Research Article

Boundedness of Oscillatory Integrals with Variable Calderón-Zygmund Kernel on Weighted Morrey Spaces

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Oscillatory integral operators play a key role in harmonic analysis. In this paper, the authors investigate the boundedness of the oscillatory singular integrals with variable Calderón-Zygmund kernel on the weighted Morrey spaces $L^{p,k}(\omega)$. Meanwhile, the corresponding results for the oscillatory singular integrals with standard Calderón-Zygmund kernel are established.

1. Introduction and Main Results

Suppose that k is the standard Calderón-Zygmund kernel. That is, $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree n , and $\int_{\Sigma} k(x) d\sigma_x = 0$, where $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$. The oscillatory integral operator T_λ is defined by

$$T_\lambda f(x) = p \cdot v \cdot \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} k(x-y) \varphi(x,y) f(y) dy, \quad (1)$$

where $\lambda \in \mathbb{R}$, $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is the space of infinitely differentiable functions on $\mathbb{R}^n \times \mathbb{R}^n$ with compact supports, and Φ is a real-analytic function or a real- $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that, for any $(x_0, y_0) \in \text{supp } \varphi$, there exists (j_0, k_0) , $1 \leq j_0, k_0 \leq n$, such that $\partial^2 \Phi(x_0, y_0) / \partial x_{j_0} \partial y_{k_0}$ does not vanish up to infinite order. These operators have arisen in the study of singular integrals supported on lower dimensional varieties and the singular Radon transform. In [1], Pan proved that T_λ are uniform in λ bounded on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$). Lu et al. [2] proved the weighted L^p boundedness of T_λ defined by (1).

Let $k(x, y)$ be a variable Calderón-Zygmund kernel. That means, for a.e. $x \in \mathbb{R}^n$, $k(x, \cdot)$ is a standard Calderón-Zygmund kernel and

$$\max_{|j| \leq 2n, j \in \mathbb{Z}} \left\| \frac{\partial^{|j|} k}{\partial y^j} \right\|_{L^\infty(\mathbb{R}^n \times \Sigma)} = A < \infty. \quad (2)$$

Define the oscillatory integral operator with variable Calderón-Zygmund kernel T_λ^* by

$$T_\lambda^* f(x) = p \cdot v \cdot \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} k(x, x-y) \varphi(x,y) f(y) dy, \quad (3)$$

where λ , φ , and Φ satisfy the same assumptions as those in the operator defined by (1).

Lu et al. [2] investigated the L^p and weighted L^p boundedness about this class of oscillatory integral operators.

The classical Morrey space $L^{p,\lambda}$ was first introduced by Morrey in [3] to study the local behavior of solutions to second order elliptic partial differential equations. In 2009, Komori and Shirai [4] first defined the weighted Morrey spaces $L^{p,k}(\omega)$ which could be viewed as an extension of weighted Lebesgue spaces. They studied the boundedness of the fractional integral operator, the Hardy-Littlewood maximal operator, and the Calderón-Zygmund singular integral operator on the space. The boundedness results about some operators on these spaces can be see in ([5–17]). Recently, Shi et al. [18] obtained the boundedness of a class of oscillatory integrals with Calderón-Zygmund kernel and polynomial phase on weighted Morrey spaces. Their results are stated as follows.

Let $P(x, y)$ be a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$ and let k satisfy the following hypotheses:

$$|k(x, y)| \leq \frac{C}{|x - y|^n}, \quad \forall x \neq y, \quad (4)$$

$$|\nabla_x k(x, y)| + |\nabla_y k(x, y)| \leq \frac{C}{|x - y|^{n+1}}, \quad \forall x \neq y.$$

We define

$$Sf(x) = p \cdot v \cdot \int_{\mathbb{R}^n} k(x, y) f(y) dy, \quad (5)$$

$$Rf(x) = p \cdot v \cdot \int_{\mathbb{R}^n} e^{iP(x, y)} k(x, y) f(y) dy.$$

Theorem A (see [18]). *Let $1 < p < \infty$, $0 < \kappa < 1$, and $\omega \in A_p$. If S is of type (L^2, L^2) , then, for any real polynomial $P(x, y)$, there exists a constant $C > 0$ such that*

$$\|Rf\|_{L^{p, \kappa}(\omega)} \leq C \|f\|_{L^{p, \kappa}(\omega)}. \quad (6)$$

The purpose of this paper is to generalize the above results to the case with real- C^∞ or analytic phase functions. Our main results in this paper are formulated as follows.

Theorem 1. *Let $\lambda \in \mathbb{R}$, $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and Φ a real- $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that, for any $(x_0, y_0) \in \text{supp } \varphi$, there exists (j_0, k_0) , $1 \leq j_0, k_0 \leq n$, such that $\partial^2 \Phi(x_0, y_0) / \partial x_{j_0} \partial y_{k_0}$ does not vanish up to infinite order. Assume that k is a standard Calderón-Zygmund kernel and T_λ is defined as in (1). Then for any $1 < p < \infty$, $0 < \kappa < 1$, and $\omega \in A_p$, T_λ is bounded on $L^{p, \kappa}(\omega)$.*

Theorem 2. *Let $\lambda \in \mathbb{R}$, $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and Φ a real- $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that, for any $(x_0, y_0) \in \text{supp } \varphi$, there exists (j_0, k_0) , $1 \leq j_0, k_0 \leq n$, such that $\partial^2 \Phi(x_0, y_0) / \partial x_{j_0} \partial y_{k_0}$ does not vanish up to infinite order. Assume that k is a variable Calderón-Zygmund kernel and T_λ^* is defined as in (3). Then for any $1 < p < \infty$, $0 < \kappa < 1$, and $\omega \in A_p$, T_λ^* is bounded on $L^{p, \kappa}(\omega)$.*

2. Notations and Preliminary Lemmas

Let $B = B(x_0, r)$ be the ball with the center x_0 and radius r . Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B .

The classical A_p weighted theory was first introduced by Muckenhoupt in [19]. A weight ω is a locally integrable function on $\mathbb{R}^n$, which takes values in $(0, \infty)$ a.e. For a given weight function ω , we denote the Lebesgue measure of B by $|B|$ and the weighted measure of E by $\omega(E)$; that is, $\omega(E) = \int_E \omega(x) dx$. Given a weight ω , we say that ω satisfies the doubling condition if there exists a constant $D > 0$ such that, for any ball B , we have $\omega(2B) \leq D\omega(B)$.

We say $\omega \in A_p$ with $1 < p < \infty$, if there exists a constant $C > 0$, such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-1/(p-1)} dx \right)^{p-1} \leq C, \quad (7)$$

for every ball $B \subseteq \mathbb{R}^n$. When $p = 1$, $\omega \in A_1$ if there exists $C > 0$, such that

$$\frac{1}{|B|} \int_B \omega(x) dx \leq C \operatorname{ess\,inf}_{x \in B} \omega(x), \quad (8)$$

for almost every $x \in \mathbb{R}^n$. We define $A_\infty = \bigcup_{p \geq 1} A_p$. A weight function ω is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 0$ and $C > 0$ such that the following reverse Hölder inequality holds:

$$\left(\frac{1}{|B|} \int_B \omega(x)^r dx \right)^{1/r} \leq C \left(\frac{1}{|B|} \int_B \omega(x) dx \right), \quad (9)$$

for every ball $B \subseteq \mathbb{R}^n$.

It is well known that, if $\omega \in A_p$ with $1 \leq p < \infty$, then there exists $r > 1$ such that $\omega \in RH_r$.

Lemma 3 (see [20]). *Let $\omega \in A_p$, $p \geq 1$, and $r > 0$. Then for any ball B and $\lambda > 1$,*

$$\omega(2B) \leq C\omega(B), \quad (10)$$

$$\omega(\lambda B) \leq C\lambda^{np}\omega(B),$$

where C does not depend on B nor on λ .

Lemma 4 (see [21]). *Let $\omega \in RH_r$ with $r > 1$. Then there exists a constant C such that*

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^{(r-1)/r}, \quad (11)$$

for any measurable subset E of a ball B .

The weighted Morrey spaces were defined as follows.

Definition 5 (see [4]). Let $1 \leq p < \infty$, $0 < \kappa < 1$, and ω a weight function. Then the weighted Morrey space is defined by

$$L^{p, \kappa}(\omega) = \{f \in L_{\text{loc}}^p(\omega) : \|f\|_{L^{p, \kappa}(\omega)} < \infty\}, \quad (12)$$

where

$$\|f\|_{L^{p, \kappa}(\omega)} = \sup_B \left(\frac{1}{\omega(B)^\kappa} \int_B |f(x)|^p \omega(x) dx \right)^{1/p}, \quad (13)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. The space $L_{\text{loc}}^p(\omega)$ is defined by

$$L_{\text{loc}}^p(\omega) = \{f : f\chi_K \in L^p(\omega), \text{ for every compact set } K \subseteq \mathbb{R}^n\}. \quad (14)$$

Our argument is based heavily on the following results.

Lemma 6 (see [2]). *Assume that T_λ is defined as in (1). Then for any $1 < p < \infty$ and $\omega \in A_p$, one has*

$$\|T_\lambda f\|_{L^p(\omega)} \leq C(n, p, \Phi, \varphi, C_{p, \omega}) B \|f\|_{L^p(\omega)}, \quad (15)$$

where $C(n, p, \Phi, \varphi, C_{p, \omega})$ is independent of λ , k , and f and $B = \|k\|_{C^1(\Sigma)}$.

Lemma 7 (see [2]). Assume that T_λ^* is defined as in (3). Then for any $1 < p < \infty$ and $\omega \in A_{p,\omega}$, one has

$$\|T_\lambda^* f\|_{L^p(\omega)} \leq C(n, p, \Phi, \varphi, C_{p,\omega}) A \|f\|_{L^p(\omega)}, \quad (16)$$

where $C(n, p, \Phi, \varphi, C_{p,\omega})$ is independent of λ , k , and f . A is defined in (2).

Definition 8 (see [4]). The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad f \in L_{\text{loc}}(\mathbb{R}^n). \quad (17)$$

Lemma 9 (see [4]). If $1 < p < \infty$, $0 < \kappa < 1$, and $\omega \in A_p$ then the Hardy-Littlewood maximal operator M is bounded on $L^{p,\kappa}(\omega)$.

Lemma 10 (see [22]). Denote by $\mathcal{H}_m$ the spaces of spherical harmonic functions of degree m . Then

- (a) $L^2(\Sigma) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$, and $g_m = \dim \mathcal{H}_m \leq C(n)m^{n-2}$ for any $m \in \mathbb{N}$;
- (b) for any $m = 0, 1, 2, \dots$, there exists an orthogonal system $\{Y_{jm}\}_{j=1}^{g_m}$ of $\mathcal{H}_m$ such that $\|Y_{jm}\|_{L^\infty(\Sigma)} \leq C(n)m^{n/2-1}$, $Y_{jm} = (-m)^{-n}(m+n-2)^{-n} \Lambda^n Y_{jm}$, $j = 1, \dots, g_m$, and Λ is the Beltrami-Laplace operator on Σ .

In the following the letter C will denote a constant which may vary at each occurrence.

3. Proof of Theorems

Proof of Theorem 1. It is sufficient to prove that there exists a constant $C > 0$ such that

$$\frac{1}{\omega(B)^\kappa} \int_B |T_\lambda f(x)|^p \omega(x) dx \leq C \|f\|_{L^{p,\kappa}(\omega)}^p. \quad (18)$$

Fix a ball $B = B(x_0, r_B)$ and decompose $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$. Then we have

$$\begin{aligned} & \frac{1}{\omega(B)^\kappa} \int_B |T_\lambda f(x)|^p \omega(x) dx \\ & \leq C \left\{ \frac{1}{\omega(B)^\kappa} \int_B |T_\lambda f_1(x)|^p \omega(x) dx \right. \\ & \quad \left. + \frac{1}{\omega(B)^\kappa} \int_B |T_\lambda f_2(x)|^p \omega(x) dx \right\} \\ & = C \{I_1 + I_2\}. \end{aligned} \quad (19)$$

Using Lemmas 3 and 6, we get

$$\begin{aligned} I_1 & \leq C \frac{1}{\omega(B)^\kappa} \int_{2B} |f(x)|^p \omega(x) dx \\ & \leq C \|f\|_{L^{p,\kappa}(\omega)}^p \cdot \frac{\omega(2B)^\kappa}{\omega(B)^\kappa} \\ & \leq C \|f\|_{L^{p,\kappa}(\omega)}^p. \end{aligned} \quad (20)$$

We now estimate I_2 . We can write

$$|T_\lambda f_2(x)| = \left| \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k(x-y) \varphi(x,y) f(y) dy \right|. \quad (21)$$

Now by an argument similar to the proof of Lemma 6 in [2], we choose $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_1(x) \equiv 1$, when $|x| \leq 1$, and $\phi_1(x) \equiv 0$ when $|x| > 2$. Let $\phi_2 = 1 - \phi_1$ and $N \in \mathbb{N}$ which is large enough and will be determined later. Write

$$k(x) = k_\lambda^1(x) + k_\lambda^2(x), \quad (22)$$

where

$$k_\lambda^j(x) = k(x) \phi_j(\lambda^{1/N} x), \quad j = 1, 2. \quad (23)$$

Then

$$\begin{aligned} T_\lambda f_2(x) & = p \cdot v \cdot \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k_\lambda^1(x-y) \\ & \quad \times \varphi(x,y) f(y) dy \\ & \quad + p \cdot v \cdot \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k_\lambda^2(x-y) \\ & \quad \times \varphi(x,y) f(y) dy \\ & := T_\lambda^1 f_2(x) + T_\lambda^2 f_2(x). \end{aligned} \quad (24)$$

Let us first estimate $T_\lambda^1 f_2(x)$. To do so, using Taylor's expansion and the compactness of $\text{supp } \varphi$, we write

$$\Phi(x,y) = \Phi(x,x) + P(x,y) + r_N(x,y) \quad (25)$$

for $(x,y) \in \text{supp } \varphi$, where $P(x,y)$ is a polynomial with $\deg P < N$ and $|r_N(x,y)| \leq C|x-y|^N$ with C independent of x and y . Define

$$Rf(x) = p \cdot v \cdot \int_{(2B)^c} e^{i\lambda P(x,y)} k_\lambda^1(x-y) \varphi(x,y) f(y) dy. \quad (26)$$

Therefore

$$\begin{aligned} & e^{-i\lambda\Phi(x,x)} T_\lambda^1 f_2(x) - Rf(x) \\ & = \int_{|x-y| \leq 2\lambda^{-1/N}} e^{i\lambda P(x,y)} [e^{i\lambda r_N(x,y)} - 1] \\ & \quad \times k_\lambda^1(x-y) \varphi(x,y) f(y) dy \\ & = \sum_{j=0}^{\infty} \int_{2^{-j}\lambda^{-1/N} < |x-y| \leq 2^{-j+1}\lambda^{-1/N}} e^{i\lambda P(x,y)} [e^{i\lambda r_N(x,y)} - 1] \\ & \quad \times k_\lambda^1(x-y) \varphi(x,y) f(y) dy \\ & \equiv \sum_{j=0}^{\infty} T_{\lambda,j}^1 f_2(x). \end{aligned} \quad (27)$$

On $T_{\lambda,j}^1 f_2(x)$, by the properties of r_N and k , we have

$$|T_{\lambda,j}^1 f_2(x)| \leq C 2^{-jN} Mf(x). \quad (28)$$

So we have

$$|T_{\lambda,j}^1 f_2(x)| \leq C \sum_{j=0}^{\infty} 2^{-jN} |Mf(x)| + C |Rf(x)|. \quad (29)$$

By Theorem A and Lemma 9, we have

$$\|T_{\lambda,j}^1 f_2\|_{L^{p,\kappa}(\omega)} \leq C \|f\|_{L^{p,\kappa}(\omega)}. \quad (30)$$

Now, let us turn to estimate $T_{\lambda,j}^2 f_2(x)$. We consider the following two cases.

Case 1 ($\lambda \leq 1$). Similar to that estimate of T_{λ}^2 in Lemma 6 in [2], we have

$$|T_{\lambda}^2 f_2(x)| \leq CM(f)(x). \quad (31)$$

By Lemma 9 we have

$$\|T_{\lambda}^2 f_2\|_{L^{p,\kappa}(\omega)} \leq C \|f\|_{L^{p,\kappa}(\omega)}. \quad (32)$$

Case 2 ($\lambda > 1$). We choose $\varphi_0 \in C_0^{\infty}(\mathbb{R}^n)$ such that

$$\text{supp } \varphi_0 \subseteq \{x \in \mathbb{R}^n : 1 < |x| \leq 2\},$$

$$\phi_2(x) = \sum_{j=0}^{\infty} \varphi_0(2^{-j}x). \quad (33)$$

Let

$$k_{\lambda,j}^2(x) = k(x) \varphi_0(2^{-j} \lambda^{1/N} x). \quad (34)$$

Then

$$\begin{aligned} T_{\lambda,j}^2 f_2(x) &= \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k_{\lambda,j}^2(x-y) \varphi(x,y) f(y) dy, \\ &\sum_{j=0}^{\infty} \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k_{\lambda,j}^2(x-y) \varphi(x,y) f(y) dy \\ &\equiv \sum_{j=0}^{\infty} T_{\lambda,j}^2 f_2(x). \end{aligned} \quad (35)$$

For $T_{\lambda,j}^2$, by its definition, we can get

$$\begin{aligned} |T_{\lambda,j}^2 f_2(x)| &\leq C \int_{2^j \lambda^{-1/N} < |x-y| \leq 2^{j+1} \lambda^{-1/N}} \\ &\quad \times c \frac{1}{|x-y|^n} |f(y)| dy \leq CM(f)(x). \end{aligned} \quad (36)$$

The inequality (36) also can be seen in [2]; we omit the details here.

By Lemma 9, we have

$$\|T_{\lambda,j}^2 f_2\|_{L^{p,\kappa}(\omega)} \leq C \|f\|_{L^{p,\kappa}(\omega)}. \quad (37)$$

Therefore

$$I_2 \leq C \|f\|_{L^{p,\kappa}(\omega)}^p. \quad (38)$$

This finishes the proof of Theorem 1. $\square$

Proof of Theorem 2. It is sufficient to prove that there exists a constant $C > 0$ such that

$$\frac{1}{\omega(B)^{\kappa}} \int_B |T_{\lambda}^* f(x)|^p \omega(x) dx \leq C \|f\|_{L^{p,\kappa}(\omega)}^p. \quad (39)$$

Fix a ball $B = B(x_0, r_B)$ and decompose $f = f_1 + f_2$, with $f_1 = f \chi_{2B}$. Then we have

$$\begin{aligned} &\frac{1}{\omega(B)^{\kappa}} \int_B |T_{\lambda}^* f(x)|^p \omega(x) dx \\ &\leq C \left\{ \frac{1}{\omega(B)^{\kappa}} \int_B |T_{\lambda}^* f_1(x)|^p \omega(x) dx \right. \\ &\quad \left. + \frac{1}{\omega(B)^{\kappa}} \int_B |T_{\lambda}^* f_2(x)|^p \omega(x) dx \right\} \\ &= C \{J_1 + J_2\}. \end{aligned} \quad (40)$$

Using Lemmas 3 and 7, we get

$$\begin{aligned} J_1 &\leq C \frac{1}{\omega(B)^{\kappa}} \int_{2B} |f(x)|^p \omega(x) dx \\ &\leq C \|f\|_{L^{p,\kappa}(\omega)}^p \cdot \frac{\omega(2B)^{\kappa}}{\omega(B)^{\kappa}} \\ &\leq C \|f\|_{L^{p,\kappa}(\omega)}^p. \end{aligned} \quad (41)$$

We now estimate J_2 .

For each $m \in \mathbb{N}$ and $j = 1, \dots, g_m$, we get

$$a_{jm}(x) = \int_{\Sigma} \Omega(x, z) Y_{jm}(z) d\sigma_z, \quad (42)$$

where $\Omega(x, z) = |z|^n k(x, z)$. Then for a.e. $x \in \mathbb{R}^n$,

$$\Omega(x, z) = \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} a_{jm}(x) Y_{jm}(z'), \quad (43)$$

where $z' = z/|z|$ for any $z \in \mathbb{R}^n \setminus \{0\}$. By Lemma 10, we have that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |a_{jm}(x)| &= m^{-n}(m+n-2)^{-n} \left| \int_{\Sigma} \Omega(x, z) \Lambda^n Y_{jm}(z) d\sigma_z \right| \\ &= m^{-n}(m+n-2)^{-n} \left| \int_{\Sigma} \Lambda^n \Omega(x, z) Y_{jm}(z) d\sigma_z \right| \\ &\leq C(n) A m^{-2n}. \end{aligned} \quad (44)$$

By Lemma 10 again, we can verify that, for any $\epsilon > 0$, $N \in \mathbb{N}$, and a.e. $x \in \mathbb{R}^n$, if $|y-x| \geq \epsilon$, then

$$\begin{aligned} &\left| \sum_{m=1}^N \sum_{j=1}^{g_m} e^{i\lambda\Phi(x,y)} \frac{a_{jm}(x) Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) \right| \\ &\leq C(\epsilon) A |f_2(y)|. \end{aligned} \quad (45)$$

Therefore, from (43), (45), and the Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned}
 T_{\lambda}^* f_2(x) &= \lim_{\epsilon \rightarrow 0} \int_{|x-y| \geq \epsilon} e^{i\lambda\Phi(x,y)} k(x, x-y) \varphi(x, y) f_2(y) dy \\
 &= \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} \int_{|x-y| \geq \epsilon} e^{i\lambda\Phi(x,y)} \frac{a_{jm}(x) Y_{jm}((x-y)')}{|x-y|^n} \\
 &\quad \times \varphi(x, y) f_2(y) dy \\
 &= \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} a_{jm}(x) \int_{|x-y| \geq \epsilon} e^{i\lambda\Phi(x,y)} \frac{Y_{jm}((x-y)')}{|x-y|^n} \\
 &\quad \times \varphi(x, y) f_2(y) dy.
 \end{aligned} \tag{46}$$

We write

$$\begin{aligned}
 R_{jm} f_2(x) &= \int_{|x-y| \geq \epsilon} e^{i\lambda\Phi(x,y)} \frac{Y_{jm}((x-y)')}{|x-y|^n} \varphi(x, y) f_2(y) dy.
 \end{aligned} \tag{47}$$

It is easy to see that $R_{jm} f_2(x)$ is the oscillatory integral operator defined by (1). By Theorem 1 we have that R_{jm} is bounded on weighted Morrey spaces. Therefore, by (44) and the above discussion we have

$$J_2 \leq C \|f\|_{L^{p,\kappa}(\omega)}^p. \tag{48}$$

This finishes the proof of Theorem 2. $\square$

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