Research Article

On Linear and Nonlinear Fourth-Order Eigenvalue Problems with Nonlocal Boundary Condition

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We determine the principal eigenvalue of the linear problem $u^{(4)}(t) + \beta u''(t) = \mu[u(t) - u''(t)]$, $t \in (0, 1)$, $u(0) = u(1) = \int_0^1 p(s)u(s)ds$, $u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds$, where $0 < \beta < \pi^2$ and $p, q \in L[0, 1]$. Moreover, we investigate the existence of positive solutions for the corresponding nonlinear problem. The proofs of our main results are based upon the Krein-Rutman theorem and fixed point index theory.

1. Introduction

The deformations of an elastic beam can be described by the boundary value problems of the fourth-order ordinary differential equations. For example, an elastic beam in an equilibrium state whose both ends are simply supported can be described by the fourth-order boundary value problem of the form

$$\begin{aligned} u^{(4)}(t) &= f(t, u(t), u''(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \end{aligned} \quad (1)$$

see Gupta [1, 2]. Owing to its significance in physics, it has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point index theory, and the method of lower and upper solutions; see, for example, [3–16].

Recently, Bai [3] investigated the existence of positive solutions for more general fourth-order nonlocal boundary value problem

$$\begin{aligned} u^{(4)}(t) + \beta u''(t) &= \lambda f(t, u(t), u''(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) &= u''(1) = \int_0^1 q(s)u''(s)ds. \end{aligned} \quad (2)$$

By using the Krasnoselskii fixed point theorem, the sufficient conditions for the existence of positive solutions of (2) are obtained. We can find that, however, those conditions do not involve the eigenvalues with respect to the relevant linear operator, and those sufficient conditions are not optimal.

The likely reason is that the spectrum structure of the linear eigenvalue problem

$$\begin{aligned} u^{(4)}(t) + \beta u''(t) &= \mu[u(t) - u''(t)], \quad t \in (0, 1), \\ u(0) &= u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) &= u''(1) = \int_0^1 q(s)u''(s)ds \end{aligned} \quad (3)$$

is not clear.

It is the purpose of this paper to investigate the first eigenvalue of (3) by using the Krein-Rutman theorem [17, 18], and then we use this spectrum result to establish the existence of positive solutions of nonlinear nonlocal problem (2). The existence of positive solution is obtained by means of fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear operator. The obtained sufficient conditions in this paper are optimal. For the concepts and properties of fixed point index theory, we refer the reader to [19].

The rest of the paper is arranged as follows: in Section 2, we study the spectrum of (3). Finally, in Section 3, we give an application of our main result.

2. Spectrum of (3)

Let us assume that

$$(A1) \quad 0 < \beta < \pi^2, \quad p, q \in L[0, 1], \quad p(t) \geq 0, \quad q(t) \geq 0, \\ t \in [0, 1], \quad \int_0^1 p(s) ds < 1, \quad \int_0^1 q(s) \sin \sqrt{\beta} s ds + \int_0^1 q(s) \sin \sqrt{\beta}(1-s) ds < \sin \sqrt{\beta}.$$

$$(A2) \quad f \in C([0, 1] \times [0, \infty) \times (-\infty, 0], [0, \infty)).$$

Definition 1. We say μ is an *eigenvalue* of linear problem (see (3)), if (3) has nontrivial solutions.

To study the spectrum of (3), we need several preliminary results.

Let

$$G_1(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1; \\ s(1-t), & 0 \leq s \leq t \leq 1; \end{cases} \\ G_2(t, s) = \begin{cases} \frac{\sin \sqrt{\beta} t \sin \sqrt{\beta}(1-s)}{\sqrt{\beta} \sin \sqrt{\beta}}, & 0 \leq t \leq s \leq 1; \\ \frac{\sin \sqrt{\beta} s \sin \sqrt{\beta}(1-t)}{\sqrt{\beta} \sin \sqrt{\beta}}, & 0 \leq s \leq t \leq 1. \end{cases} \quad (4)$$

Lemma 2 (see [7]). Assume (A1) holds. Then one has

$$(i) \quad G_i(t, s) \geq 0, \text{ for } t, s \in [0, 1]; \quad G_i(t, s) > 0, \text{ for } t, s \in (0, 1);$$

$$(ii) \quad b_i G_i(t, t) G_i(s, s) \leq G_i(t, s) \leq C_i G_i(t, t), \text{ for } t, s \in [0, 1],$$

where $C_1 = 1, b_1 = 1; C_2 = 1/\sin \sqrt{\beta}, b_2 = \sqrt{\beta} \sin \sqrt{\beta}$.

Lemma 3 (see [3]). Assume (A1) holds. Then for any $g \in C[0, 1]$, u solves the problem

$$u^{(4)}(t) + \beta u''(t) = g(t), \quad t \in (0, 1), \\ u(0) = u(1) = \int_0^1 p(s) u(s) ds, \\ u''(0) = u''(1) = \int_0^1 q(s) u''(s) ds, \quad (5)$$

if and only if

$$u(t) = \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) g(\tau) d\tau ds, \quad (6)$$

where

$$K_1(t, s) = G_1(t, s) + \frac{1}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(s, x) p(x) dx,$$

$$K_2(s, \tau) = G_2(s, \tau) + \rho(s) \int_0^1 G_2(\tau, x) q(x) dx,$$

$$\rho(s) =$$

$$= \frac{\sin \sqrt{\beta} s + \sin \sqrt{\beta}(1-s)}{\sin \sqrt{\beta} - \int_0^1 q(x) \sin \sqrt{\beta} x dx - \int_0^1 q(x) \sin \sqrt{\beta}(1-x) dx}. \quad (7)$$

Let

$$e(t) := -t^2 + t + \frac{\int_0^1 p(s)(s-s^2) ds}{1 - \int_0^1 p(s) ds}, \quad t \in [0, 1]. \quad (8)$$

Then, there exist $c, \tilde{c} > 0$ such that

$$\int_0^1 K_1(t, s) e(s) ds \leq ce(t), \quad t \in [0, 1], \quad (9)$$

$$\int_0^1 K_2(t, s) e(s) ds \leq \tilde{c}e(t), \quad t \in [0, 1], \quad (10)$$

respectively.

In fact, from (ii) of Lemma 2, we can obtain

$$\int_0^1 K_1(t, s) e(s) ds \\ = \int_0^1 \left[G_1(t, s) + \frac{1}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(s, x) p(x) dx \right] e(s) ds$$

$$\begin{aligned}
& \leq \int_0^1 \left[G_1(t, t) + \frac{1}{1 - \int_0^1 p(x) dx} \right. \\
& \quad \left. \times \int_0^1 G_1(s, s) p(x) dx \right] e(s) ds \\
& = G_1(t, t) \int_0^1 e(s) ds + \frac{\int_0^1 p(x) dx}{1 - \int_0^1 p(x) dx} \\
& \quad \times \int_0^1 G_1(s, s) e(s) ds \\
& = (-t^2 + t) \int_0^1 e(s) ds + \frac{\int_0^1 p(x) dx}{1 - \int_0^1 p(x) dx} \\
& \quad \times \int_0^1 (s - s^2) e(s) ds \\
& \leq ce(t), \quad t \in [0, 1],
\end{aligned} \tag{11}$$

where we can take

$$\begin{aligned}
c = \max \left\{ \int_0^1 e(s) ds, \frac{\int_0^1 p(s) ds}{\int_0^1 (s - s^2) p(s) ds} \right. \\
\left. \times \int_0^1 (s - s^2) e(s) ds \right\},
\end{aligned} \tag{12}$$

and also we obtain

$$\begin{aligned}
& \int_0^1 K_2(t, s) e(s) ds \\
& = \int_0^1 \left[G_2(t, s) + \rho(t) \int_0^1 G_2(s, x) q(x) dx \right] e(s) ds \\
& \leq \frac{1}{\sin \sqrt{\beta}} \int_0^1 \left[G_2(t, t) + \rho(t) \int_0^1 G_2(x, x) q(x) dx \right] \\
& \quad \times e(s) ds \\
& \leq c_1 \sin \sqrt{\beta} t \sin \sqrt{\beta} (1 - t) + c_2 \\
& \leq \tilde{c} e(t),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
c_1 &= \frac{1}{\sqrt{\beta} \sin^2 \sqrt{\beta}} \int_0^1 e(s) ds, \\
c_2 &= \frac{1}{\sin \sqrt{\beta}} \int_0^1 e(s) ds \int_0^1 G_2(s, s) q(s) ds \max_{0 \leq t \leq 1} \rho(t), \\
\tilde{c} &= \frac{1 - \int_0^1 p(s) ds}{\int_0^1 (s - s^2) p(s) ds} (c_1 + c_2).
\end{aligned} \tag{14}$$

Let

$$X = \left\{ u \in C^2[0, 1] : u(0) = u(1) = \int_0^1 p(s) u(s) ds, \right. \\
\left. u''(0) = u''(1) = \int_0^1 q(s) u''(s) ds, \right. \\
\left. -\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t) \right\}, \tag{15}$$

for a certain $\varepsilon > 0$, and let Y be the Banach space $C[0, 1]$ with the norm $\|u\|_\infty = \max_{0 \leq t \leq 1} |u(t)|$. For $u \in X$, we have

$$u(t) = \int_0^1 K_1(t, s) (-u''(s)) ds. \tag{16}$$

Combining (16) and (9) with the fact that $-\varepsilon e(s) \leq -u''(s) \leq \varepsilon e(s)$, we conclude that

$$-\varepsilon e(t) \leq u(t) \leq \varepsilon e(t), \quad t \in [0, 1]. \tag{17}$$

So, we may define the norm of $u \in X$ by

$$\|u\|_X := \inf \left\{ \varepsilon \mid \varepsilon > 0, -\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t), t \in [0, 1] \right\}. \tag{18}$$

This norm is so called u_0 -norm.

It is easy to get the following lemmas.

Lemma 4. $(X, \|\cdot\|_X)$ is a Banach space.

Proof. Let $\{u_n\}$ be Cauchy sequence of $(X, \|\cdot\|_X)$. Then, we have

$$\|u_n - u_m\|_X \rightarrow 0, \quad (n, m \rightarrow \infty). \tag{19}$$

From (18) and (19), for all $\varepsilon > 0$, there exists $N > 0$ such that

$$-\varepsilon e(t) \leq -(u_n''(t) - u_m''(t)) \leq \varepsilon e(t), \tag{20}$$

$$t \in [0, 1], \quad \forall n, m > N.$$

Hence,

$$\|u_n'' - u_m''\|_\infty \leq \varepsilon \|e\|_\infty \rightarrow 0, \quad (n, m \rightarrow \infty), \tag{21}$$

which implies that $\{u_n''\}$ is a Cauchy sequence of $(C[0, 1], \|\cdot\|_\infty)$. According to the completeness of $(C[0, 1], \|\cdot\|_\infty)$, there exists $u_0 \in C^2[0, 1]$ such that

$$\|u_n'' - u_0''\|_\infty \rightarrow 0, \quad (n \rightarrow \infty). \tag{22}$$

Let $m \rightarrow \infty$ in (20). Then, we have

$$-\varepsilon e(t) \leq -(u_n''(t) - u_0''(t)) \leq \varepsilon e(t), \tag{23}$$

$$t \in [0, 1], \quad (n > N),$$

which implies that

$$\|u_n - u_0\|_X \leq \varepsilon, \quad (n > N). \tag{24}$$

Hence, for Cauchy sequence $\{u_n\} \subset (X, \|\cdot\|_X)$, there exists $u_0 \in X$ satisfying

$$\|u_n - u_0\|_X \rightarrow 0, \quad (n \rightarrow \infty). \tag{25}$$

Therefore, $(X, \|\cdot\|_X)$ is a Banach space. $\square$

Lemma 5. $X \cap C^4[0, 1]$ is compactly embedded in X .

Proof. Let $\{u_{n_j}\} \subset X \cap C^4[0, 1]$ be bounded. Then, according to the fact that $C^4[0, 1]$ is compactly embedded in $C^2[0, 1]$, there exist $\{u_{n_j}\} \subset \{u_n\}$ and $u_0 \in C^2[0, 1]$ such that

$$\|u_{n_j}'' - u_0''\|_\infty \rightarrow 0, \quad j \rightarrow \infty. \quad (26)$$

From (26), for all $\varepsilon > 0$, there exists $N > 0$ such that

$$-\varepsilon \leq -(u_{n_j}''(t) - u_0''(t)) \leq \varepsilon, \quad t \in [0, 1], \quad n_j > N. \quad (27)$$

Since

$$e(t) \geq \frac{\int_0^1 (s-s^2) p(s) ds}{1 - \int_0^1 p(s) ds}, \quad t \in [0, 1], \quad (28)$$

combining this fact with (27) we have

$$\begin{aligned} & -\frac{1 - \int_0^1 p(s) ds}{\int_0^1 (s-s^2) p(s) ds} \varepsilon e(t) \\ & \leq -(u_{n_j}''(t) - u_0''(t)) \\ & \leq \frac{1 - \int_0^1 p(s) ds}{\int_0^1 (s-s^2) p(s) ds} \varepsilon e(t), \quad t \in [0, 1], \quad n_j > N, \end{aligned} \quad (29)$$

which implies

$$\|u_{n_j} - u_0\|_X \leq \frac{1 - \int_0^1 p(s) ds}{\int_0^1 (s-s^2) p(s) ds} \varepsilon \rightarrow 0, \quad n_j \rightarrow \infty. \quad (30)$$

Therefore, $X \cap C^4[0, 1]$ is compactly embedded in X . $\square$

Let

$$P := \{u \in X \mid u(t) \geq 0, \quad u''(t) \leq 0, \quad t \in [0, 1]\}. \quad (31)$$

Then, the cone P is normal and has nonempty interior $\text{int } P$.

In the rest of this section, we will prove the existence of the first eigenvalue of (3). To wit, we get the following.

Theorem 6. Assume (A1) holds. Then (3) has an algebraically simple eigenvalue μ_1 , with an eigenfunction $\varphi_1 \in \text{int } P$, and there is no other eigenvalue with a positive eigenfunction.

Remark 7. If $p = q \equiv 0$, then μ_1 can be explicitly given by

$$\mu_1 = \frac{\pi^4 - \beta\pi^2}{1 + \pi^2} \quad (32)$$

and the corresponding eigenfunction $\varphi_1(t) = \sin \pi t$, $t \in [0, 1]$.

Proof of Theorem 6. For $u \in X$, define a linear operator $T : X \rightarrow Y$ by

$$(Tu)(t) := \int_0^1 K_1(t, s) K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau ds. \quad (33)$$

Then, by the definition of K_1 , we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 \left\{ \left[G_1(t, s) + \frac{1}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(s, x) p(x) dx \right] \right. \\ &\quad \times \left. \int_0^1 K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau \right\} ds \\ &= \int_0^t \left\{ \left[s(1-t) + \frac{1}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(s, x) p(x) dx \right] \right. \\ &\quad \times \left. \int_0^1 K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau \right\} ds \\ &\quad + \int_t^1 \left\{ \left[t(1-s) + \frac{1}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(s, x) p(x) dx \right] \right. \\ &\quad \times \left. \int_0^1 K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau \right\} ds. \end{aligned} \quad (34)$$

Therefore, we can obtain

$$\begin{aligned} (Tu)'(t) &= \int_0^t \left\{ -s \int_0^1 K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau \right\} ds \\ &\quad + \int_t^1 \left\{ (1-s) \int_0^1 K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau \right\} ds, \end{aligned} \quad (35)$$

$$\begin{aligned} (Tu)''(t) &= -t \int_0^1 K_2(t, \tau) [u(\tau) - u''(\tau)] d\tau \\ &\quad + (t-1) \int_0^1 K_2(t, \tau) [u(\tau) - u''(\tau)] d\tau \\ &= - \int_0^1 K_2(t, \tau) [u(\tau) - u''(\tau)] d\tau. \end{aligned} \quad (36)$$

We claim that $T : P \rightarrow P$. In fact, for $u \in X$, let $\|u\|_X = \alpha$. Then, from (18) and (17) we have, for $t \in [0, 1]$,

$$\begin{aligned} -\alpha e(t) &\leq -u''(t) \leq \alpha e(t), \\ -\alpha e(t) &\leq u(t) \leq \alpha e(t). \end{aligned} \quad (37)$$

On the one hand, from (36)-(37) and (10) we have

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 K_2(t, s) [u(s) - u''(s)] ds \\ &\leq \int_0^1 K_2(t, s) [c\alpha + \alpha] e(s) ds \\ &\leq \bar{c}(c\alpha + \alpha) e(t). \end{aligned} \quad (38)$$

On the other hand, from (36)-(37) and (10) we have

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 K_2(t, s) [u(s) - u''(s)] ds \\ &\geq - \int_0^1 K_2(t, s) [c\alpha + \alpha] e(s) ds \\ &\geq -\bar{c}(c\alpha + \alpha) e(t). \end{aligned} \quad (39)$$

Therefore, $T(X) \subseteq X$. If $u \in P$, then $u(s) \geq 0$, $u''(s) \leq 0$ on $[0, 1]$. According to (36) and the fact that

$$K_2(t, s) \geq 0, \quad 0 \leq t, s \leq 1, \quad (40)$$

we have

$$(Tu)''(t) = - \int_0^1 K_2(t, s) [u(s) - u''(s)] ds \leq 0. \quad (41)$$

Furthermore, according to (41), the definition of T , and the fact that

$$K_1(t, s) \geq 0, \quad 0 \leq t, s \leq 1, \quad (42)$$

we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 \left\{ K_1(t, s) \int_0^1 K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau \right\} ds \\ &= \int_0^1 K_1(t, s) [-(Tu)''(s)] ds \geq 0. \end{aligned} \quad (43)$$

Then $u \in P$, and accordingly, $T(P) \subseteq P$.

Now, since $T(X) \subset X \cap C^4[0, 1]$ and $X \cap C^4[0, 1]$ is compactly embedded in X , we have that $T : X \rightarrow X$ is compact.

Next, we show that $T : P \rightarrow P$ is strongly positive.

For $u \in P \setminus \{\theta\}$, it is easy to check that there exist $\rho_{1u}, \rho_{2u} > 0$, such that for $t \in [0, 1]$

$$\begin{aligned} (Tu)(t) &\geq \rho_{1u} e(t), \\ -(Tu)''(t) &\geq \rho_{2u} e(t). \end{aligned} \quad (44)$$

In fact, for every $u \in P \setminus \{\theta\}$, it follows from Lemma 2 that

$$\begin{aligned} (Tu)(t) &= \iint_0^1 \left[G_1(t, s) + \frac{1}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(s, x) p(x) dx \right] \\ &\quad \times \left[G_2(s, \tau) + \rho(s) \int_0^1 G_2(\tau, x) q(x) dx \right] \\ &\quad \times [u(\tau) - u''(\tau)] d\tau ds \\ &\geq \iint_0^1 \left[G_1(t, s) G_1(s, s) \right. \\ &\quad \left. + \frac{G_1(s, s)}{1 - \int_0^1 p(x) dx} \int_0^1 G_1(x, x) p(x) dx \right] \\ &\quad \times \left[G_2(s, \tau) + \rho(s) \int_0^1 G_2(\tau, x) q(x) dx \right] \\ &\quad \times [u(\tau) - u''(\tau)] d\tau ds \\ &= \rho_{1u} e(t), \end{aligned} \quad (45)$$

where

$$\begin{aligned} \rho_{1u} &= \iint_0^1 G_1(s, s) \\ &\quad \times \left[G_2(s, \tau) + \rho(s) \int_0^1 G_2(\tau, x) q(x) dx \right] \\ &\quad \times [u(\tau) - u''(\tau)] d\tau ds > 0, \end{aligned} \quad (46)$$

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 \left[G_2(t, s) + \rho(t) \int_0^1 G_2(s, x) q(x) dx \right] \\ &\quad \times [u(s) - u''(s)] ds \\ &\geq \sqrt{\beta} \sin \sqrt{\beta} \int_0^1 \left[G_2(t, t) G_2(s, s) \right. \\ &\quad \left. + \rho(t) G_2(s, s) \int_0^1 G_2(x, x) q(x) dx \right] \\ &\quad \times [u(s) - u''(s)] ds \\ &= \sqrt{\beta} \sin \sqrt{\beta} \left[G_2(t, t) + \rho(t) \int_0^1 G_2(x, x) q(x) dx \right] \\ &\quad \times \left[\int_0^1 G_2(s, s) [u(s) - u''(s)] ds \right] \end{aligned}$$

$$\begin{aligned}
&\geq \rho(t) \left[\int_0^1 \sin \sqrt{\beta} x \sin \sqrt{\beta} (1-x) q(x) dx \right] \\
&\quad \times \left[\int_0^1 G_2(s, s) [u(s) - u''(s)] ds \right] \\
&\geq \rho_{2u} e(t),
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
\rho_{2u} &= \min_{t \in [0,1]} \rho(t) \left[\int_0^1 \sin \sqrt{\beta} x \sin \sqrt{\beta} (1-x) q(x) dx \right. \\
&\quad \times \left. \int_0^1 G_2(s, s) [u(s) - u''(s)] ds \right] \\
&\quad \times \left(\frac{1}{4} + \frac{\int_0^1 p(s) (s-s^2) ds}{1 - \int_0^1 p(s) ds} \right)^{-1} \\
&> 0.
\end{aligned} \tag{48}$$

Therefore, it follows from (44) that $Tu \in \text{int } P$.

Now, on the one hand, by the Krein-Rutman theorem ([17, Theorem 7.C] and [18, Theorem 19.3]), T has an algebraically simple eigenvalue $r(T) > 0$ with an eigenfunction $\varphi_1 \in \text{int } P$. Moreover, there is no other eigenvalue with a positive eigenfunction. On the other hand, we have from the definition of T and Lemma 3 that (3) is equivalent to the integral equation

$$u(t) = \mu(Tu)(t). \tag{49}$$

Therefore, $\mu_1 = (r(T))^{-1}$ with a positive eigenfunction φ_1 is a simple eigenvalue of (3). Moreover, for (3), there is no other eigenvalue with a positive eigenfunction. $\square$

3. An Application of Theorem 6

For convenience, we introduce the following notations:

$$\begin{aligned}
\overline{f}^\infty &= \limsup_{|u|+|v| \rightarrow +\infty} \max_{t \in [0,1]} \frac{f(t, u, v)}{|u| + |v|}, \\
\underline{f}_0 &= \liminf_{|u|+|v| \rightarrow 0^+} \min_{t \in [0,1]} \frac{f(t, u, v)}{|u| + |v|}, \\
\overline{f}^0 &= \limsup_{|u|+|v| \rightarrow 0^+} \max_{t \in [0,1]} \frac{f(t, u, v)}{|u| + |v|},
\end{aligned} \tag{50}$$

$$\underline{f}_{-\infty} = \liminf_{|u|+|v| \rightarrow +\infty} \min_{t \in [0,1]} \frac{f(t, u, v)}{|u| + |v|},$$

$$B_r = \{u \in X \mid \|u\|_X < r\},$$

$$\partial B_r = \{u \in X \mid \|u\|_X = r\} \quad (r > 0).$$

Theorem 8. Assume that (A1) and (A2) hold, and $0 \leq \overline{f}^\infty < \underline{f}_0 \leq +\infty$. Then, for each λ satisfying

$$\frac{\mu_1}{\underline{f}_0} < \lambda < \frac{\mu_1}{\overline{f}^\infty}, \tag{51}$$

there exists at least one positive solution of (2).

Theorem 9. Assume that (A1) and (A2) hold, and $0 \leq \overline{f}^0 < \underline{f}_{-\infty} \leq +\infty$. Then, for each λ satisfying

$$\frac{\mu_1}{\underline{f}_{-\infty}} < \lambda < \frac{\mu_1}{\overline{f}^0}, \tag{52}$$

there exists at least one positive solution of (2).

Remark 10. Bai [3] proved existence of positive solutions via Guo-Krasnoselskii fixed point theorem under some conditions which do not involve the eigenvalue of (3). While our Theorems 8 and 9 are established under (51) or (52) which are related to the eigenvalue of (3). Our Theorems 8 and 9 cover an undefined case in [3]. Consider the following boundary value problem:

$$\begin{aligned}
&u^{(4)}(t) + \frac{\pi^2}{4} u''(t) \\
&= \frac{3\pi^4 + 1}{4(1 + \pi^2)} [u(t) - u''(t)] \\
&\quad - \frac{2}{4(1 + \pi^2)} \sin[u(t) - u''(t)], \quad t \in (0, 1), \\
&u(0) = u(1) = u''(0) = u''(1) = 0.
\end{aligned} \tag{53}$$

In this case, $\mu_1 = 3\pi^4/4(1 + \pi^2)$, $f(t, u, v) = ((3\pi^4 + 1)/4(1 + \pi^2))(u - v) - (2/4(1 + \pi^2)) \sin(u - v)$, $\lambda = 1$, and

$$\frac{\mu_1}{\underline{f}_{-\infty}} = \frac{3\pi^4}{3\pi^4 + 1} < 1 = \lambda < \frac{3\pi^4}{3\pi^4 - 1} = \frac{\mu_1}{\overline{f}^0}. \tag{54}$$

According to Theorem 9, the above boundary value problem has at least one positive solution. For the above boundary value problem, however, we cannot obtain the above conclusion by [3, Theorem 3.1] since

$$\underline{f}_{-\infty} = \frac{3\pi^4 + 1}{4(1 + \pi^2)} < 20 < 36 < \frac{\pi^3 \left(\sqrt{4 + 2\sqrt{2}} + 2 \right)}{2\sqrt{2}} = \frac{\eta_1}{\lambda}. \tag{55}$$

Moreover, (51) and (52) are optimal. In order to illustrate this point, consider the problem

$$\begin{aligned}
&u^{(4)}(t) + \beta u''(t) = \frac{\pi^4 - \beta\pi^2}{1 + \pi^2} [u(t) - u''(t)] \\
&\quad + \rho(u(t), u''(t)), \quad t \in (0, 1), \\
&u(0) = u(1) = u''(0) = u''(1) = 0,
\end{aligned} \tag{56}$$

where

$$\rho(x, y) = \begin{cases} (x - y)^2, & 0 \leq x - y \leq 1, \\ \sqrt{x - y}, & 1 \leq x - y < \infty. \end{cases} \quad (57)$$

In this case, $\mu_1 = (\pi^4 - \beta\pi^2)/(1 + \pi^2)$, $f(t, u, v) = ((\pi^4 - \beta\pi^2)/(1 + \pi^2))[u - v] + \rho(u, v)$, $\lambda = 1$, and

$$\frac{\mu_1}{f_0} = \lambda = \frac{\mu_1}{f_\infty}, \quad \frac{\mu_1}{f_0} = \lambda = \frac{\mu_1}{f_\infty}. \quad (58)$$

However, (56) has no positive solution. In fact, suppose on the contrary that (56) has a positive solution u . Multiplying the first equation of (56) with $\sin \pi t$ and integrating from 0 to 1, we get

$$0 = \int_0^1 \rho(u(t), u''(t)) \sin \pi t \, dt > 0, \quad (59)$$

which is a contradiction.

To prove Theorems 8 and 9, we need the following preliminary results.

Lemma 11. For every $u \in X$, there exist $L_0, L_1 > 0$ such that

$$L_0 \|u\|_\infty \leq \|u'\|_\infty \leq \|u''\|_\infty \leq L_1 \|u\|_X. \quad (60)$$

Proof. (1) By $u(0) = \int_0^1 p(s)u(s)ds$, there is $u(t) = \int_0^t u'(s)ds + \int_0^1 p(s)u(s)ds$, so

$$\begin{aligned} |u(t)| &\leq \int_0^1 |u'(s)| \, ds + \int_0^1 p(s) |u(s)| \, ds \\ &\leq \|u'\|_\infty + \|u\|_\infty \int_0^1 p(s) \, ds. \end{aligned} \quad (61)$$

Hence

$$L_0 \|u\|_\infty \leq \|u'\|_\infty, \quad (62)$$

where $L_0 = 1 - \int_0^1 p(s)ds$.

(2) By $u(0) = u(1)$, there is a $\xi \in (0, 1)$ such that $u'(\xi) = 0$, and so, for $t \in [0, 1]$

$$-u'(t) = \int_t^\xi u''(s) \, ds. \quad (63)$$

Hence $|u'(t)| \leq \int_t^\xi |u''(s)| \, ds \leq \int_0^1 |u''(s)| \, ds \leq \|u''\|_\infty$, and accordingly,

$$\|u'\|_\infty \leq \|u''\|_\infty. \quad (64)$$

(3) We have from (18) that $|u''(t)| \leq \|u\|_X e(t)$, which implies that

$$|u''(t)| \leq \left(\frac{1}{4} + \frac{\int_0^1 p(s)(s-s^2) \, ds}{1 - \int_0^1 p(s) \, ds} \right) \|u\|_X, \quad (65)$$

and consequently

$$\|u''\|_\infty \leq L_1 \|u\|_X, \quad (66)$$

where $L_1 = (1/4) + (\int_0^1 p(s)(s-s^2)ds)/(\int_0^1 p(s)ds)$. $\square$

Let

$$\begin{aligned} (Au)(t) &:= \lambda \iint_0^1 K_1(t, s) K_2(s, \tau) f(\tau, u(\tau), u''(\tau)) \, d\tau ds, \\ &t \in [0, 1], \quad \lambda > 0. \end{aligned} \quad (67)$$

It is easy to show that $A : P \rightarrow P$ is a completely continuous operator. In addition, we can verify that the nonzero fixed points of the operator A are positive solutions of the problem (2).

Lemma 12 (see [19]). Let E be Banach space, P a cone in E , and $\Omega(P)$ a bounded open set in P . Suppose that $A : \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega(P), \quad \mu \geq 0, \quad (68)$$

then the fixed point index $i(A, \Omega(P), P) = 0$.

Lemma 13 (see [19]). Let E be Banach space, P a cone in E , and $\Omega(P)$ a bounded open set in P with $\theta \in \Omega(P)$. Suppose that $A : \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. If

$$Au \neq \mu u, \quad \forall u \in \partial\Omega(P), \quad \mu \geq 1, \quad (69)$$

then the fixed point index $i(A, \Omega(P), P) = 1$.

Proof of Theorem 8. It follows from the first inequality of (51) that there exists $r_0 > 0$, such that

$$\lambda f(t, u, v) \geq \mu_1 (|u| + |v|), \quad \forall t \in [0, 1], \quad 0 \leq |u| + |v| \leq r_0. \quad (70)$$

Let φ_1 be the positive eigenfunction of T (see (33)) corresponding to μ_1 . Thus $\varphi_1 = \mu_1 T\varphi_1$.

Let $r_1 = L_0 r_0 / (1 + L_0) L_1$. Then for every $u \in \partial B_{r_1} \cap P$, we have from Lemma 11 that

$$\begin{aligned} 0 &\leq u(t) - u''(t) \leq \|u\|_\infty + \|u''\|_\infty \\ &\leq \frac{(1 + L_0)}{L_0} \|u\|_X \leq r_0. \end{aligned} \quad (71)$$

It follows from (70) that

$$\begin{aligned} (Au)(t) &\geq \mu_1 \iint_0^1 K_1(t, s) K_2(s, \tau) [u(\tau) - u''(\tau)] \, d\tau ds \\ &= \mu_1 (Tu)(t), \quad t \in [0, 1]. \end{aligned} \quad (72)$$

We may suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). Now we show that

$$u - Au \neq \mu \varphi_1, \quad \forall u \in \partial B_{r_1} \cap P, \quad \mu \geq 0. \quad (73)$$

Suppose the contrary, that there exists $u_1 \in \partial B_{r_1} \cap P$ and $\tau_1 \geq 0$ such that $u_1 - Au_1 = \tau_1 \varphi_1$. Hence $\tau_1 > 0$ and

$$u_1 = Au_1 + \tau_1 \varphi_1 \geq \tau_1 \varphi_1. \quad (74)$$

Put

$$\tau^* := \sup \{ \tau \mid u_1 \geq \tau \varphi_1 \}. \quad (75)$$

It is easy to see that $\tau^* \geq \tau_1 > 0$ and $u_1 \geq \tau^* \varphi_1$. We find from $T(P) \subset P$ that

$$\mu_1 T u_1 \geq \tau^* \mu_1 T \varphi_1 = \tau^* \varphi_1. \quad (76)$$

Therefore, by (72), we have

$$u_1 = Au_1 + \tau_1 \varphi_1 \geq \mu_1 T u_1 + \tau_1 \varphi_1 \geq \tau^* \varphi_1 + \tau_1 \varphi_1, \quad (77)$$

which contradicts the definition of τ^* . Hence (73) is true, and we have from Lemma 12 that

$$i(A, B_{r_1} \cap P, P) = 0. \quad (78)$$

It follows from the second inequality of (51) that there exists $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$\lambda f(t, u, v) \leq \sigma \mu_1 (|u| + |v|), \quad \forall t \in [0, 1], \quad |u| + |v| \geq r_2. \quad (79)$$

Let $T_1 u = \sigma \mu_1 T u$, $u \in X$. Then $T_1 : X \rightarrow X$ is a bounded linear operator and $T_1(P) \subset P$.

Let

$$M = \left[\max_{0 \leq t, s, \tau \leq 1} K_1(t, s) K_2(s, \tau) \right] \times \left[\sup_{u \in \bar{B}_{r_3} \cap P} \lambda \int_0^1 f(\tau, u(\tau), u''(\tau)) d\tau \right], \quad (80)$$

where $r_3 = L_0 r_2 / (1 + L_0) L_1$. It is clear that $M < +\infty$.

Let

$$W = \{ u \in P \mid u = \mu A u, \quad 0 \leq \mu \leq 1 \}. \quad (81)$$

In the following, we prove that W is bounded.

For any $u \in W$, set $E(\tau) = \{ \tau \in [0, 1] \mid u(\tau) - u''(\tau) \geq r_2 \}$. Then,

$$\begin{aligned} u(t) &= \mu(Au)(t) \\ &\leq \lambda \int_0^1 K_1(t, s) K_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &= \lambda \int_0^1 \int_{E(\tau)} K_1(t, s) K_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &\quad + \lambda \int_0^1 \int_{[0, 1] \setminus E(\tau)} K_1(t, s) K_2(s, \tau) f \\ &\quad \times (\tau, u(\tau), u''(\tau)) d\tau ds \\ &\leq \sigma \mu_1 \int_0^1 K_1(t, s) K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau ds \\ &\quad + \lambda \int_0^1 \int_{[0, 1] \setminus E(\tau)} K_1(t, s) K_2(s, \tau) f \\ &\quad \times (\tau, u(\tau), u''(\tau)) d\tau ds \\ &\leq (T_1 u)(t) + M, \quad t \in [0, 1]. \end{aligned} \quad (82)$$

Thus, $((I - T_1)u)(t) \leq M$, for every $t \in [0, 1]$. Since μ_1 is the first eigenvalue of T and $0 < \sigma < 1$, the first eigenvalue of T_1 , $(r(T_1))^{-1} > 1$. Therefore, the inverse operator $(I - T_1)^{-1}$ exists and

$$(I - T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots. \quad (83)$$

It follows from $T_1(P) \subset P$ that $(I - T_1)^{-1}(P) \subset P$. So, we have $u(t) \leq (I - T_1)^{-1}M$, $t \in [0, 1]$, and we conclude that W is bounded.

Select $r_4 > \max\{r_1, \sup W\}$. Then, from the invariance property of the fixed point index, we have

$$i(A, B_{r_4} \cap P, P) = i(\theta, B_{r_4} \cap P, P) = 1. \quad (84)$$

By (78) and (84), we have that

$$\begin{aligned} i(A, (B_{r_4} \cap P) \setminus (\bar{B}_{r_1} \cap P), P) \\ = i(A, B_{r_4} \cap P, P) - i(A, B_{r_1} \cap P, P) = 1. \end{aligned} \quad (85)$$

Then A has at least one fixed point on $(B_{r_4} \cap P) \setminus (\bar{B}_{r_1} \cap P)$. This means that the boundary value problem (2) has at least one positive solution. $\square$

Proof of Theorem 9. It follows from the second inequality of (52) that there exists $r_0 > 0$, such that

$$\lambda f(t, u, v) \leq \mu_1 (|u| + |v|), \quad \forall t \in [0, 1], \quad 0 \leq |u| + |v| \leq r_0. \quad (86)$$

Let $r_1 = L_0 r_0 / (1 + L_0) L_1$. Then, for every $u \in \partial B_{r_1} \cap P$, we have from Lemma 11 that

$$\begin{aligned} 0 \leq u(t) - u''(t) &\leq \|u\|_\infty + \|u''\|_\infty \\ &\leq \frac{(1 + L_0) L_1}{L_0} \|u\|_X \leq r_0. \end{aligned} \quad (87)$$

It follows from (86) that

$$\begin{aligned} (Au)(t) &\leq \mu_1 \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) [u(\tau) - u''(\tau)] d\tau ds \\ &= \mu_1 (Tu)(t), \quad t \in [0, 1]. \end{aligned} \quad (88)$$

Suppose there exists $u_2 \in \partial B_{r_1} \cap P$ and $\tau_2 \geq 1$ satisfying $Au_2 = \tau_2 u_2$. We may suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). So $\tau_2 > 1$. By (88), we have $\tau_2 u_2 = Au_2 \leq \mu_1 T u_2$. By induction, we have $\tau_2^n u_2 \leq \mu_1^n T^n u_2$, for all $n \in \mathbb{N}$. Thus

$$\|T^n\| \geq \frac{\|T^n u_2\|_X}{\|u_2\|_X} \geq \frac{\tau_2^n \|u_2\|_X}{\mu_1^n \|u_2\|_X} = \frac{\tau_2^n}{\mu_1^n}. \quad (89)$$

By Gelfand's formula, we have

$$r(T) = \lim_{n \rightarrow \infty} \sqrt[n]{\|T^n\|} \geq \frac{\tau_2}{\mu_1} > \frac{1}{\mu_1}, \quad (90)$$

which is a contradiction with $r(T) = 1/\mu_1$. Hence

$$Au \neq \tau u, \quad \forall u \in \partial B_{r_1} \cap P, \quad \tau \geq 1, \quad (91)$$

and we have from Lemma 13 that

$$i(A, B_{r_1} \cap P, P) = 1. \quad (92)$$

It follows from the first inequality of (52) that there exist $\varepsilon > 0$ and $R_0 > 0$ such that

$$\lambda f(t, u, v) \geq (\mu_1 + \varepsilon)(|u| + |v|), \quad |u| + |v| \geq R_0. \quad (93)$$

From (A2), we know that there exists $d \geq 0$ such that

$$\begin{aligned} \lambda f(t, u, v) &\geq (\mu_1 + \varepsilon)(|u| + |v|) - d, \\ t &\in [0, 1], \quad 0 \leq |u| + |v| < +\infty. \end{aligned} \quad (94)$$

We can take $R > r_1$ large enough, such that

$$\rho_{1u} \geq \frac{d \int_0^1 \int_0^1 K_1(s, s) K_2(s, \tau) d\tau ds}{\varepsilon \left(\int_0^1 (s - s^2) p(s) ds / \left(1 - \int_0^1 p(s) ds \right) \right)}, \quad (95)$$

where ρ_{1u} is defined as in (46). Then by (94) and (33), for all $u \in P$, $\|u\|_X \geq R$, $t \in [0, 1]$, we have

$$\begin{aligned} (Au)(t) &= \lambda \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &\geq (\mu_1 + \varepsilon) \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) \\ &\quad \times [u(\tau) - u''(\tau)] d\tau ds \\ &\quad - d \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) d\tau ds \\ &\geq \mu_1 (Tu)(t) + \varepsilon \rho_{1u} e(t) \\ &\quad - d \int_0^1 \int_0^1 K_1(s, s) K_2(s, \tau) d\tau ds \\ &\geq \mu_1 (Tu)(t) + \varepsilon \rho_{1u} \frac{\int_0^1 (s - s^2) p(s) ds}{1 - \int_0^1 p(s) ds} \\ &\quad - d \int_0^1 \int_0^1 K_1(s, s) K_2(s, \tau) d\tau ds \\ &\geq \mu_1 (Tu)(t). \end{aligned} \quad (96)$$

Hence, for every $u \in P$, $\|u\|_X \geq R$, we have

$$(Au)(t) \geq \mu_1 (Tu)(t), \quad t \in [0, 1]. \quad (97)$$

Let φ_1 be the positive eigenfunction of T corresponding to μ_1 . Thus $\varphi_1 = \mu_1 T \varphi_1$.

We may suppose that A has no fixed points on $\partial B_R \cap P$ (otherwise, the proof is finished). Now, we show that

$$u - Au \neq \tau \varphi_1, \quad \forall u \in \partial B_R \cap P, \quad \tau \geq 0. \quad (98)$$

Suppose the contrary, that there exists $u_3 \in \partial B_R \cap P$ and $\tau_3 \geq 0$ such that

$$u_3 - Au_3 = \tau_3 \varphi_1. \quad (99)$$

Hence $\tau_3 > 0$ and

$$u_3 = Au_3 + \tau_3 \varphi_1 \geq \tau_3 \varphi_1. \quad (100)$$

Put

$$\tau^* := \sup \{ \tau \mid u_3 \geq \tau \varphi_1 \}. \quad (101)$$

It is easy to see that $\tau^* \geq \tau_3 > 0$ and $u_3 \geq \tau^* \varphi_1$. We find from $T(P) \subset P$ that

$$\mu_1 T u_3 \geq \tau^* \mu_1 T \varphi_1 = \tau^* \varphi_1. \quad (102)$$

Therefore by (97), we have

$$u_3 = Au_3 + \tau_3 \varphi_1 \geq \mu_1 T u_3 + \tau_3 \varphi_1 \geq \tau^* \varphi_1 + \tau_3 \varphi_1, \quad (103)$$

which contradicts the definition of τ^* . Hence (98) is true, and we have from Lemma 12 that

$$i(A, B_R \cap P, P) = 0. \quad (104)$$

By (92) and (104), we have that

$$\begin{aligned} i(A, (B_R \cap P) \setminus (\overline{B}_{r_1} \cap P), P) \\ = i(A, B_R \cap P, P) - i(A, \overline{B}_{r_1} \cap P, P) = -1. \end{aligned} \quad (105)$$

Then, A has at least one fixed point on $(B_R \cap P) \setminus (\overline{B}_{r_1} \cap P)$. This means that the boundary value problem (2) has at least one positive solution. $\square$

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