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Research Article

Oscillatory Criteria for Higher Order Functional Differential Equations with Damping

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We investigate a class of higher order functional differential equations with damping. By using a generalized Riccati transformation and integral averaging technique, some oscillation criteria for the differential equations are established.

1. Introduction

In this paper, we consider the following higher order functional differential equations with distributed deviating arguments of the form as follows:

$$x^{(n)}(t) + p(t)x^{(n-1)}(t) + \int_a^b q(t,\xi) \times f\left(x\left[g_1(t,\xi)\right],\ldots,x\left[g_m(t,\xi)\right]\right)d\mu(\xi) = 0, \quad t \geq t_0, \tag{1}$$

where $n \geq 2$ is an even number, $p(t) \in C^1([t_0,\infty),R_+)$, $q(t,\xi) \in C([t_0,\infty)\times[a,b],R_+)$, $g_i(t,\xi) \in C([t_0,\infty)\times[a,b],R)$, $\liminf_{t\to\infty,\xi\in[a,b]} g_i(t,\xi) = \infty$ for $i \in I_m = \{1,2,\ldots,m\}$, and $f(u_1,\ldots,u_m) \in C(R^m,R)$ has the same sign as $u_1,\ldots,u_m$; when they have the same sign, $\mu(\xi) \in C([a,b],R)$ is nondecreasing, and the integral of (1) is a Stieltjes one.

We restrict our attention to those solutions $x(t)$ of (1) which exist on same half liner $[t_\mu,\infty)$ with $\sup\{x(t) : t \geq T\} \neq 0$ for any $T \geq t_\mu$ and satisfy (1). As usual, a solution $x(t)$ of (1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all solutions are oscillatory.

In recent years, there has been an increasing interest in studying the oscillation behavior of solutions for the differential equations with distributed deviating arguments, and a number of results have been obtained (refer to [1–3] and their references). However, to the best of our knowledge, very little is known for the case of higher order differential equations with damping. The purpose of this paper is to establish some new oscillation criteria for (1) by introducing a class of functions $\Phi(t,s,r)$ defined in [2] and a generalized Riccati technique.

Firstly, we define the following two class functions.

We say that a function $\Phi = \Phi(t,u,v)$ belongs to the function class X, denoted by $\Phi \in X$, if $\Phi \in C(E,R)$, where $E = \{(t,u,v) : t_0 \leq v \leq u \leq t < \infty\}$, which satisfies $\Phi(t,t,v) = \Phi(t,v,v) = 0$, $\Phi(t,u,v) \neq 0$, $v < u < t$ and has the partial derivative $\partial\Phi/\partial u$ on E that is locally integrable with respect to u in E.

Let $D_0 = \{(t,u) : t_0 \leq u < t\}$, $D = \{(t,u) : t_0 \leq u \leq t\}$. We say that a function $H = H(t,u)$ belongs to the function class Y, denoted by $H \in Y$, if $H(t,t) = 0$ for $t \geq t_0$, $H(t,u) \neq 0$ in D_0, H has continuous partial derivative in D_0 with respect to t and s.

In order to prove the main theorems, we need the following lemmas.

Lemma 1 (see [4]). *Let $x(t) \in C^n([t_0,\infty),R_+)$, if $x^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_\mu,+\infty)$ for $t_\mu \geq t_0$, then there exists a $t_\sigma \geq t_\mu$, an integer $l(0 \leq l \leq n)$, with $n+l$ even for $x(t)x^{(n)}(t) \geq 0$ or $n+l$ odd for $x(t)x^{(n)}(t) \leq 0$; and for $t \geq t_\sigma \geq t_\mu$, $x(t)x^{(k)}(t) > 0$, $0 \leq k \leq l$, and $(-1)^{k+l}x(t)x^{(k)}(t) > 0$, $l \leq k \leq n$.*

Lemma 2 (see [5]). *If the function $x(t)$ is as in Lemma 1 and $x^{(n-1)}(t)x^{(n)}(t) \le 0$ for $t \ge t_\sigma \ge t_\mu$, then there exists a constant $\theta \in (0,1)$ such that for sufficiently large t, there exists a constant $M_\theta > 0$, satisfying*

$$x'(\theta t) \ge M_\theta t^{n-2} x^{(n-1)}(t). \tag{2}$$

Lemma 3 (see [3]). *Suppose that $x(t)$ is a nonoscillatory solution of* (1). *If*

$$\lim_{t\to\infty} \int_{T_0}^{t} \exp\left\{-\int_{T_0}^{s} p(\tau)\,d\tau\right\} ds = \infty, \quad T_0 \ge t_0, \tag{3}$$

then $x(t)x^{(n-1)}(t) > 0$ for any large t.

2. Main Results

Theorem 4. *Assume that* (3) *holds, and*

(A_1) *there exists a function $\sigma(t) \in C^n([t_0,\infty), R_+)$ such that $\sigma(t) = \min_{1\le i\le m}\{t, \inf_{\xi\in[a,b]} g_i(t,\xi)\}$, $\lim_{t\to\infty}\sigma(t) = \infty$. $\sigma'(t) \ge \sigma > 0$, $\sigma^{n-2}(t) \ge \rho^{n-2} > 0$, $(M_\theta/2)\sigma\rho^{n-2} =: k > 0$, where σ, ρ, and k are constants.*

(A_2) *$f(u_1,\dots,u_m)$ is nondecreasing with u_i, $i \in I_m$, and there exist constants $N > 0$ and $\lambda > 0$ such that*

$$\liminf_{u_{i_0}\to\infty} \frac{f(u_1,\dots,u_m)}{u_{i_0}} \ge \lambda, \quad u_i \ge N,\ i \ne i_0. \tag{4}$$

If there exists a function $\Phi(t,u,v) \in X$, such that for any $l(t) \in C^1([t_0,\infty), R_+)$, $r(t) \in C^1([t_0,\infty), R)$ and $T_0 \ge t_0$,

$$\limsup_{t\to\infty} \int_{T_0}^{t} \left\{\Phi^2(s,u,v)\psi(s) - \frac{\phi(s)}{k} \times \left[\Phi_s'(s,u,v) + \frac{l'(s)\Phi(s,u,v)}{2l(s)}\right]^2\right\} ds > 0, \tag{5}$$

where

$$\phi(t) = l(t)\exp\left\{-2k\int_{T_0}^{t} r(s)\,ds\right\},$$
$$\psi(t) = \phi(t)\left\{\lambda\int_a^b q(t,\xi)\,d\mu(\xi) + kr^2(t) - r'(t) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}\right\}. \tag{6}$$

Then (1) *is oscillatory.*

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. From the conditions of $g_i(t,\xi)$ and $f(u_1,\dots,u_m)$, there exists a $T_0 \ge t_0$, such that

$$x(t) > 0, \quad x[g_i(t,\xi)] > 0,$$
$$f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)]) > 0, \quad t \ge T_0,\ i \in I_m. \tag{7}$$

By Lemma 3, there exists a $t_1 \ge T_0$ such that $x^{(n-1)}(t) > 0$, $t \ge t_1$. Thus, we have

$$x^{(n)}(t) = -p(t)x^{(n-1)}(t) - \int_a^b q(t,\xi) f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)]) \times d\mu(\xi) \le 0, \quad t \ge t_1. \tag{8}$$

By Lemma 1, there exists a $t_2 > t_1$ such that $x'(t) > 0$, $t \ge t_2$. Further, by Lemma 2, there exist constant $M_\theta > 0$ and a $t_3 \ge t_2$, such that

$$x'\left[\frac{\sigma(t)}{2}\right] \ge M_\theta \sigma^{n-2}(t) x^{(n-1)}[\sigma(t)] \ge M_\theta \sigma^{n-2}(t) x^{(n-1)}(t), \quad t \ge t_3. \tag{9}$$

Set

$$y(t) = \phi(t)\left\{\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]} + r(t) + \frac{p(t)}{2k}\right\}, \tag{10}$$

then

$$y'(t) = \frac{\phi'(t)}{\phi(t)} y(t) + \phi(t) \times \left\{-\int_a^b q(t,\xi)\frac{f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)])}{x[\sigma(t)/2]} \times d\mu(\xi) - p(t)\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]} - \frac{\sigma'(t)x^{(n-1)}(t)x'[\sigma(t)/2]}{2x^2[\sigma(t)/2]} + r'(t) + \frac{p'(t)}{2k}\right\}. \tag{11}$$

In view of (A_1), (A_2) and the definition of $y(t)$, $\phi(t)$, we have

$$\begin{aligned} y'(t) &\le \frac{\phi'(t)}{\phi(t)} y(t) + \phi(t) \times \left\{-p(t)\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]} - \lambda\int_a^b q(t,\xi)\,d\mu(\xi) - k\left[\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]}\right]^2 + r'(t) + \frac{p'(t)}{2k}\right\} \\ &= -\frac{ky^2(t)}{\phi(t)} + \frac{l'(t)y(t)}{l(t)} - \psi(t), \end{aligned} \tag{12}$$

where $\psi(t) = \phi(t)\{\lambda\int_a^b q(t,\xi)d\mu(\xi) + kr^2(t) - r'(t) - p'(t)/2k - p^2(t)/4k\}$.

Multiplying (12) by $\Phi^2(t,u,v)$ and integrating from t_3 to t, we have

$$\begin{aligned}\int_{t_3}^{t} \Phi^2(s,u,v)\,\psi(s)\,ds &\le \int_{t_3}^{t} \Phi^2(s,u,v)\left[-y'(s)+\frac{l'(s)}{l(s)}y(s)\right]ds \\ &\quad -\int_{t_3}^{t} \Phi^2(s,u,v)\frac{ky^2(s)}{\phi(s)}ds.\end{aligned} \tag{13}$$

Integrating by parts and using integral averaging technique, we have

$$\begin{aligned}\int_{t_3}^{t} \Phi^2(s,u,v)\,\psi(s)\,ds &\le \int_{t_3}^{t} \frac{\phi(s)}{k}\left[\Phi_s'(s,u,v)+\frac{l'(s)\,\Phi(s,u,v)}{2l(s)}\right]^2 ds \\ &\quad -\int_{t_3}^{t}\left\{\Phi(s,u,v)\,y(s)\sqrt{\frac{k}{\phi(s)}}-\sqrt{\frac{\phi(s)}{k}}\right. \\ &\qquad \left.\times\left[\Phi_s'(s,u,v)+\frac{l'(s)\,\Phi(s,u,v)}{2l(s)}\right]\right\}^2 ds \\ &\le \int_{t_3}^{t} \frac{\phi(s)}{k}\left[\Phi_s'(s,u,v)+\frac{l'(s)\,\Phi(s,u,v)}{2l(s)}\right]^2 ds,\end{aligned} \tag{14}$$

thus

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_3}^{t}\left\{\Phi^2(s,u,v)\,\psi(s)-\frac{\phi(s)}{k}\right. \\ \left.\times\left[\Phi_s'(s,u,v)+\frac{l'(s)\,\Phi(s,u,v)}{2l(s)}\right]^2\right\}ds \le 0,\end{aligned} \tag{15}$$

which contradicts (5). This completes the proof of Theorem 4. □

If we choose $\Phi(t,u,v) = \sqrt{H_1(t,u)H_2(u,v)}$, where H_1, $H_2 \in Y$. By Theorem 4, we have the following results.

Corollary 5. *Assume that* (3), (A_1), *and* (A_2) *hold. If there exist* $H_1, H_2 \in Y$ *such that for each* $T_0 \ge t_0$,

$$\begin{aligned}\limsup_{t\to\infty}\int_{T_0}^{t} H_1(s,u)\,H_2(u,v) \\ \times\left\{\psi(s)-\frac{\left[h_1(s,u)+h_2(u,v)\right]^2}{4k}\right\}ds > 0,\end{aligned} \tag{16}$$

where h_1 *and* h_2 *are defined by* $\partial H_1(t,u)/\partial u = h_1(t,u)H_1(t,u)$, $\partial H_2(u,v)/\partial u = h_2(u,v)H_2(u,v)$, *and*

$$\psi(t) = \lambda\int_a^b q(t,\xi)\,d\mu(\xi) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}. \tag{17}$$

Then (1) *is oscillatory.*

If we choose $l(t) = 1$, $r(t) = 0$, and let $\Phi(t,u,v) = (t-u)(u-v)^{\alpha}$, $\alpha > 1/2$, by Theorem 4, we have the following corollary.

Corollary 6. *Assume that* (3), (A_1), *and* (A_2) *hold. If there exists a constant* $\alpha > 1/2$ *such that for each* $T_0 \ge t_0$,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_{T_0}^{t}(s-u)^2(u-v)^{2\alpha} \\ \times k\psi(s)\,ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)},\end{aligned} \tag{18}$$

where $\psi(t)$ *is defined as in Corollary 5. Then* (1) *is oscillatory.*

Theorem 7. *Assume that* (3) *holds, and*

(A_3) *there exist functions* $\gamma_i(t) \in C^n([t_0,\infty),R_+)$, *such that* $\gamma_i(t) \le \min_{1\le i\le m}\{t, \inf_{\xi\in[a,b]} g_i(t,\xi)\}$, $\lim_{t\to\infty}\gamma_i(t) = \infty$, $\gamma_i'(t) \ge \gamma_i > 0$, *where* γ_i *are constants,* $i \in I_m$;

(A_4) *there exist constants* $\lambda_i \in [0,1]$ *and* $\overline{\lambda} > 0$, *such that*

$$\sum_{i=1}^{m}\lambda_i\gamma_i(t) > 0, \qquad \frac{f(u_1,\ldots,u_m)}{\lambda_1u_1+\cdots+\lambda_mu_m} \ge \overline{\lambda}. \tag{19}$$

$\gamma_i^{n-2}(t) \ge \rho_i^{n-2} \ge 0$, $(M_\theta/2)\sum_{i=1}^{m}\lambda_i\gamma_i\rho_i^{n-2}(t) =: k > 0$, *where* ρ_i *and* k *are constants,* $i \in I_m$.

If there exists a function $\Phi \in X$, such that for any $l(t) \in C^1([t_0,\infty),R_+)$, $r(t) \in C^1([t_0,\infty),R)$, and $T_0 \ge t_0$, and (5) holds, where $\phi(t)$ is defined as in Theorem 4:

$$\begin{aligned}\psi(t) = \phi(t)\left\{\overline{\lambda}\int_a^b q(t,\xi)\,d\mu(\xi)\right. \\ \left. + kr^2(t) - r'(t) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}\right\}.\end{aligned} \tag{20}$$

Then (1) is oscillatory.

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. Similar to the proof of Theorem 4, there exists a $T_0 \ge t_0$, such that $x[g_i(t,\xi)] > 0$, $x[\gamma_i(t)] > 0$, $x'(t) > 0$, $f(x[g_1(t,\xi)],\ldots,x[g_m(t,\xi)]) > 0$, $x^{(n-1)}(t) > 0$, and $x^{(n)}(t) \le 0$, for $t \ge T_0$, $i \in I_m$. Set

$$y(t) = \phi(t)\left\{\frac{x^{(n-1)}(t)}{\sum_{i=1}^{m}\lambda_i x\left[\gamma_i(t)/2\right]} + r(t) + \frac{p(t)}{2k}\right\}, \tag{21}$$

then

$$y'(t) = \frac{\phi'(t)}{\phi(t)} y(t) + \phi(t) \times \left\{ -p(t) \frac{x^{(n-1)}(t)}{\sum_{i=1}^{m} \lambda_i x[\gamma_i(t)/2]} - \int_a^b q(t,\xi) \frac{f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)])}{\sum_{i=1}^{m} \lambda_i x[\gamma_i(t)/2]} \times d\mu(\xi) - \frac{x^{(n-1)}(t)}{2\{\sum_{i=1}^{m} \lambda_i x[\gamma_i(t)/2]\}^2} \times \sum_{i=1}^{m} \lambda_i x'[\gamma_i(t)] \gamma_i'(t) + r'(t) + \frac{p'(t)}{2k} \right\}. \tag{22}$$

In view of (A_3), (A_4) and the definition of $y(t)$, $\phi(t)$, we have

$$y'(t) \le -\frac{ky^2(t)}{\phi(t)} + \frac{l'(t)\, y(t)}{l(t)} - \psi(t). \tag{23}$$

The following proof is similar to Theorem 4, and we omit the details. This completes the proof of Theorem 7. □

Similar to Corollaries 5 and 6, we have the following corollaries.

Corollary 8. *Assume that* (3), (A_3), *and* (A_4) *hold. If there exist* $H_1, H_2 \in Y$ *such that for each* $T_0 \ge t_0$, *and* (16) *holds, where* h_1, h_2 *are defined as in Corollary 5:*

$$\psi(t) = \overline{\lambda} \int_a^b q(t,\xi)\, d\mu(\xi) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}. \tag{24}$$

Then (1) *is oscillatory.*

Corollary 9. *Assume that* (3), (A_3), *and* (A_4) *hold. If there exists a constant* $\alpha > 1/2$ *such that for each* $T_0 \ge t_0$, *and* (18) *holds. where* $\psi(t)$ *is defined as in Corollary 8, then* (1) *is oscillatory.*

For the case of the function $f(u_1, \ldots, u_m)$ with monotonicity, we have the following theorem.

Theorem 10. *Assume that* (3), (A_3) *hold, and*

(A_5) *there exist* $\partial f/\partial u_i$ *and* $\partial f/\partial u_i \ge \lambda_i \ge 0$, *such that* $\sum_{i=1}^{m} \lambda_i \gamma_i'(t) > 0$, *where* λ_i *is constants.* $\gamma_i^{n-2} \ge \rho_i^{n-2} \ge 0$, $(M_\theta/2) \sum_{i=1}^{m} \lambda_i \gamma_i \rho_i^{n-2}(t) =: k > 0$, *in which* ρ_i *and* k *are constants,* $i \in I_m$.

If there exists a function $\Phi \in X$, *such that for any* $l(t) \in C^1([t_0,\infty), R_+)$, $r(t) \in C^1([t_0,\infty), R)$, *and* (5) *holds, where* $\phi(t)$ *is defined as in Theorem 4:*

$$\psi(t) = \phi(t) \left\{ \int_a^b q(t,\xi)\, d\mu(\xi) + kr^2(t) - r'(t) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k} \right\}. \tag{25}$$

Then (1) *is oscillatory.*

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. Similar to the proof of Theorem 4, there exists a $T_0 \ge t_0$, when $t \ge T_0$, and we have $x[g_i(t,\xi)] > 0$, $x[\gamma_i(t)] > 0$, $x'(t) > 0$, $f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)]) > 0$, $x^{(n-1)}(t) > 0$, and $x^{(n)}(t) \le 0$, $i \in I_m$. Set

$$y(t) = \phi(t) \left\{ \frac{x^{(n-1)}(t)}{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])} + r(t) + \frac{p(t)}{2k} \right\}, \tag{26}$$

then

$$y'(t) = \frac{\phi'(t)}{\phi(t)} y(t) + \phi(t) \times \left\{ -p(t) \frac{x^{(n-1)}(t)}{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])} - \int_a^b q(t,\xi) \frac{f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)])}{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])} \times d\mu(\xi) - \frac{x^{(n-1)}(t)}{2\{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])\}^2} \times \sum_{i=1}^{m} \lambda_i x'[\gamma_i(t)] \gamma_i'(t) + r'(t) + \frac{p'(t)}{2k} \right\}. \tag{27}$$

In view of (A_3), (A_5) and the definition of $y(t)$, $\phi(t)$, we have

$$y'(t) \le -\frac{ky^2(t)}{\phi(t)} + \frac{l'(t)\, y(t)}{l(t)} - \psi(t). \tag{28}$$

The following proof is similar to Theorem 4, we omit the details. This completes the proof of Theorem 10. □

Similar to Corollaries 5 and 6, we have the following corollaries.

Corollary 11. *Assume that* (3), (A_3), *and* (A_5) *hold. If there exist* $H_1, H_2 \in Y$ *such that for each* $T_0 \geq t_0$, *and* (16) *holds, where* h_1, h_2 *are defined as in Corollary 5:*

$$\psi(t) = \int_a^b q(t,\xi)\, d\mu(\xi) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}. \tag{29}$$

Then (1) *is oscillatory.*

Corollary 12. *Assume that* (3), (A_3), *and* (A_5) *hold. If there exists a constant* $\alpha > 1/2$ *such that for each* $T_0 \geq t_0$, *and* (18) *holds. where* $\psi(t)$ *is defined as in Corollary 11. Then* (1) *is oscillatory.*

3. Examples

Example 13. Consider the following equation

$$x^{(4)}(t) + \frac{1}{t}x^{(3)}(t)\int_0^{\pi/2} \frac{\xi}{t^2}\frac{2x(t+\sin\xi)}{2-\exp\left(-x^2(t+\cos\xi)\right)}d\xi = 0, \quad t \geq 1, \tag{30}$$

where $u_1 = x(t+\cos\xi)$, $u_2 = x(t+\sin\xi)$, obviously $f(u_1,u_2)/u_2 \geq 1 = \lambda$. Choosing $\sigma = 1$, $\rho = 1$, then $k = M_\theta/2$, and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_0^t (s-u)^2(u-v)^{2\alpha}\left(\frac{M_\theta}{2s^2}+\frac{1}{4s^2}\right)ds \\
&\quad = \left(\frac{M_\theta}{2}+\frac{1}{4}\right)\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_0^t (s-u)^2(u-v)^{2\alpha}\frac{1}{s^2}ds \\
&\quad = \left(\frac{M_\theta}{2}+\frac{1}{4}\right)\frac{1}{\alpha(2\alpha+1)(2\alpha-1)}.
\end{aligned} \tag{31}$$

Thus, there exists a constant $\alpha > 1/2$, such that $(M_\theta/2+1/4) > \alpha^2$, that is,

$$\left(\frac{M_\theta}{2}+\frac{1}{4}\right)\frac{1}{\alpha(2\alpha+1)(2\alpha-1)} > \frac{\alpha}{(2\alpha+1)(2\alpha-1)}. \tag{32}$$

By Corollary 6, then (30) is oscillatory.

Example 14. Consider the following equation

$$\begin{aligned}
&x^{(4)}(t) + \frac{1}{t}x^{(3)}(t) + \int_0^1 \frac{\xi}{t^2} \\
&\quad \times \left[x(t-\xi) + x(t+\xi) + x^3(t-\xi) + x^5(t+\xi)\right]d\xi = 0,
\end{aligned} \tag{33}$$

where $f(u_1,u_2) = u_1 + u_2 + u_1^3 + u_2^5$, obviously $\partial f/\partial u_1 = 1 + 3u_1^2 \geq 1$, $\partial f/\partial u_2 = 1 + 5u_2^4 \geq 1$. Choosing $\lambda_i = 1$, $\gamma_i = 1$, $\rho_i = 1$, and $i = 1,2$, then $k = M_\theta$. By Corollary 12, then (33) is oscillatory.

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