Research Article

Estimation of Approximation with Jacobi Weights by Multivariate Baskakov Operator

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Received 7 May 2013; Accepted 14 July 2013

Academic Editor: Yongsheng S. Han

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We first give the unboundedness of multivariate Baskakov operators with the normal weighted norm. By introducing new norms, using the multivariate decomposition technique and the modulus of smoothness with Jacobi weight, the upper bound estimation of multivariate Baskakov operators is obtained. The obtained results not only generalize the corresponding ones for multivariate Baskakov operators without weights, but also give the approximation accuracy with the Jacobi weights approximation.

1. Introduction and Main Results

Let \( C_{\omega} [0, +\infty) \) be the set of bounded continuous functions on \([0, +\infty);\) then the univariate Baskakov operator is defined by

\[
V_{n,1} (f, x) = \sum_{k=0}^{\infty} v_{n,k} (x) f \left( \frac{k}{n} \right),
\]

(1)

where

\[
v_{n,k} (x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \quad x \in [0, \infty].
\]

(2)

There are many papers to study the univariate Baskakov-type operator (see [1–4]). In [5], the authors discussed the convergence property of Baskakov operator’s approximation with weights by introducing weighted smoothness modulus and obtained the characteristics of approximation. And in [6], using the weighted smoothness modulus, the close connection between the derivative of Baskakov operator and the smoothness of function approximated which has been investigated, the upper bound estimation has been established with the \( J_{\text{Jacobi}} \) weight; that is, for any \( f \in C_{\omega} \) \( 0 \leq \lambda \leq 1, \)

\[
| \omega (x) (V_{n,1} (f, x) - f (x)) | \leq C_{\omega}^2 \left( f; n^{-1/2} \varphi_{\lambda}^{\text{Jacobi}} (x) \right)_{\omega},
\]

(3)

where \( C_{\omega} = \{ f \mid f \in C[0, \infty], a f \in L_{\infty}[0, +\infty] \} \). In this paper, the letter \( C \), appearing in various formulas, denotes a positive constant independent of \( n, x, \) and \( f \). Its value may be different at different occurrences, even within the same formula.

Let \( T \in R^d (d \in N) \), let \( T = x = (x_1, x_2, \ldots, x_d) \), and let \( 0 \leq x_i < \infty, 0 \leq i \leq d \),

\[
x = (x_1, x_2, \cdots, x_d), \quad k = (k_1, k_2, \ldots, k_d),
\]

\[
x^k = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \quad |k| = k_1 + k_2 + \cdots + k_d |x| = x_1 + x_2 + \cdots + x_d,
\]

\[
\binom{n}{k} = \frac{n!}{k! (n-|k|)!}, \quad \sum_{k=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_d=0}^{\infty},
\]

(4)

\[
C_{\text{\varphi}} (T) = \{ f \in C_B (T) : f \in C_B (T), \quad \text{and} \quad \varphi_i^T D^i f \in C_B (T), 1 \leq i \leq d \},
\]

where \( C_B (T) \) represents the set of bounded continuous functions in \( T \). The norm defined on \( T \) is \( \| f \|_{\text{\varphi}} = \max_{x \in T} |f (x)| \).

Using the previous notations, the multivariate Baskakov operator is defined by

\[
V_{n,d} (f; x) = \sum_{k=0}^{\infty} v_{n,k} (x) f \left( \frac{k}{n} \right),
\]

(5)
where \( v_{n,k}(x) = \binom{n+|k|-1}{k} x^k (1+x)^{-n-|k|} \). In [7], the authors gave the equivalent relation between \( K \) -functional and modulus of smoothness and obtained the following result: for any \( f \in C_B(T) \),

\[
\| V_{n,d} f - f \|_\infty \leq C \omega^2_{\psi} \left( f, \frac{1}{\sqrt{n}} \right)_\infty , \tag{6}
\]

where

\[
\omega^r_{\psi}(f,t) = \sup_{0<h,t} \sum_{j=1}^d \| \Delta_{h^r} f \|_\infty ,
\]

\[
\Delta_{h^r} f(x) = \left\{ \begin{array}{ll}
0, & \text{else,}
\end{array} \right.
\]

\[
\sum_{i=0}^r \left( \begin{array}{c}
r \iota \end{array} \right) (-1)^i f(x + i h e) , \quad x+h \in T .
\]

The \( r \) th \( K \) -functional is defined as follows

\[
K^r_{\psi}(f;t) = \inf_g \left\{ \| f - g \|_\infty + t \sum_{i=1}^d \| \psi_i^r D_i^r g \|_\infty , g \in C^2_p(T) \right\} . \tag{8}
\]

And the \( r \) th modulus of smoothness is

\[
\omega^r_{\psi}(f,t) = \sup_{0<h,t} \sum_{j=1}^d \| \Delta_{h^r} f \|_\infty . \tag{9}
\]

Naturally, we will consider the following problem: “are there similar results in the approximation with weights by multivariate Baskakov operator?” However, approximation with Jacobi weights \((\omega(x) = x^\alpha (1+|x|)^{-b} \quad (0 < |\alpha| < 1, b>0)\) is not a simple generalization of normal approximation means. In the norm \( \| \omega f \|_\omega \), the multivariate Baskakov operator is not bounded. By introducing the weighted norm

\[
\| f \|_\omega = \| \omega f \|_\infty + f(0) , \tag{10}
\]

we find that the multivariate Baskakov operator is bounded, and thus we can investigate approximation capability of the operators. In the previous weighted norm, the \( r \) th weighted \( K \) -functional is defined as follows:

\[
K^r_{\psi}(f;t)_\omega = \inf_g \left\{ \| f - g \|_\omega + t \sum_{i=1}^d \| \psi_i^r D_i^r g \|_\omega , g \in C^2_p(T) \right\} , \tag{11}
\]

\[
f \in C_B(T) ,
\]

and \( r \) th weighted Ditzian-Totik modulus of smoothness is

\[
\omega^r_{\psi}(f,t)_\omega = \sup_{0<h,t} \sum_{j=1}^d \| \Delta_{h^r} f \|_\omega , \quad g \in C_B(T) , \tag{12}
\]

where \( \psi_i^2(x) = x_i (1+|x|) \).

Using the previous notations, we will present our results. Firstly, we study the unboundedness of multivariate Baskakov operator in the normal norm.

**Theorem 1.** Let \( f \in C_B(T) \); then the operator \( V_{n,d}(f;\omega) \) is unbounded in the norm \( \| \omega f \|_\omega \).

In addition, introducing the new norm, we establish the upper bound estimation of weighted approximation by multivariate Baskakov operator.

**Theorem 2.** Let \( f \in C_B(T) \); then one has

\[
\| V_{n,d} f - f \|_\omega \leq C \omega^2_{\psi} \left( f, \frac{1}{\sqrt{n}} \right)_\omega . \tag{13}
\]

**Remark 3.** Our result reveals two things: (i) for any multivariate bounded continuous function \( f \in C_B(T) \), there is a multivariate polynomial \( V_{n,d}(f) \) that approximates \( f \) arbitrarily well (when \( n \) is sufficiently large) in the continuous space. (ii) Quantitatively, the approximation accuracy of a polynomial \( V_{n,d}(f) \) can be controlled by the \( \omega^2_{\psi}(f,1/\sqrt{n})_\omega \); here \( \omega^2_{\psi}(f,1/\sqrt{n})_\omega \) is the weighted smoothness modulus of the function \( f \).

### 2. Some Lemmas

In order to prove our results, we will show some lemmas in this section.

**Lemma 4 (see [6]).** For \( f \in C_\omega , 0 \leq \lambda \leq 1, \) one has

\[
\| V_{n,1} f - f \|_\omega \leq C \omega^2_{\psi} \left( f, \frac{1}{\sqrt{n}} \right)_\omega . \tag{14}
\]

**Lemma 5 (see [5]).** For \( c,d \geq 0, \)

\[
\sum_{k=1}^{\infty} \left( \frac{k}{n} \right)^c (1 + \frac{k}{n})^d V_{n,k}(x) \leq C x^c (1 + x)^d , \quad x > 0 . \tag{15}
\]

**Lemma 6.** For \( f \in C_B(T) \), then one has

\[
\left| \omega(x) \sum_{k=1}^{\infty} v_{n,k}(x) f \left( \frac{k}{n} \right) \right| \leq C \| \omega f \|_\infty . \tag{16}
\]
Proof. For convenience, we only prove the case of \( d = 2 \); for a general \( d \), we can prove similarly. For \( d = 2 \), we have
\[
\left| \omega (x) \sum_{k=1}^{\infty} v_{n,k} (x) f \left( \frac{k}{n} \right) \right|
\]
\[
= \left| \omega (x) \sum_{k_1=1}^{\infty} v_{n,k_1} (x_1) \sum_{k_2=1}^{\infty} v_{n+k_1,k_2} \left( \frac{x_2}{1 + x_1} \right) f \left( \frac{k_1}{n} , \frac{k_2}{n} \right) \right|
\]
\[
= \omega (x) \sum_{k_1=1}^{\infty} v_{n,k_1} (x_1) \sum_{k_2=1}^{\infty} v_{n+k_1,k_2} \left( \frac{x_2}{1 + x_1} \right) f \left( \frac{k_1}{n} , \frac{k_2}{n} \right)
\]
\[
\times \omega \left( \frac{k_1}{n} \times \frac{k_2}{n} \right) \left( \frac{1}{n} \right)^{-\alpha_1} \left( \frac{1}{n} \right)^{-\alpha_2} \times \left( 1 + \frac{k_1}{n} + \frac{k_2}{n} \right)^{-b}
\]
\[
\leq C \| \omega f \|_\infty \| \omega (x) \sum_{k_1=1}^{\infty} v_{n,k_1} (x_1) \left( \frac{k_1}{n} \right)^{-\alpha_1} \right\}
\]
\[
\times \left\{ \sum_{k_2=1}^{\infty} v_{n+k_1,k_2} \left( \frac{x_2}{1 + x_1} \right) \left( \frac{k_2}{n} \right)^{-\alpha_2} \times \left( 1 + \frac{k_1}{n} + \frac{k_2}{n} \right)^{-b} \right\}
\]
\[
= C \| \omega f \|_\infty \| \omega (x) \sum_{k_1=1}^{\infty} v_{n,k_1} (x_1) \left( \frac{k_1}{n} \right)^{-\alpha_1} \right\}
\]
\[
\times \left\{ \sum_{k_2=1}^{\infty} v_{n+k_1,k_2} \left( \frac{x_2}{1 + x_1} \right) \left( 1 + \frac{k_2}{n + k_1} \right)^{-b} \right\}
\]
\[
\times \left( \frac{k_2}{n + k_1} \right)^{-\alpha_2} \right\}
\]
\[
\leq C \| \omega f \|_\infty \| \omega (x) \sum_{k=0}^{\infty} v_{n,k} (x) \left( \frac{n}{k+1} \right)^{l} \right\} \leq Cx^{-l}, \quad l \in \mathbb{N}.
\]

Proof of Theorem 1. Let
\[
f_m (x_1, x_2) = \frac{1}{x_1^{\alpha_1}x_2^{\alpha_2} + (1/m)}.
\]

Then we have
\[
\| \omega f \|_\infty = \max_{x>0} \left| x_1^{-\alpha_1}x_2^{-\alpha_2} \left( 1 + x_1 + x_2 \right)^{-b} \right| \frac{1}{x_1^{\alpha_1}x_2^{\alpha_2} + (1/m)} \leq \left( \frac{1}{1 + x_1 + x_2} \right)^b \leq 1.
\]

On the other hand,
\[
\| \omega f \|_\infty \| \omega V_{n,d} f \|_\infty = \max_{x>0} \left| \omega (x) \sum_{k=0}^{\infty} V_{n,k} (x) f \left( \frac{k}{n} \right) \right|
\]
\[
= \max_{x>0} \left| \omega (x) \sum_{k=0}^{\infty} v_{n,k} (x) f \left( \frac{k}{n} \right) \right|
\]
\[
\geq \max_{x>0} \left| \omega (x) \right| f_m (0,0) v_{n,0} (x) \]
\[
\geq \| \omega (x) \| f_m (0,0) \left( 1 + x_1 + x_2 \right)^{\alpha_1} \leq \| \omega (x) \| \left( \frac{1}{2} \right)^{\alpha_1}
\]
\[
\geq \| \omega (x) \| \left( \frac{1}{2} \right)^{\alpha_1} - C \rightarrow \infty (m \rightarrow \infty).
\]

\[\square\]

3. The Proof of Theorems

In this section, we will show the unboundedness of multivariate Baskakov operator in the general weighted norm and then give the proof of Theorem 2.

(i) For \( d = 1 \), from Lemma 4, we have
\[
\| V_{n,d} (f) - f \|_\omega \leq n^{-1} \| \phi_0^2 D_1^2 f \|_\omega.
\]

(ii) When \( d = r \), suppose that
\[
\| V_{n,d} (f) - f \|_\omega \leq \| \phi_0^2 D_1^2 f \|_\omega.
\]
is established. So when \( d = r + 1 \), using the multivariate decomposition technique in [7], we have

\[
V_{n,d}(f, x) = \sum_{k_i=0}^{\infty} v_{nk_i}(x) \sum_{k^*=0}^{\infty} v_{n+k^*, k^*} \left( \frac{x^*}{1+x_1} \right),
\]

(25)

where

\[
x^* = (x_2, x_3, \ldots, x_n), \quad x = (x_1, x^*),
\]

\[
k^* = (k_2, k_3, \ldots, k_d), \quad k = (k_1, K^*).
\]

(26)

Let

\[
g_{k_1}(u) = f \left( \frac{k_1}{n}, 1 + \frac{k_1}{n} u \right), \quad u \in T_{d-1},
\]

\[
h(t) = h(t, x) = f \left( t, 1 + t \frac{x^*}{1+x_1} \right).
\]

(27)

We get

\[
\omega(x)(V_{n,d}(f, x) - f(x)) = \omega(x) \left( \sum_{k_i=0}^{\infty} v_{nk_i}(x_1) \left( V_{n+k_i,d-1}(g_{k_1}(\cdot), z) - g_{k_1}(z) \right) \right)
\]

\[
+ (V_{n,1}(h(\cdot), x_1) - h(x_1))
\]

\[
= P + Q,
\]

(28)

where \( z = x^*/(1+x_1) \), and we have

\[
|P| \leq \left| \omega(x) \left( \sum_{k_i=0}^{\infty} v_{nk_i}(x_1) \left( V_{n+k_i,d-1}(g_{k_1}(\cdot), z) - g_{k_1}(z) \right) \right) \right|
\]

\[
= x_1^{\alpha_i}(1+x_1)^{-b+|\alpha|} \sum_{k_i=0}^{\infty} v_{nk_i}(x_1) \omega(z)
\]

\[
\times \left( V_{n+k_i,d-1}(g_{k_1}(\cdot), z) - g_{k_1}(z) \right)
\]

\[
\leq x_1^{\alpha_i}(1+x_1)^{-b+|\alpha|} \sum_{k_i=0}^{\infty} \frac{1}{n+k_1} v_{nk_i}(x_1) \sum_{i=1}^{r} \left\| \phi_i^2 D_i^2 f \right\|_\omega.
\]

(29)

The previous derivation used the following inequality:

\[
\omega(x) = x^a(1+|x|)^b
\]

\[
= x_1^{\alpha_i} \left( \frac{x^*}{1+x_1} \right)^{\alpha_i} (1+x_1)^{-b+|\alpha|} \left( 1 + \frac{x^*}{1+x_1} \right)^{-b}
\]

\[
= x_1^{\alpha_i} (1+x_1)^{-b+|\alpha|} \omega(z),
\]

\[
\phi_i^2 (u) D_i^2 (u) g_{k_1}(u) \omega(u)
\]

\[
= u_i (1+|u|) \left( 1 + \frac{k_1}{n} \right) D_i^2 f \left( \frac{k_1}{n}, 1 + \frac{k_1}{n} u \right)
\]

\[
\times u^{|\alpha|}(1+|u|)^{-b}
\]

\[
= \left( \frac{n}{k_1} \right)^{\alpha_i} \left( \frac{n}{n+k_1} \right)^{-b+|\alpha|} \omega \phi_i^2 D_i^2 f \left( \frac{k_1}{n}, 1 + \frac{k_1}{n} u \right).
\]

(30)

So we get

\[
\left\| \phi_i^2 D_i^2 g_k \right\|_\omega \leq \left( \frac{n}{k_1} \right)^{\alpha_i} \left( \frac{n}{n+k_1} \right)^{-b+|\alpha|} \left\| \phi_i^2 D_i^2 f \right\|_\omega.
\]

(31)

From Lemma 6, we have

\[
|P| \leq x_1^{\alpha_i} (1+x_1)^{-b+|\alpha|} \sum_{k_i=0}^{\infty} \frac{1}{n+k_1} p_{nk_i}(x_1)
\]

\[
\times \left( \frac{n}{k_1} \right)^{\alpha_i} \left( \frac{n}{n+k_1} \right)^{-b+|\alpha|} \left\| \phi_i^2 D_i^2 f \right\|_\omega.
\]

(32)

\[
\leq C \left\| \phi_i^2 D_i^2 f \right\|_\omega.
\]

In the following, we will establish the estimation of \( Q \):

\[
|Q| = \left| (V_{n,1}(h(\cdot), x_1) - h(x_1)) \omega(x) \right|
\]

\[
= x_1^{\alpha_i} (x^a)^b (B_{n,1}(h(\cdot), x_1) - h(x_1))
\]

\[
= \left( x^a \right)^b \left( 1 + \frac{|x|}{1+x_1} \right)^{-b} x_1^{\alpha_i}
\]

\[
\times (V_{n,1}(h(\cdot), x_1) - h(x_1))
\]

\[
\leq C \left( x^a \right)^b \left( 1 + \frac{|x|}{1+x_1} \right)^{-b} \left\| \phi_i^2 H'' \right\|_\omega,
\]

(33)
and for the term $\|q^2h''\|_\omega$, we have the following estimation:

$$
\|q^2h''\|_\omega = \max_{0 \leq t < \infty} |(1 + t) t^a (1 + t)^b h''(t, (1 + t) \frac{x^*}{1 + x_1})|
$$

$$
= \max_{0 \leq t < \infty} |t (1 + t) t^a (1 + t)^b h''(t, (1 + t) \frac{x^*}{1 + x_1})|
$$

$$
= \max_{0 \leq t < \infty} |t (1 + t) t^{a+1} (1 + t)^b h''(t, (1 + t) \frac{x^*}{1 + x_1})|
$$

$$
+ \sum_{i=2}^{d} \frac{x_i}{1 + x_1} D_{i1}^2 f + \sum_{i=2}^{d} \frac{x_i}{1 + x_1} D_{i1}^2 f
$$

$$
+ \sum_{i,j=2,i \neq j}^{d} x_i x_j (1 + x_1)^2 D_{i,j}^2 f
$$

$$
= \max_{0 \leq t < \infty} \left| t (1 + t) t^a (1 + t)^b L_1 \right|,
$$

where

$$
L_1 = \frac{1 + x_1}{1 + |x_1|} q_1^2 D_{11}^2 f + \sum_{i=2}^{d} q_1^2 D_{i1}^2 f + \sum_{i=2}^{d} q_1^2 D_{i1}^2 f
$$

$$
+ \sum_{i=2}^{d} \frac{t}{1 + t} \frac{x_i}{1 + |x_1|} q_1^2 D_{i1}^2 f
$$

$$
+ \sum_{i,j=2,i \neq j}^{d} q_1^2 D_{i,j}^2 f \left( t, (1 + t) \frac{x^*}{1 + x_1} \right).
$$

Note that

$$
\omega \left( t, (1 + t) \frac{x^*}{1 + x_1} \right) = \left( x^* \right)^{a'} \left( \frac{1 + t}{1 + x_1} \right)^{a'} \left( \frac{x^*}{1 + x_1} \right)^{b'}.
$$

We have

$$
\|q^2h''\|_\omega = (x^*)^{a'} \left( \frac{1 + t}{1 + x_1} \right)^{a'} \left( \frac{x^*}{1 + x_1} \right)^{b'}. \left( 1 + |x_1| \right)^b \left( 1 + t \right)^b.
$$

Here

$$
\left| L_2 \right| = \frac{1 + x_1}{1 + |x_1|} \omega_1 q_1^2 D_{11}^2 f + \sum_{i=2}^{d} \omega_1 q_1^2 D_{i1}^2 f + \sum_{i=2}^{d} \omega_1 q_1^2 D_{i1}^2 f
$$

$$
+ \sum_{i=2}^{d} \frac{t}{1 + t} \frac{x_i}{1 + |x_1|} \omega_1 q_1^2 D_{i1}^2 f
$$

$$
+ \sum_{i,j=2,i \neq j}^{d} \omega_1 q_1^2 D_{i,j}^2 f \left( t, (1 + t) \frac{x^*}{1 + x_1} \right).
$$

From [7], we have

$$
|D_{i1}^2 f(x)| \leq \sup_{t \in T} \left| \int_{x_1}^t (x^* - s) \left( \frac{1 + |x_1|}{1 + x_1} \right)^b \left( 1 + t \right)^b \omega_1^2 \right|.
$$

$$
\phi_{i,j}(x) \leq \min \{ \phi_i(x), \phi_j(x) \},
$$

$$
\|q^2h''\|_\omega = (x^*)^{-a'} \left( \frac{1 + |x_1|}{1 + x_1} \right)^{-b'} \left( 1 + t \right)^{-b'} \left( 1 + x_1 \right)^{-b'} \left\| q_1^2 D_{i1}^2 f \right\|_\omega.
$$

Hence

$$
|Q| \leq \frac{C}{n} \left\| q_1^2 D_{i1}^2 f \right\|_\omega.
$$

Thus we have

$$
\|V_{n,d}f - f\|_\omega = \|V_{n,d}f - V_{n,d}g + V_{n,d}g - g + f\|_\omega
$$

$$
\leq C \|V_{n,d}f - V_{n,d}g\|_\omega + \|f - g\|_\omega
$$

$$
\leq \frac{C}{n} \left\| q_1^2 D_{i1}^2 f \right\|_\omega + \|f - g\|_\omega
$$

$$
\leq CK_1(f, \frac{1}{1 + t}) \omega
$$

$$
\leq C \omega_1^2 (f, \frac{1}{1 + t}) \omega
.$$
