Gamma_j : j \in N\}$ is a g -frame for $H \oplus H$ with respect to $\{H_i : i \in N\}$, then they are called *disjoint*.

In general, let $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ be normalized tight g -frames for Hilbert space H , if $\{\Lambda_{1j} \oplus \dots \oplus \Lambda_{kj} : j \in N\}$ is a normalized g -frame for $H \oplus H \oplus \dots \oplus H$, which is the inner direct sum of k copies of H , then we call them *strongly disjoint*. If the k number of sequences are g -frames and their inner direct sum is a g -frame for the inner direct sum of k copies of H , then we call them *disjoint*.

Definition 20. Let $\{\Lambda_j : j \in N\}, \{\Gamma_j : j \in N\}$ be normalized tight g -frames for Hilbert space H , if $\{\Lambda_j \oplus \Gamma_j : j \in N\}$ is a g -orthonormal basis of $H \oplus H$, then we call them *strongly complementary*.

In general, let $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ be normalized tight g -frames for Hilbert space H , if $\{\Lambda_{1j} \oplus \dots \oplus \Lambda_{kj} : j \in N\}$ is a g -orthonormal basis for the inner direct sum of k copies of H , then we call them *strongly complementary*.

In [8], the authors characterize sums of frames by the topological and algebraic properties of $P(H)$, the set of orthogonal projections on H . In this section, we always assume that there exists a g -orthonormal basis $\{\theta_i : i \in N\}$ for H with respect to $\{H_i : i \in N\}$ and we will characterize all kinds of sums of g -frames through the properties of g -preframe operators.

Theorem 21. Suppose $\{\Lambda_j : j \in N\}, \{\Gamma_j : j \in N\}$ are normalized tight g -frames for Hilbert space H , T_1 and T_2 are the g -preframe operators associated with $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$, respectively; then $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are strongly disjoint if and only if $T_2 T_1^* = 0$.

Proof. $\Rightarrow$: Since $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are strongly disjoint, so $\{\Lambda_j \oplus \Gamma_j : j \in N\}$ is a normalized tight g -frame for $H \oplus H$. Since T_1, T_2 are g -preframe operators associated with $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$, respectively, so T_1 and T_2 are coisometries by Theorem 12 and $\Lambda_n = \theta_n T_1^*, \Gamma_n = \theta_n T_2^*$ for any $n \in N$. So for any $x, y \in H$, we have

$$\begin{aligned} \|x\|^2 + \|y\|^2 &= \|x \oplus y\|^2 \\ &= \sum_{i \in N} \|(\Lambda_i \oplus \Gamma_i)(x \oplus y)\|^2 \\ &= \sum_{i \in N} \|\Lambda_i x + \Gamma_i y\|^2 \\ &= \sum_{i \in N} \|\theta_i T_1^* x + \theta_i T_2^* y\|^2 \\ &= \sum_{i \in N} \|\theta_i (T_1^* x + T_2^* y)\|^2 \\ &= \|T_1^* x + T_2^* y\|^2 \\ &= \|x\|^2 + \|y\|^2 + 2 \operatorname{Re} \langle T_2 T_1^* x, y \rangle. \end{aligned} \quad (25)$$

So $\operatorname{Re} \langle T_2 T_1^* x, y \rangle = 0$. Replace y by iy in the above argument; then we get $\operatorname{Im} \langle T_2 T_1^* x, y \rangle = 0$. Hence $\langle T_2 T_1^* x, y \rangle = 0$ for any $x, y \in H$. Therefore $T_2 T_1^* = 0$.

$\Leftarrow$: Suppose $T_2 T_1^* = 0$; then for any $x, y \in H$, we have

$$\begin{aligned} \sum_{i \in N} \|(\Lambda_i \oplus \Gamma_i)(x \oplus y)\|^2 &= \sum_{i \in N} \|\Lambda_i x + \Gamma_i y\|^2 \\ &= \sum_{i \in N} \|\theta_i T_1^* x + \theta_i T_2^* y\|^2 \\ &= \sum_{i \in N} \|\theta_i (T_1^* x + T_2^* y)\|^2 \end{aligned}$$

$$\begin{aligned} &= \|T_1^* x + T_2^* y\|^2 \\ &= \|T_1^* x\|^2 + \|T_2^* y\|^2 \\ &\quad + 2 \operatorname{Re} \langle T_1^* x, T_2^* y \rangle \\ &= \|x\|^2 + \|y\|^2 + 2 \operatorname{Re} \langle T_2 T_1^* x, y \rangle \\ &= \|x\|^2 + \|y\|^2 = \|x \oplus y\|^2. \end{aligned} \quad (26)$$

So $\{\Lambda_j \oplus \Gamma_j : j \in N\}$ is a normalized tight g -frame for $H \oplus H$; that is, $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are strongly disjoint. $\square$

It is easy to see that for any finite number of normalized tight g -frames, we have the following result.

Corollary 22. Suppose $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are normalized tight g -frames for Hilbert space H , $T_1, \dots, T_k$ are g -preframe operators associated with $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$, respectively; then $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are strongly disjoint if and only if $T_m T_n^* = 0, m, n = 1, \dots, k$.

Theorem 23. Suppose $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are g -frames for Hilbert space H with respect to $\{H_i : i \in N\}$, T_1 and T_2 are g -preframe operators associated with $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$, respectively. If $T_2 T_1^* = 0$, then $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are disjoint.

Proof. Since T_1 and T_2 are g -preframe operators associated with g -frames $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$, respectively, so T_1 and T_2 are onto operators by Theorem 12. So there exist $T_1^\dagger$ and $T_2^\dagger$ such that $T_1 T_1^\dagger = I$ and $T_2 T_2^\dagger = I$ by Lemma 11. So $(T_1^\dagger)^* T_1^* = I$ and $(T_2^\dagger)^* T_2^* = I$. Hence $\|x\| = \|(T_1^\dagger)^* T_1^* x\| \leq \|(T_1^\dagger)^*\| \|T_1^* x\|$, so $\|T_1^* x\| \geq (1/\|(T_1^\dagger)^*\|) \|x\|$. Similarly, $\|T_2^* x\| \geq (1/\|(T_2^\dagger)^*\|) \|x\|$. Since $T_2 T_1^* = 0$, we have

$$\begin{aligned} \sum_{i \in N} \|(\Lambda_i \oplus \Gamma_i)(x \oplus y)\|^2 &= \sum_{i \in N} \|\Lambda_i x + \Gamma_i y\|^2 \\ &= \sum_{i \in N} \|\theta_i T_1^* x + \theta_i T_2^* y\|^2 \\ &= \sum_{i \in N} \|\theta_i (T_1^* x + T_2^* y)\|^2 \\ &= \|T_1^* x + T_2^* y\|^2 \\ &= \|T_1^* x\|^2 + \|T_2^* y\|^2 \\ &\quad + 2 \operatorname{Re} \langle T_1^* x, T_2^* y \rangle \\ &= \|T_1^* x\|^2 + \|T_2^* y\|^2 \\ &\quad + 2 \operatorname{Re} \langle T_2 T_1^* x, y \rangle \\ &= \|T_1^* x\|^2 + \|T_2^* y\|^2. \end{aligned} \quad (27)$$

Since

$$\begin{aligned} \|T_1^* x\|^2 + \|T_2^* y\|^2 &\leq \|T_1^*\|^2 \cdot \|x\|^2 + \|T_2^*\|^2 \cdot \|y\|^2 \\ &\leq B \cdot (\|x\|^2 + \|y\|^2) = B \cdot \|x \oplus y\|^2, \end{aligned} \quad (28)$$

where $B = \max\{\|T_1^*\|^2, \|T_2^*\|^2\}$. Hence $\sum_{i \in N} \|(\Lambda_i \oplus \Gamma_i)(x \oplus y)\|^2 \leq B \cdot \|x \oplus y\|^2$. Since

$$\begin{aligned} \|T_1^* x\|^2 + \|T_2^* y\|^2 &\geq \frac{1}{\|(T_1^\dagger)^*\|^2} \|x\|^2 + \frac{1}{\|(T_2^\dagger)^*\|^2} \|y\|^2 \\ &\geq A \cdot (\|x\|^2 + \|y\|^2) = A \cdot \|x \oplus y\|^2, \end{aligned} \quad (29)$$

where $A = \min\{1/\|(T_1^\dagger)^*\|^2, 1/\|(T_2^\dagger)^*\|^2\}$. Hence

$$\sum_{i \in N} \|(\Lambda_i \oplus \Gamma_i)(x \oplus y)\|^2 \geq A \cdot \|x \oplus y\|^2. \quad (30)$$

So $\{\Lambda_n \oplus \Gamma_n : n \in N\}$ is a g -frame for $H \oplus H$; that is, $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are disjoint. $\square$

Similarly, for any finite number of g -frames, we have the following.

Corollary 24. Suppose $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are g -frames for Hilbert space H , $T_1, \dots, T_k$ are g -preframe operators associated with $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$, respectively; If $T_m T_n^* = 0, m, n = 1, \dots, k$, then $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are disjoint.

Theorem 25. Let $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ be normalized tight g -frames for H , T_1 and T_2 are g -preframe operators associated with $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively. If $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are strongly complementary, then $T_2 T_1^* = 0$ and $T_1^* T_1 + T_2^* T_2 = I$.

Proof. If $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are strongly complementary, then $\{\Lambda_n \oplus \Gamma_n : n \in N\}$ is a g -orthonormal basis for $H \oplus H$. So they are of course strongly disjoint, therefore $T_2 T_1^* = 0$ by Theorem 21. Since $\{\theta_i : i \in N\}$ is a g -orthonormal basis for H , for any $n, k \in N$ and $g_n \in H_n, g_k \in H_k$, we have

$$\begin{aligned} &\langle (\Lambda_n \oplus \Gamma_n)^* g_n, (\Lambda_k \oplus \Gamma_k)^* g_k \rangle \\ &= \delta_{nk} \langle g_n, g_k \rangle = \langle \theta_n^* g_n, \theta_k^* g_k \rangle. \end{aligned} \quad (31)$$

But

$$\begin{aligned} &\langle (\Lambda_n \oplus \Gamma_n)^* g_n, (\Lambda_k \oplus \Gamma_k)^* g_k \rangle \\ &= \langle \Lambda_n^* g_n \oplus \Gamma_n^* g_n, \Lambda_k^* g_k \oplus \Gamma_k^* g_k \rangle \end{aligned}$$

$$\begin{aligned} &= \langle T_1 \theta_n^* g_n \oplus T_2 \theta_n^* g_n, T_1 \theta_k^* g_k \oplus T_2 \theta_k^* g_k \rangle \\ &= \langle T_1 \theta_n^* g_n, T_1 \theta_k^* g_k \rangle + \langle T_2 \theta_n^* g_n, T_2 \theta_k^* g_k \rangle \\ &= \langle T_1^* T_1 \theta_n^* g_n, \theta_k^* g_k \rangle + \langle T_2^* T_2 \theta_n^* g_n, \theta_k^* g_k \rangle \\ &= \langle (T_1^* T_1 + T_2^* T_2) \theta_n^* g_n, \theta_k^* g_k \rangle. \end{aligned} \quad (32)$$

Since $\overline{\text{span}}\{\theta_i^* g_i : i \in N\} = H$, we have that $T_1^* T_1 + T_2^* T_2 = I$. $\square$

Similarly, for any finite number of normalized tight g -frames, we have

Corollary 26. Suppose $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are normalized tight g -frames for Hilbert space H , $T_1, \dots, T_k$ are g -preframe operators associated with $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$, respectively, if $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are strongly complementary, then $T_m T_n^* = 0, m, n = 1, \dots, k$, and $\sum_{i=1}^k T_i^* T_i = I$.

The above theorems are related with the inner direct sums of g -frames or normalized tight g -frames. In the following, we will consider the usual sums of g -frames or normalized tight g -frames.

Theorem 27. Suppose that $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are g -frames for Hilbert space H , T_1 and T_2 are g -preframe operators associated with $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively. If $T_2 T_1^* = 0$, then $\{\Lambda_n + \Gamma_n : n \in N\}$ is a g -frame for H . Moreover, if $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are normalized tight g -frames and $T_2 T_1^* = 0$, then $\{\Lambda_n + \Gamma_n : n \in N\}$ is a tight g -frame for H with bounds 2.

Proof. Since T_1 and T_2 are g -preframe operators associated with g -frames $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively, so $\Lambda_n = \theta_n T_1^*$ and $\Gamma_n = \theta_n T_2^*$ for any $n \in N$, where $\{\theta_n : n \in N\}$ is a g -orthonormal basis for H . Hence $\Lambda_n + \Gamma_n = \theta_n (T_1 + T_2)^*$ for any $n \in N$. To show $\{\Lambda_n + \Gamma_n : n \in N\}$ is a g -frame, it is sufficient to show that $T_1 + T_2$ is onto by Theorem 12. Since $T_2 T_1^* = 0$, so

$$(T_1 + T_2) T_1^* = T_1 T_1^* + T_2 T_1^* = T_1 T_1^*. \quad (33)$$

Since $T_1 T_1^*$ is invertible, so for any $z \in H$, there exists $x = T_1^* (T_1 T_1^*)^{-1} z \in H$ such that

$$\begin{aligned} (T_1 + T_2) x &= (T_1 + T_2) T_1^* (T_1 T_1^*)^{-1} z \\ &= (T_1 T_1^*) (T_1 T_1^*)^{-1} z = z. \end{aligned} \quad (34)$$

So $T_1 + T_2$ is onto.

If $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are normalized tight g -frames and $T_2 T_1^* = 0$, then $\{\Lambda_n + \Gamma_n : n \in N\}$ is a g -frame and for any $x \in H$, we have

$$\begin{aligned} &\sum_{n \in N} \|(\Lambda_n + \Gamma_n) x\|^2 \\ &= \sum_{n \in N} \|\Lambda_n x\|^2 + \sum_{n \in N} \|\Gamma_n x\|^2 \\ &\quad + \sum_{n \in N} \langle \Lambda_n x, \Gamma_n x \rangle + \sum_{n \in N} \langle \Gamma_n x, \Lambda_n x \rangle \end{aligned}$$

$$\begin{aligned}
&= 2\|x\|^2 + \sum_{n \in N} \langle \theta_n T_1^* x, \theta_n T_2^* x \rangle \\
&\quad + \sum_{n \in N} \langle \theta_n T_2^* x, \theta_n T_1^* x \rangle \\
&= 2\|x\|^2 + \sum_{n \in N} \langle T_2 \theta_n^* \theta_n T_1^* x, x \rangle \\
&\quad + \sum_{n \in N} \langle T_1 \theta_n^* \theta_n T_2^* x, x \rangle \\
&= 2\|x\|^2 + \left\langle T_2 \sum_{n \in N} \theta_n^* \theta_n T_1^* x, x \right\rangle \\
&\quad + \left\langle T_1 \sum_{n \in N} \theta_n^* \theta_n T_2^* x, x \right\rangle \\
&= 2\|x\|^2 + \langle T_2 T_1^* x, x \rangle + \langle T_1 T_2^* x, x \rangle \\
&= 2\|x\|^2.
\end{aligned} \tag{35}$$

So $\{\Lambda_n + \Gamma_n : n \in N\}$ is a tight g -frame with bound 2. $\square$

Similarly, for any finite number of g -frames, we have the following.

Corollary 28. Suppose $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are g -frames for Hilbert space H , $T_1, \dots, T_k$ are g -preframe operators associated with $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$, respectively. If $T_m T_n^* = 0, m, n = 1, \dots, k$, then $\{\Lambda_{1j} + \dots + \Lambda_{kj} : j \in N\}$ is a g -frame for H . Moreover, if $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are normalized tight g -frames for Hilbert space H and $T_m T_n^* = 0, m, n = 1, \dots, k$, then $\{\Lambda_{1j} + \dots + \Lambda_{kj} : j \in N\}$ is a tight g -frame for H with bound k .

The above theorem can be generalized further as follows.

Theorem 29. Suppose that $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are g -frames for Hilbert space H , T_1 and T_2 are g -preframe operators associated with $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively, such that $T_2 T_1^* = 0$ and $A, B \in B(H)$. If either A or B is onto, then $\{\Lambda_n A^* + \Gamma_n B^* : n \in N\}$ is a g -frame for H .

Proof. Since T_1 and T_2 are g -preframe operators associated with $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively, $\Lambda_n = \theta_n T_1^*$ and $\Gamma_n = \theta_n T_2^*$. So

$$\begin{aligned}
\Lambda_n A^* + \Gamma_n B^* &= \theta_n T_1^* A^* + \theta_n T_2^* B^* \\
&= \theta_n (T_1^* A^* + T_2^* B^*) = \theta_n (AT_1 + BT_2)^*.
\end{aligned} \tag{36}$$

To show $\{\Lambda_n A^* + \Gamma_n B^* : n \in N\}$ is a g -frame it is sufficient to show that $AT_1 + BT_2$ is onto by Theorem 12. It is well known that $T_1 T_1^*$ is invertible. If A is onto, then for any $z \in H$, there

exists $x \in H$ such that $Ax = z$. So for any $z \in H$, there exists $f = T_1^* (T_1 T_1^*)^{-1} x \in H$ such that

$$\begin{aligned}
(AT_1 + BT_2)f &= (AT_1 + BT_2)T_1^* (T_1 T_1^*)^{-1} x \\
&= AT_1 T_1^* (T_1 T_1^*)^{-1} x = Ax = z.
\end{aligned} \tag{37}$$

It follows that $AT_1 + BT_2$ is onto. In the case that B is onto, the result can be proved similarly. $\square$

Similarly, for any finite number of g -frames, we have the following.

Corollary 30. Suppose $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$ are g -frames for Hilbert space H , and $T_1, \dots, T_k$ are g -preframe operators associated with $\{\Lambda_{1j} : j \in N\}, \dots, \{\Lambda_{kj} : j \in N\}$, respectively, and $T_n T_m^* = 0, m, n = 1, 2, \dots, k, A_i \in B(H), i = 1, \dots, k$. If there exists $i \in \{1, 2, \dots, k\}$ such that A_i is surjective, then $\{\sum_{i=1}^k \Lambda_{ij} A_i^*\}$ is a g -frame for H .

4. Dual g -Frames of g -Frames

In this section, we study the dual g -frames by using the properties of g -preframe operators.

Theorem 31. Suppose g -frames $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are alternate dual g -frame pair for Hilbert space H , T is a coisometry in $B(H)$. Then $\{\Lambda_n T^*\}$ and $\{\Gamma_n T^* : n \in N\}$ are alternate dual g -frame pair for H .

Proof. Since $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are g -frames for H and T is a coisometry, so $\{\Lambda_n T^*\}$ and $\{\Gamma_n T^* : n \in N\}$ are also g -frames for H by Theorem 14. Since $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are alternate dual g -frame pair, we have, for any $x \in H$

$$x = \sum_{n \in N} \Lambda_n^* \Gamma_n x. \tag{38}$$

So

$$Tx = \sum_{n \in N} T \Lambda_n^* \Gamma_n x = \sum_{n \in N} (\Lambda_n T^*)^* \Gamma_n x. \tag{39}$$

Replacing x by $T^* x$ and applying $TT^* = I$, we have

$$x = TT^* x = \sum_{n \in N} (\Lambda_n T^*)^* \Gamma_n T^* x. \tag{40}$$

So $\{\Lambda_n T^*\}$ and $\{\Gamma_n T^* : n \in N\}$ are alternate dual g -frame pair for H . $\square$

More general, we have the following result.

Theorem 32. Suppose $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are alternate dual g -frame pair for Hilbert space H , T is a surjective operator in $B(H)$. Then $\{\Lambda_n T^* : n \in N\}$ and $\{\Gamma_n T^\dagger : n \in N\}$ are alternate dual g -frame pair for H . Similarly, $\{\Lambda_n T^\dagger\}$ and $\{\Gamma_n T^* : n \in N\}$ are alternate dual g -frame pair for H .

Proof. Since T is onto, $TT^\dagger = I$ by Lemma 11, so $(T^\dagger)^*T^* = I$. It follows that $(T^\dagger)^*$ is onto. Since $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are g -frames for H , so $\{\Lambda_n T^* : n \in N\}$, $\{\Gamma_n T^* : n \in N\}$, $\{\Lambda_n T^\dagger : n \in N\}$ and $\{\Gamma_n T^\dagger : n \in N\}$ are g -frames for H by Theorem 14. Since $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are alternate dual g -frame pair, we have, for any $x \in H$

$$x = \sum_{n \in N} \Lambda_n^* \Gamma_n x. \quad (41)$$

So

$$Tx = \sum_{n \in N} T \Lambda_n^* \Gamma_n x = \sum_{n \in N} (\Lambda_n T^*)^* \Gamma_n x. \quad (42)$$

Replaced x by $T^\dagger x$ and applying $TT^\dagger = I$, we have

$$x = TT^\dagger x = \sum_{n \in N} (\Lambda_n T^*)^* \Gamma_n T^\dagger x. \quad (43)$$

So $\{\Lambda_n T^*\}$ and $\{\Gamma_n T^\dagger : n \in N\}$ are alternate dual g -frame pair for H . Interchanging Λ_n and Γ_n for any $n \in N$, we get that $\{\Lambda_n T^\dagger\}$ and $\{\Gamma_n T^* : n \in N\}$ are alternate dual g -frame pair for H . $\square$

Theorem 33. Let $\{\Lambda_n : n \in N\}$ be a g -frame for H . Let $\{\tilde{\Lambda}_n : n \in N\}$ denote the canonical dual g -frame of $\{\Lambda_n : n \in N\}$ and T_x denote the g -preframe operator associated with $\{\Lambda_n : n \in N\}$. If P is an orthogonal projection with $PT_x = T_x P$, then $\{\Lambda_n P : n \in N\}$ is a g -frame for PH and the canonical dual g -frame for $\{\Lambda_n P : n \in N\}$ is $\{\tilde{\Lambda}_n P : n \in N\}$.

Proof. It is easy to see that $\{\Lambda_n P : n \in N\}$ is a g -frame for PH and $\{\theta_n P : n \in N\}$ is a g -orthonormal basis for PH with respect to $\{H_i : i \in N\}$. We only need to prove the second part. By definition, the canonical dual g -frame for $\{\Lambda_n : n \in N\}$ is $\{\Lambda_n S^{-1} : n \in N\}$, where $S = T_x T_x^*$. So $\tilde{\Lambda}_n = \Lambda_n S^{-1}$. Since $PT_x = T_x P$, so $PT_x^* = T_x^* P$. Since $\Lambda_n = \theta_n T_x^*$, so $\Lambda_n P = \theta_n T_x^* P = \theta_n P T_x^*$. Hence the g -preframe operator associated with $\{\Lambda_n P : n \in N\}$ is T_x as well. So the canonical dual g -frame for $\{\Lambda_n P\}$ is $\{\Lambda_n P (T_x T_x^*)^{-1} = \{\Lambda_n (T_x T_x^*)^{-1} P\} = \{\tilde{\Lambda}_n P : n \in N\}$. $\square$

The following theorem characterizes the alternate dual g -frames in terms of g -preframe operators.

Theorem 34. Let $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ be g -frames for Hilbert space H , and let T_x and T_y be the g -preframe operators associated with $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively. Then $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are alternate dual g -frame pair if and only if $T_y T_x^* = I$ or $T_x T_y^* = I$.

Proof. Since T_x and T_y are g -preframe operators associated with $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$, respectively, so $\theta_n T_x^* = \Lambda_n$ and $\theta_n T_y^* = \Gamma_n$ for any $n \in N$. Hence for any $x \in H$,

$$\sum_{n \in N} \Lambda_n^* \Gamma_n x = \sum_{n \in N} T_x \theta_n^* \theta_n T_y^* x = T_x T_y^* x. \quad (44)$$

Similarly, $\sum_{n \in N} \Gamma_n^* \Lambda_n x = T_y T_x^* x$. It follows that $\{\Lambda_n : n \in N\}$ and $\{\Gamma_n : n \in N\}$ are alternate dual g -frame pair if and only if $T_y T_x^* = I$ or $T_x T_y^* = I$. $\square$

Remark 35. Throughout this paper we assume that there exists g -orthonormal basis for Hilbert space H . It is well known that any Hilbert space has orthonormal basis. A natural question is that given Hilbert space H , K and sequence of Hilbert space $\{H_i : i \in J\} \subset K$, Do the g -orthonormal bases for H with respect to $\{H_i : i \in J\}$ always exist? if not, what is the condition such that the g -orthonormal bases exist? These interesting questions will be our next research subjects.

Acknowledgment

This work was partially supported by SWUFE's Key Subjects Construction Items Funds of 211 Project.

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