Research Article

Energy Scattering for Schrödinger Equation with Exponential Nonlinearity in Two Dimensions

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Received 9 January 2013; Accepted 24 February 2013

Academic Editor: Baoxiang Wang

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When the spatial dimensions \( n = 2 \), the initial data \( u_0 \in H^1 \), and the Hamiltonian \( H(u_0) \leq 1 \), we prove that the scattering operator is well defined in the whole energy space \( H^1(\mathbb{R}^2) \) for nonlinear Schrödinger equation with exponential nonlinearity \( (e^{\lambda |u|^2} - 1)u \), where \( 0 < \lambda < 4\pi \).

1. Introduction

We consider the Cauchy problem for the following nonlinear Schrödinger equation:

\[
\begin{align*}
    iu_t + \Delta u &= f(u), \\
    f(u) &= (e^{\lambda |u|^2} - 1)u,
\end{align*}
\]

in two spatial dimensions with initial data \( u_0 \in H^1 \) and \( 0 < \lambda < 4\pi \). Solutions of the above problem satisfy the conservation of mass and Hamiltonian:

\[
\begin{align*}
    M(u_0) &:= \int_{\mathbb{R}^2} |u|^2 \, dx = M(u_0), \\
    H(u_0) &:= \int_{\mathbb{R}^2} \left( |\nabla u|^2 + F(u) \right) \, dx = H(u_0),
\end{align*}
\]

where

\[
F(u) = \frac{1}{\lambda} \left( e^{\lambda |u|^2} - \lambda |u|^2 - 1 \right).
\]

Nakamura and Ozawa [1] showed the existence and uniqueness of the scattering operator of (1) with (2). Then, Wang [2] proved the smoothness of this scattering operator. However, both of these results are based on the assumption of small initial data \( u_0 \). In this paper, we remove this assumption and show that for arbitrary initial data \( u_0 \in H^1(\mathbb{R}^2) \) and \( H(u_0) \leq 1 \), the scattering operator is always well defined.

Wang et al. [3] proved the energy scattering theory of (1) with \( f(u) = (e^{\lambda |u|^2} - 1 - \lambda |u|^2 - (\lambda^2/2)|u|^4)u \), where \( \lambda \in \mathbb{R} \) and the spatial dimension \( n = 1 \). Ibrahim et al. [4] showed the existence and asymptotic completeness of the wave operators for (1) with \( f(u) = (e^{\lambda |u|^2} - 1 - \lambda |u|^2)u \) when the spatial dimensions \( n = 2, \lambda = 4\pi \), and \( H(u_0) \leq 1 \). Under the same assumptions as [4], Colliander et al. [5] proved the global well-posedness of (1) with (2).

Theorem 1. Assume that \( u_0 \in H^1(\mathbb{R}^2), H(u_0) \leq 1 \), and \( \lambda = 4\pi \). Then problem (1) with (2) has a unique global solution \( u \) in the class \( C(\mathbb{R}, H^1(\mathbb{R}^2)) \).

Remark 2. In fact, by the proof in [5], the global well-posedness of (1) with (2) is also true for \( 0 < \lambda \leq 4\pi \).

In this paper, we further study the scattering of this problem. Note that \( f(u) = (e^{\lambda |u|^2} - 1)u = \sum_{k=1}^{\infty} (\lambda^k/k!) |u|^{2k} u \). Nakanoishi [6] proved the existence of the scattering operators in the whole energy space \( H^1(\mathbb{R}^2) \) for (1) with \( f(u) = |u|^p u \) when \( p > 2 \). Then, Killip et al. [7] and Dodson [8] proved the existence of the scattering operators in \( L^2(\mathbb{R}^2) \) for (1) with \( f(u) = |u|^2 u \). Inspired by these two works, we use the concentration compactness method, which was introduced by Kenig and Merle in [9], to prove the existence of the scattering operators for (1) with (2).
For convenience, we write (1) and (2) together; that is,
\[ iu_t + \Delta u = f(u) := (e^{\lambda|u|^2} - 1)u, \ u(0, x) = u_0, \]  
where \( u_0 \in H^1(\mathbb{R}^2) \) and \( 0 < \lambda < 4\pi \). Our main result is as follows.

**Theorem 3.** Assume that the initial data \( u_0 \in H^1(\mathbb{R}^2) \), \( H(u_0) \leq 1 \), and \( 0 < \lambda < 4\pi \). Let \( u \) be a global solution of (5). Then
\[ \|u\|_{L^2_tL^q_x(\mathbb{R} \times \mathbb{R}^2)} < \infty. \]  

In Section 2, Lemma 9 will show us that Theorem 3 implies the following scattering result.

**Theorem 4.** Assume that the initial data \( u_0 \in H^1(\mathbb{R}^2) \), \( H(u_0) \leq 1 \), and \( 0 < \lambda < 4\pi \). Then the solution of (5) is scattering in the energy space \( H^1(\mathbb{R}^2) \).

We will prove Theorem 3 by contradiction in Section 5. In Section 2, we give some nonlinear estimates. In Section 3, we prove the stability of solutions. In Section 4, we give a new profile decomposition for \( H^1 \) sequence which will be used to prove concentration compactness.

Now, we introduce some notations:
\[ G(u) := \bar{u}f(u) - F(u) = e^{\lambda|u|^2}|u|^2 - \frac{1}{\lambda}(e^{\lambda|u|^2} - 1) \]
\[ = \sum_{k=1}^{\infty} \lambda^{2k+1} \frac{\lambda^k|u|^{2k+2}}{(k+1)!}, \]  
\[ E = E(u; t) := M(u; t) + H(u; t). \]  

We define
\[ \|u\|_{H^s_x(\mathbb{R}^2)} := \left\| (I - \Delta)^{s/2} u \right\|_{L^2_x(\mathbb{R}^2)}, \]
\[ \|u\|_{H^1(\mathbb{R}^2)} := \left\| -\Delta^{1/2} u \right\|_{L^2_x(\mathbb{R}^2)}. \]  

For Banach space \( X = H^s_x(\mathbb{R}^2) \), \( L^p_x(\mathbb{R}^2) \), or \( L^q(\mathbb{R}^2) \), we denote
\[ \|u\|_{L^p(\mathbb{R}^2)} := \left( \int_{\mathbb{R}^2} |u(x)|^p \, dx \right)^{1/p}. \]  

When \( q = r \), \( L^q_xL^r_x \) is abbreviated to \( L^q_x \). When \( q \neq r \) or when the domain \( \mathbb{R} \times \mathbb{R}^2 \) is replaced by \( \mathbb{R} \times \mathbb{R}^2 \), we make the usual modifications. Specially, we denote
\[ S(u) := \|u\|_{L^q_xL^r_x(\mathbb{R} \times \mathbb{R}^2)}. \]

For \( t_0 \in \mathbb{R} \), we split \( S(u) = S_{\leq t_0} + S_{> t_0} \), where
\[ S_{< t_0} := \int_{-\infty}^{t_0} \left| \int_{\mathbb{R}^2} |u(t, x)|^4 \, dx \right| \, dt, \]
\[ S_{> t_0} := \int_{t_0}^{\infty} \left| \int_{\mathbb{R}^2} |u(t, x)|^4 \, dx \right| \, dt. \]  

For any two Banach spaces \( X \) and \( Y \), \( \|\cdot\|_{X \rightarrow Y} := \max\{\|\cdot\|_X, \|\cdot\|_Y\} \). \( C \) denotes positive constant. If \( C \) depends upon some parameters, such as \( \lambda \), we will indicate this with \( C(\lambda) \).

**Remark 5.** Note that \( 0 < \lambda < 4\pi \) in Theorem 3; we only need to prove the result for \( 0 < \lambda < 4(1-4\epsilon)^2\pi \), \( \epsilon \in (0, 1/8) \). Hence, we always suppose that \( 0 < \lambda < 4(1-4\epsilon)^2\pi \) in the context.

Moreover, we always suppose that the initial data \( u_0 \) of (5) satisfies \( u_0 \in H^1(\mathbb{R}^2) \) and \( H(u_0) \leq 1 \).

**2. Nonlinear Estimates**

In order to estimate (2), we need the following Trudinger-type inequality.

**Lemma 6** (see [10]). Let \( \lambda \in [0, 4\pi) \). Then for all \( u \in H^1(\mathbb{R}^2) \) satisfying \( \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1 \), one has
\[ \int_{\mathbb{R}^2} \left( e^{\lambda|u|^2} - 1 \right) \, dx \leq C(\lambda) \|u\|_{L^2(\mathbb{R}^2)}^2. \]  

Note that for all \( \alpha \geq 1 \),
\[ \left( e^{\lambda|u|^2} - 1 \right)^\alpha \leq e^{\alpha \lambda|u|^2} - 1. \]  

By Lemma 6 and Hölder inequality, for \( \lambda \in [0, 4\pi) \) and for all \( \beta \geq 0 \), we have
\[ \int_{\mathbb{R}^2} \left( e^{\lambda|u|^2} - 1 \right) |u|^\beta \, dx \leq \left\| e^{\lambda|u|^2} - 1 \right\|_{L^{1/\beta}(\mathbb{R}^2)} \|u\|_{L^\beta_x(\mathbb{R}^2)}^\beta \]
\[ \leq \|u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^\beta_x(\mathbb{R}^2)}^\beta \]
\[ \leq C(\lambda) \|u\|_{L^2(\mathbb{R}^2)}^2. \]

**Lemma 7** (Strichartz estimates). For \( s = 0 \) or 1,
\[ \|e^{-\lambda|u|^2} u(t_0)\|_{L^{1/\gamma}(\mathbb{R} \times \mathbb{R}^2)} \leq C \|u(t_0)\|_{H^{1/\gamma}(\mathbb{R}^2)}, \]
\[ \|e^{-\lambda|u|^2} \int_t^{t_0} f(u) \, dr\|_{L^{1/\gamma}(\mathbb{R} \times \mathbb{R}^2)} \leq C \|f(u)\|_{L^{1/\gamma}(\mathbb{R} \times \mathbb{R}^2)}. \]  

**Lemma 8** (see [3, Proposition 2.3]). Let \( 1 < r < p < \infty \) be fixed indices. Then for any \( q + p = 1 \),
\[ \|u\|_{L^q(\mathbb{R}^2)} \leq C(\lambda) \|u\|_{L^p(\mathbb{R}^2)}^q \|u\|_{E^\lambda(\mathbb{R}^2)}^{1-p/q}. \]
As shown in [6, 11], to obtain the scattering result, it suffices to show that any finite energy solution has a finite global space-time norm. In fact, if Theorem 3 is true, we have the following theorem.

**Lemma 9** (Theorem 3 implies Theorem 4). Let $u$ be a global solution of (5), $H(u) \leq 1$, and $\|u\|_{L^6_x([0,\infty) \times \mathbb{R}^2)} < \infty$. Then, for all admissible pairs, we have

$$\|u\|_{L^6([\mathbb{R}^2 \times \mathbb{R}^2])} < \infty.$$  \hspace{1cm} (19)

Moreover, there exist $u_k \in H^1$ such that

$$\lim_{k \to \infty} \|u_k - e^{i\lambda \Delta} u_k\|_{H^1(\mathbb{R}^2)} = 0.$$  \hspace{1cm} (20)

**Proof.** Defining $X(I) = L^{2/(1-2\epsilon)}(I; H^1(\mathbb{R}^2))$, $Y(I) = L^4(I; H^1(\mathbb{R}^2))$, by Strichartz estimates, (14) and (15),

$$\|u\|_{X(I) \cap Y(I)} \leq C(E) + \|u\|_{L^6_x(L^8_{t,x})}^{1/2} \|u\|_{Y(I)}$$

Proof. Defining $X(I) = L^{2/(1-2\epsilon)}(I; H^1(\mathbb{R}^2))$, $Y(I) = L^4(I; H^1(\mathbb{R}^2))$, by Strichartz estimates, (14) and (15),

$$\|u\|_{X(I) \cap Y(I)} \leq C(E) + \|u\|_{L^6_x(L^8_{t,x})}^{1/2} \|u\|_{Y(I)}$$

Using the same way as in Bourgain [12], one can split $\mathbb{R}$ into finitely many pairwise disjoint intervals:

$$\mathbb{R} = \bigcup_{j=1}^{J} I_j, \quad \|u\|_{L^6_x(I_1 \times \mathbb{R}^2)} \leq \eta, \quad C(E) \left(\eta^2 + \eta^8\right) \leq \frac{1}{2}.$$  \hspace{1cm} (22)

By (21),

$$\|u\|_{L^{2/(1-2\epsilon)}(I_1; H^1(\mathbb{R}^2))} \leq C(E).$$  \hspace{1cm} (23)

Since $\|u\|_{L^{2}(0,\infty; H^1)} \leq C(E)$ and $\epsilon \in (0, 1/8)$ can be chosen small arbitrarily, by interpolation,

$$\|u\|_{L^{p_1}(0,\infty; H^{1} ; x)} \leq C(E),$$  \hspace{1cm} (24)

for all admissible pairs $j = 1, 2, \ldots, J$. The desired result (19) follows.

Let

$$u_\pm = u_0 - i \int_0^{\infty} e^{-i\tau \Delta} f(u(\tau)) \, d\tau.$$  \hspace{1cm} (25)

By (19) and (21),

$$\|u_\pm\|_{H^1(\mathbb{R}^2)} \leq 1 + \|u\|_{Y(0,\infty)}^3 + \|u\|_{X(0,\infty)}^{1+8\epsilon} Y(0,\infty) \leq 1.$$  \hspace{1cm} (26)

Thus, $u_k$ were well defined and belong to $H^1$. Since

$$\|u - e^{i\lambda \Delta} u_k\|_{H^1(\mathbb{R}^2)} \leq 1,$$

we must have

$$\lim_{k \to \infty} \|u - e^{i\lambda \Delta} u_k\|_{H^1(\mathbb{R}^2)} \leq \lim_{k \to \infty} \left(\|u\|_{Y(0,\infty)}^3 + \|u\|_{X(0,\infty)}^{1+8\epsilon} Y(0,\infty)\right) = 0.$$  \hspace{1cm} (28)

(20) was proved. \hfill \Box

### 3. Stability

**Lemma 10** (stability). For any $A > 0$ and $\sigma > 0$, there exists $\delta > 0$ with the following property: suppose that $u : [0, \infty) \times \mathbb{R}^2 \to C$ satisfies $\|V u\|_{L^2_{t,x}} \leq 1$ for all $t \in [0, \infty)$, $\|u\|_{L^4_t([0,\infty) \times \mathbb{R}^2)} \leq A$ and approximately solves (5) in the sense that

$$\int_0^t e^{(t-\tau)\Delta} \left(\Delta u + \Delta u - f(u)\right)(\tau) \, d\tau \|_{L^2_{t,x}(L^4_{t,x})([0,\infty) \times \mathbb{R}^2)} \leq \delta.$$  \hspace{1cm} (29)

Then for any initial data $v(0) \in H^1(\mathbb{R}^2)$ satisfying $H(v(0)) \leq 1$ and $\|u(0) - v(0)\|_{L^2_{t,x}} \leq \delta$, there is a unique global solution $v$ to (5) satisfying $\|u - v\|_{L^4_t([0,\infty) \times \mathbb{R}^2)} \leq \sigma$.

**Proof.** Denote $v = u + w$, then

$$i w + \Delta w = (f(u + w) - f(u)) - (i u + \Delta u - f(u))$$  \hspace{1cm} (30)

and $\|w(0)\|_{L^2_{t,x}} \leq \delta$. Let $X = L^{\infty}_{t,x} \cap L^4_{t,x} \cap L^{2/(1-2\epsilon)}_{t,x}$. By the similar estimates as (21), we have

$$\|w\|_{X} \leq \delta + \|\int_0^t e^{(t-\tau)\Delta} (f(u + w) - f(u))(\tau) \, d\tau\|_{X} + \delta$$

$$\leq 2\delta + C \|u + w\|^2 \leq 2\delta + C \left(\|e^{i\lambda \Delta} u\|^2 - \lambda (u + w)^2\right)$$

$$+ C \left(\|e^{i\lambda \Delta} w\|^2 - \lambda u^2\right)$$

$$\leq 1 + \|u\|_{Y(0,\infty)}^2 + \|u\|_{X(0,\infty)}^{1+8\epsilon} Y(0,\infty).$$
\[ \leq 2\delta + C\|u + w\|_2^2 |w| + |uw|_2^2 \|w\|_{L^4/3}^2 \\
+ C\left( e^{\lambda u} |u| - \lambda |u|^2 - 1 \right) |w| \\
+ \left( e^{\lambda |u|} - \lambda |u| - 1 \right) \|w\|_{L^2(1,\delta)}^2 \\
+ C\left( e^{\lambda (u^2 + w^2) - 1} \right) |w| + |u| |w| \|w\|_{L^4/3}^2 \\
\leq 2\delta + C\left( \|u\|_{L^2}^2 + \|u + w\|_{L^2}^2 \right) \|w\|_{L^2}^2 \\
+ C\left( \|u\|_{L^4/3}^2 + \|u + w\|_{L^4/3}^2 \right) \|w\|_{L^2(1,\delta)}^2 \\
\leq 2\delta + C\left( \|u\|_{L^2}^2 + \|u + w\|_{L^2}^2 \right) \|w\|_{L^2}^2 + \|w\|_{L^4/3}^2 \\
\leq 2\delta + C\left( \|u\|_{L^2}^2 + \|u + w\|_{L^2}^2 \right) \|w\|_{L^2}^2 + \|w\|_{L^4/3}^2 \\
(31) \]

Then we subdivide the time interval \([0, \infty)\) into finite sub-intervals \(I_j = [t_j, t_{j+1}]\), \(j = 1, \ldots, J\), \(J = f(A)\) such that
\[
C\left( \|u\|_{L^2}^2 + \|u + w\|_{L^2}^2 \right) \|w\|_{L^2}^2 < \frac{1}{4}
(32)

for each \(j\). Let \(\delta = \delta(A, f)\) be small such that
\[
C\delta^2 \delta \ll 1, \quad 8\delta \delta < \sigma.
(33)

Then by (31) on \(I_1 \times \mathbb{R}^2\), we have \(\|w\|_{X(I_1, \mathbb{R}^2)} \leq 8\delta\) and
\[
\|w(t_2)\|_{L^4/3(\mathbb{R}^2)} \leq \|w(t_2)\|_{L^4/3(\mathbb{R}^2)} \\
+ \int_{t_1}^{t_2} e^{it_1 - t_1} \|f(u + w) - f(u)\|_4 dt + \delta \\
\leq 2\delta + \delta + C\left( \|u\|_{L^2}^2 + \|u + w\|_{L^2}^2 \right) \|w\|_{X(I_1, \mathbb{R}^2)} \\
+ \|w\|_{L^4/3(\mathbb{R}^2)}^2 \|w\|_{X(I_1, \mathbb{R}^2)} \\
\leq 2\delta + 4 \cdot 8\delta \leq 8^2 \delta.
(34)

Using the same analysis as above, we can get \(\|w\|_{X(I_2, \mathbb{R}^2)} \leq 8^3 \delta\). Iterating this for \(I_2, I_3, \ldots, I_J\), we obtain \(\|w\|_{X([0, \infty), \mathbb{R}^2)} \leq 8^3 \delta \leq \sigma\); the desired result was obtained.

\section*{4. Linear Profile Decomposition}

In this section, we will give the linear profile decomposition for Schrödinger equation in \(H^1(\mathbb{R}^2)\). First, we give some definitions and lemmas.

\begin{definition}[symmetry group, \cite{13}]
For any phase \(\theta \in \mathbb{R}/2\pi \mathbb{Z}\), position \(x_0 \in \mathbb{R}^2\), frequency \(\xi_0 \in \mathbb{R}^2\), and scaling parameter \(\lambda > 0\), we define the unitary transformation \(g_{\theta, x_0, \lambda} \colon L^2_x(\mathbb{R}^2) \to L^2_x(\mathbb{R}^2)\) by the formula
\[
g_{\theta, x_0, \lambda} f(t) = \frac{1}{\lambda} e^{i \theta} e^{ix\xi_0} f\left( \frac{x - x_0}{\lambda} \right).
(35)
\end{definition}

We let \(G\) be the collection of such transformations; this is a group with identity \(g_{\theta, 0, 1, \lambda} = 1\), \(g_{\theta, -x_0, \lambda} = g_{\theta, x_0, \lambda\lambda}^{-1}\), and group law
\[
g_{\theta_1, x_0_1, \lambda_1} g_{\theta_2, x_0_2, \lambda_2} = g_{\theta_1 + \theta_2 - x_0_2, \lambda_1 + \lambda_2} (\lambda_1, \lambda_2).
(36)

If \(u : I \times \mathbb{R}^2 \to C\) is a function, we define \(T_{g_{\theta, x_0, \lambda}} u : \lambda^2 I \times \mathbb{R}^2 \to C\), where \(\lambda^2 I := \{\lambda^2 t : t \in I\}\) by the formula
\[
T_{g_{\theta, x_0, \lambda}} u(t, x) := \frac{1}{\lambda} e^{i \theta} e^{ix\xi_0} e^{-i\lambda t} u\left( \frac{t}{\lambda^2}, \frac{x - x_0 - 2\xi_0 t}{\lambda} \right),
(37)
\]
or equivalently
\[
T_{g_{\theta, x_0, \lambda}} u(t) = g_{\theta, -t\xi_0, \lambda} g_{\theta, x_0, \lambda} u(t),
(38)
\]
if \(g \in G\), we can easily prove that \(M(T_{g} u) = M(u)\) and \(S(T_{g} u) = S(u)\).

\begin{definition}[enlarged group, \cite{13}]
For any phase \(\theta \in \mathbb{R}/2\pi \mathbb{Z}\), position \(x_0 \in \mathbb{R}^2\), frequency \(\xi_0 \in \mathbb{R}^2\), scaling parameter \(\lambda > 0\), and time \(t_0\), we define the unitary transformation \(g_{\theta, x_0, \lambda, t_0} \colon L^2_x(\mathbb{R}^2) \to L^2_x(\mathbb{R}^2)\) by the formula
\[
g_{\theta, x_0, \lambda, t_0} = g_{\theta, x_0, \lambda} e^{it_0/\lambda},
(39)
\]
or in other words
\[
g_{\theta, x_0, \lambda, t_0} f(x) := \frac{1}{\lambda} e^{i \theta} e^{ix\xi_0} f\left( \frac{x - x_0}{\lambda} \right).
(40)
\]
Let \(G'\) be the collection of such transformations. We also let \(G'\) act on global space-time function \(u : \mathbb{R} \times \mathbb{R}^2 \to C\) by defining
\[
T_{g_{\theta, x_0, \lambda, t_0}} u(t, x) := \frac{1}{\lambda} e^{i \theta} e^{ix\xi_0} e^{-i\lambda t} u\left( \frac{t}{\lambda^2}, \frac{x - x_0 - 2\xi_0 t}{\lambda} \right),
(41)
\]
or equivalently
\[
T_{g_{\theta, x_0, \lambda, t_0}} u(t) = g_{\theta, -t\xi_0, \lambda} g_{\theta, x_0, \lambda} u(t),
(42)
\]
if \(g \in G'\).

\begin{lemma}[linear profiles for \(L^2\) sequence, \cite{14}]
Let \(u_n\) be a bounded sequence in \(L^2_x(\mathbb{R}^2)\). Then (after passing to a subsequence if necessary) there exists a family \(u_n^{(j)}\), \(j = 1, 2, \ldots\) of functions in \(L^2_x(\mathbb{R}^2)\) and group elements \(g_n^{(j)} \in G'\) for \(j, n = 1, 2, \ldots\) such that one has the decomposition
\[
u_n = \sum_{j=1}^{\nu_n} g_n^{(j)} u_n^{(j)} + w_n^{(j)}
(43)\]
for all \( l = 1, 2, \ldots \); here, \( w_n^{(l)} \in L^2_x(\mathbb{R}^2) \) is such that its linear evolution has asymptotically vanishing scattering size:

\[
\lim_{l \to \infty} \lim_{n \to \infty} S(e^{i\lambda_n^{(l)}w_n^{(l)}}) = 0. \tag{44}
\]

Moreover, for any \( j \neq j' \),

\[
\lambda_n^{(j)} + \lambda_n^{(j')} + \lambda_n^{(j)}|e^{i\xi_n^{(j)}} - e^{i\xi_n^{(j')}}|^2 + \frac{|\xi_n^{(j)} - \xi_n^{(j')}}{\lambda_n^{(j)}\lambda_n^{(j')}}^2 \sum_{|k| \leq N} \left| a_n^{(j)} a_n^{(j')} \right|^2 \to \infty. \tag{45}
\]

Furthermore, for any \( l \geq 1 \), one has the mass decoupling property

\[
\lim_{n \to \infty} \left[ M(u_n) - \sum_{j=1}^l M(\phi_n^{(j)}) - M(w_n^{(j)}) \right] = 0. \tag{46}
\]

For any \( j \leq l \), we have

\[
\left( g_n^{(j)} \right)^{-1} w_n^{(j)} \rightharpoonup 0 \quad \text{weakly in } L^2_x(\mathbb{R}^2). \tag{47}
\]

Remark 14. If the orthogonal condition (45) holds, then (see [14])

\[
\lim_{n \to \infty} \left( g_n^{(j)} \phi_n^{(j)} + g_n^{(j')} \phi_n^{(j')} \right)_{L^2(\mathbb{R}^2)} = 0, \quad j \neq j',
\]

\[
\lim_{n \to \infty} \left( g_n^{(j)} \phi_n^{(j)}, w_n^{(j)} \right)_{L^2(\mathbb{R}^2)} = 0. \tag{48}
\]

Moreover, if \( \phi^{(j)}, \phi^{(j')} \in L^4_{L^2_x}(\mathbb{R} \times \mathbb{R}^2) \), then (see [14, 15]), for any \( 0 < \theta < 1 \),

\[
\lim_{n \to \infty} \left\| T_{g_n^{(j)}} \phi^{(j)} \right\|_{L^2_{L^2_x}(\mathbb{R} \times \mathbb{R}^2)} = 0. \tag{49}
\]

If \( \phi^{(j)}, \ldots, \phi^{(j')} \in L^4_{L^2_x}(\mathbb{R} \times \mathbb{R}^2) \), then (see [16, Lemma 5.5])

\[
\lim_{n \to \infty} \left( g_n^{(j)} \phi^{(j)} \right)_{L^2_{L^2_x}(\mathbb{R} \times \mathbb{R}^2)} = 0. \tag{50}
\]

Remark 15. As each linear profile \( \phi^{(j)} \) in Lemma 13 is constructed in the sense that

\[
e^{it_n^{(j)}x} \left( e^{i\lambda_n^{(j)}x} u_n(\lambda_n^{(j)}x) \right) \to \phi^{(j)} \tag{51}
\]

weakly in \( L^2_x(\mathbb{R}^2) \) (see [14]), after passing to a subsequence in \( n \), rearrangement, translation, and refining \( \phi^{(j)} \) accordingly, we may assume that the parameters satisfy the following properties:

(i) \( t_n^{(j)} \to \pm \infty \) as \( n \to \infty \), or \( t_n^{(j)} \equiv 0 \) for all \( n, j \);  
(ii) \( \lambda_n^{(j)} \to 0 \) or \( \lambda_n^{(j)} \to \infty \) as \( n \to \infty \), or \( \lambda_n^{(j)} \equiv 1 \) for all \( n, j \);  
(iii) \( |\xi_n^{(j)}| \to \infty \) as \( n \to \infty \), or \( \xi_n^{(j)} \equiv \xi^{(j)} \) with \( |\xi^{(j)}| < \infty \);  
(iv) when \( \lambda_n^{(j)} \equiv 1 \), \( \xi_n^{(j)} \equiv \xi^{(j)} \) and \( |\xi^{(j)}| < \infty \), we can let \( \xi^{(j)} \equiv 0 \).

Our main result in this section is the following lemma.

Lemma 16 (linear profiles for \( H^1 \) sequence). Let \( u_n \) be a bounded sequence in \( H^1(\mathbb{R}^2) \). Then up to a subsequence, for any \( j \geq 1 \), there exists a sequence \( \phi_n^{(j)} \) in \( H^1(\mathbb{R}^2) \) and a sequence of group elements \( g_{n\alpha} = g_{n\alpha}^* \) such that

\[
u_n = \sum_{\alpha=1}^j g_{n\alpha} \xi_n^{(j)} + R(n, J). \tag{52}
\]

Here, for each \( \alpha, \lambda_n, \xi_n \) must satisfy

\[
\lambda_n \equiv 1 \text{ and } \xi_n = 0, \quad \text{or } \lambda_n \to \infty. \tag{53}
\]

\( R(n, J) \in H^1(\mathbb{R}^2) \) is such that

\[
\lim_{n \to \infty} \lim_{l \to \infty} S(e^{i\lambda_n^{(j)} R(n, J)}) = 0. \tag{54}
\]

Moreover, for any \( \alpha \neq \alpha' \), one has the same orthogonal conditions as (45). For any \( j \geq 1 \), one has the following decoupling properties:

\[
\lim_{n \to \infty} \left\| \left( \sum_{\alpha=1}^j g_{n\alpha} \phi_n^{(j)} \right)^2 \right\|_{L^2(\mathbb{R}^2)} = 0, \tag{55}
\]

\[
\lim_{n \to \infty} \left\| \left( \sum_{\alpha=1}^j g_{n\alpha} \phi_n^{(j)} \right)^2 \right\|_{H^1(\mathbb{R}^2)} = 0, \tag{56}
\]

\[
\lim_{n \to \infty} \lim_{l \to \infty} \left\{ H(u_n) - \sum_{\alpha=1}^j H(\phi_n\xi_n^{(j)}) - H(R(n, J)) \right\} = 0. \tag{57}
\]

Proof. Let

\[
\Box_k = \mathcal{R}^{-1} \chi_k \mathcal{F}, \quad \chi_k = \begin{cases} 1 & \text{if } |\xi| \leq 2^k, \\ 0 & \text{else}. \end{cases} \tag{58}
\]

Then, we have

\[
u_n = \sum_{k=1}^\infty \Box_k u_n := \sum_{|k| \leq N} \Box_k u_n + R_N, \tag{59}
\]

\[
\|u_n\|_{L^p(\mathbb{R}^2)} = \sum_{|k| \leq N} \|\Box_k u_n\|_{L^p(\mathbb{R}^2)} + \|R_N\|_{L^p(\mathbb{R}^2)}, \tag{59}
\]

\[
\|u_n\|_{H^1(\mathbb{R}^2)} = \sum_{|k| \leq N} \|\Box_k u_n\|_{H^1(\mathbb{R}^2)} + \|R_N\|_{H^1(\mathbb{R}^2)}, \tag{59}
\]

\[
\lim_{N \to \infty} \lim_{n \to \infty} \|R_N\|_{L^2(\mathbb{R}^2)} = 0. \tag{59}
\]
By Lemma 13, after passing to a subsequence if necessary, we can obtain

$$
\Box_k u_n = \sum_{j=1}^k g_k^{(j)} \psi_k^{(j)} + w_{nk}^{(j)}
$$

(60)

with the stated properties (i)–(iv) in Remark 15 and (43)–(47). Denote

$$
\Lambda_{1,0} = \left\{ (k, j) \mid |k| \leq N, 1 \leq j \leq l_k, \lambda_{nk}^{(j)} = 1, \xi_{nk}^{(j)} = 0 \right\},
$$

$$
\Lambda_{1,\infty} = \left\{ (k, j) \mid |k| \leq N, 1 \leq j \leq l_k, \lambda_{nk}^{(j)} = 1, \xi_{nk}^{(j)} \to \infty \right\},
$$

$$
\Lambda_0 = \left\{ (k, j) \mid |k| \leq N, 1 \leq j \leq l_k, \lambda_{nk}^{(j)} \to 0 \right\},
$$

$$
\Lambda_{\infty,0} = \left\{ (k, j) \mid |k| \leq N, 1 \leq j \leq l_k, \lambda_{nk}^{(j)} \to \infty, \xi_{nk}^{(j)} = \xi_k^{(j)}, \right\}
$$

$$
\Lambda_{\infty,\infty} = \left\{ (k, j) \mid |k| \leq N, 1 \leq j \leq l_k, \lambda_{nk}^{(j)} \to \infty, \xi_{nk}^{(j)} \to \infty, \right\}
$$

(61)

Step 1. We prove that

$$
u_n = \sum_{(k, j) \in \Lambda_{1,\infty}} g_{nk}^{(j)} \psi_k^{(j)} + R
$$

with $\psi_k^{(j)} \in H^1$ and for each fixed $N$,

$$
\lim_{n \to \infty} \left\{ \left\| \nu_n \right\|_{L^2(R^2)}^2 - \sum_{(k, j) \in \Lambda_{1,\infty}} \left\| \psi_k^{(j)} \right\|_{L^2(R^2)}^2 \right\} = 0,
$$

$$
\lim_{n \to \infty} \left\{ \left\| u_n \right\|_{H^1(R^2)}^2 - \sum_{(k, j) \in \Lambda_{1,\infty}} \left\| g_{nk}^{(j)} \psi_k^{(j)} \right\|_{H^1(R^2)}^2 \right\} = 0,
$$

$$
\lim_{N \to \infty} \lim_{k \to \infty} \limsup_{n \to \infty} S \left( e^{\Delta} R \right) = 0,
$$

(64)

where

$$
R = R_N + R_w, \quad R_w = \sum_{|k| \leq N} w_{nk}^{(j)}.
$$

(65)

By (44) and $\lim_{N \to \infty} \limsup_{n \to \infty} \left\| R_N \right\|_{L^2(R^2)} = 0$, (64) holds obviously. For (62), we prove it by induction. For every $k$, suppose that

$$
\Box_k u_n = g_{nk}^{(1)} \psi_k^{(1)} + w_{nk}^{(1)}.
$$

(66)

Case 1. If $(k, 1) \in \Lambda_{1,\infty} \cup \Lambda_0 \cup \Lambda_{\infty,0} \cup \Lambda_{\infty,\infty}$, we have $\psi_k^{(1)} = 0$.

In fact, by (66),

$$
\psi_k^{(1)} = \left( g_{nk}^{(1)} \right)^{-1} \Box_k u_n - \left( g_{nk}^{(1)} \right)^{-1} w_{nk}^{(1)}.
$$

(67)

Thus,

$$
\left\| \Phi_k^{(1)} \right\|_{L^2(R^2)}^2 = \left( \left( g_{nk}^{(1)} \right)^{-1} \Box_k u_n - \left( g_{nk}^{(1)} \right)^{-1} w_{nk}^{(1)} \right)_{L^2(R^2)}
$$

$$
- \left( \left( g_{nk}^{(1)} \right)^{-1} w_{nk}^{(1)} \right)_{L^2(R^2)}.
$$

(68)

Using (47),

$$
\left( \left( g_{nk}^{(1)} \right)^{-1} w_{nk}^{(1)} \right)_{L^2(R^2)} \to 0 \quad \text{as} \quad n \to \infty.
$$

(69)

By direct calculation,

$$
\Box_k g_{nk}^{(1)} \psi_k^{(1)}
$$

$$
= \mathcal{F}^{-1} \chi_k \left( \frac{\xi}{\lambda_{nk}^{(1)}} \right) e^{itw_{nk}^{(1)}} e^{-i\zeta_{nk}^{(1)}(\xi + \xi_{nk}^{(1)})} e^{-\frac{i}{2} \left( \zeta_{nk}^{(1)} \right)^2} \times \frac{\lambda_{nk}^{(1)}}{\left( \xi + \xi_{nk}^{(1)} \right)}
$$

$$
= \frac{1}{\lambda_{nk}^{(1)}} e^{itw_{nk}^{(1)}} \mathcal{F}^{-1} \chi_k \left( \frac{\xi}{\lambda_{nk}^{(1)}} + \frac{\xi_{nk}^{(1)}}{\lambda_{nk}^{(1)}} \right) e^{-\frac{i}{2} \left( \zeta_{nk}^{(1)} \right)^2} \times \left( \frac{x - x_{nk}^{(1)}}{\lambda_{nk}^{(1)}} \right).
$$

(70)

Let $n \to \infty$. When $(k, 1) \in \Lambda_{1,\infty}$,

$$
\left\| \Box_k g_{nk}^{(1)} \psi_k^{(1)} \right\|_{L^2(R^2)} \leq \int_{|\zeta| = \lambda_{nk}^{(1)} - \epsilon_{nk}^{(1)}} \left| \mathcal{F} \Phi_k^{(1)} \right|^2 d\xi \to 0.
$$

(71)

When $(k, 1) \in \Lambda_0$,

$$
\left\| \Box_k g_{nk}^{(1)} \psi_k^{(1)} \right\|_{L^2(R^2)} \leq \int_{\lambda_{nk}^{(1)} - \epsilon_{nk}^{(1)} < |\zeta| < \lambda_{nk}^{(1)} + \epsilon_{nk}^{(1)}} \left| \mathcal{F} \Phi_k^{(1)} \right|^2 d\xi \to 0.
$$

(72)

When $(k, 1) \in \Lambda_{\infty,0}$,

$$
\left\| \Box_k g_{nk}^{(1)} \psi_k^{(1)} \right\|_{L^2(R^2)} \leq \int_{|\zeta| = \lambda_{nk}^{(1)} - \epsilon_{nk}^{(1)}} \left| \mathcal{F} \Phi_k^{(1)} \right|^2 d\xi \to 0.
$$

(73)
When \((k, 1) \in \Lambda_{\infty, \infty}\),
\[
\|\partial_k g^{(1)}_{nk} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 \leq \int_{|\xi| \leq 2^{-k} |\xi_k| \leq 2^{-k}} |\mathcal{F} \phi_k^{(1)}|^2 \, d\xi \rightarrow 0. \tag{74}
\]
By (68)–(74), \(\|g^{(1)}_{nk} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)} = 0\) and thus \(\phi_k^{(1)} = 0\).

**Case 2.** If \((k, 1) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}\), we can prove
\[
\|g^{(1)}_{nk} \phi_k^{(1)} - \partial_k g^{(1)}_{nk} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{75}
\]

By absorbing the error into \(w^{(1)}_{nk}\), we can suppose \(g^{(1)}_{nk} \phi_k^{(1)} = \partial_k g^{(1)}_{nk} \phi_k^{(1)}\). Since \(\partial_k g^{(1)}_{nk} \phi_k^{(1)} \in H^1\) for each fixed \(n\), we must have \(\phi_k^{(1)} \in H^1\).

Now, we begin to prove (75). Let \(\chi^{(1)}_k\) be the characteristic function of the set \(A^{(1)}_k\) and \(P^{(1)}_k = \mathcal{F}^{-1} \chi^{(1)}_k \mathcal{F}\), and then
\[
g^{(1)}_{nk} \left(P^{(1)}_k \phi_k^{(1)} + P^{(1)}_k \left(g^{(1)}_{nk} \right)^{-1} w^{(1)}_{nk}\right) = P^{(1)}_k \|g^{(1)}_{nk} \phi_k^{(1)} + w^{(1)}_{nk}\| = P^{(1)}_k \|g^{(1)}_{nk} \phi_k^{(1)} + w^{(1)}_{nk}\| \rightarrow 0,
\]
from which
\[
\lim_{n \to \infty} \|P^{(1)}_k \|g^{(1)}_{nk} \phi_k^{(1)} + w^{(1)}_{nk}\|_{L^2(\mathbb{R}^2)} = 0.
\]

We have
\[
\lim_{n \to \infty} \|P^{(1)}_k \|g^{(1)}_{nk} \phi_k^{(1)} + w^{(1)}_{nk}\|_{L^2(\mathbb{R}^2)} = 0.
\]
Step 2. For arbitrary \((k_1, j_1), (k_2, j_2) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}\), we define \((k_1, j_1) \sim (k_2, j_2)\) if the orthogonal condition (45) is NOT true for any subsequence; that is,

\[
\limsup_{n \to \infty} \left( \lambda_{nk_1}^{(j_1)} \lambda_{nk_2}^{(j_2)} + \lambda_{nk_1}^{(j_2)} \lambda_{nk_2}^{(j_1)} + \lambda_{nk_1}^{(j_1)} \lambda_{nk_2}^{(j_2)} \right) < \infty. \tag{86}
\]

By the definition above, if \((k_1, j_1) \sim (k_2, j_2)\), we have

\[
\begin{align*}
\lambda_{nk_1}^{(j_1)} &\sim \lambda_{nk_1}^{(j_2)}, & \lambda_{nk_1}^{(j_1)} \xi_{nk_1}^{(j_1)} &\sim \lambda_{nk_1}^{(j_2)} \xi_{nk_1}^{(j_2)}, \\
\lambda_{nk_2}^{(j_1)} &\sim \lambda_{nk_2}^{(j_2)}, & \lambda_{nk_2}^{(j_1)} \xi_{nk_2}^{(j_1)} &\sim \lambda_{nk_2}^{(j_2)} \xi_{nk_2}^{(j_2)}.
\end{align*}
\tag{87}
\]

Note that

\[
g_{\theta, \xi_0, X_{\alpha}, \Lambda_\alpha} f(x) := \frac{1}{\lambda} e^{\theta (x/\lambda) \cdot \xi} \left( e^{\lambda \theta} f \left( \frac{x}{\lambda} - \frac{X_0}{\lambda} \right) \right). \tag{88}
\]

By Remark 15, we can put these two profiles together as one profile. Then, by denoting \((\Lambda_{1,0} \cup \Lambda_{\infty,1})/\sim = \{1, 2, \ldots, J\}\), we can obtain the sequence \(\phi_\alpha^J, \alpha = 1, 2, \ldots, J\); and (52)–(56) were proved.

Specially, since \(\phi_{k}^{(j)} \in H^1\) for each \((k, j) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}\), we have

\[
\sum_{(k, j) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}} \phi_{k}^{(j)} \in H^1
\tag{89}
\]

for fixed \(N\) and \(I^\alpha\), and hence \(\phi_\alpha^J \in H^1(\mathbb{R}^2)\) for any fixed \(J\) and \(1 \leq \alpha \leq J\).

Step 3. We prove (57) now. By (56), we only need to prove that for all \(m \in \mathbb{N}, m \geq 2\),

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\{ \left\| \mathcal{N}_m \right\|_{L^m(\mathbb{R}^2)}^2 - \frac{1}{m} \sum_{\alpha = 1}^{J} \left\| g_{\alpha} \phi_\alpha^J \right\|_{L^m(\mathbb{R}^2)}^2 - \left\| R(n, J) \right\|_{L^m(\mathbb{R}^2)}^m \right\} = 0.
\tag{90}
\]

As

\[
\left\| R(n, J) \right\|_{L^2(\mathbb{R}^2)} \leq \left\| R(n, J) \right\|_{L^4(\mathbb{R}^2)}^{1/2} \left\| R(n, J) \right\|_{H^1(\mathbb{R}^2)}^{(m-2)/(m-1)} \tag{91}
\]

and for \(1/4 < \theta < 1/2\),

\[
\begin{align*}
\left\| R(n, J) \right\|_{L^4(\mathbb{R}^2)} &\leq \left\| e^{\theta \mathcal{N}_m} R(n, J) \right\|_{L^4(\mathbb{R}^2)} \leq \left\| e^{\theta \mathcal{N}_m} R(n, J) \right\|_{H^1(\mathbb{R}^2)}^{1/2} \left\| e^{\theta \mathcal{N}_m} R(n, J) \right\|_{L^4(\mathbb{R}^2)}^{(m-2)/(m-1)} \tag{92}
\end{align*}
\]

By (54), we have

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\| R(n, J) \right\|_{L^m(\mathbb{R}^2)}^{2m} = 0. \tag{93}
\]

We separate the set \(1 \leq \alpha \leq J\) into two subsets:

\[
\Lambda_1 = \{ \alpha \mid 1 \leq \alpha \leq J, \lambda_{\alpha} = 1, \xi_{\alpha} = 0 \}, \quad \Lambda_\infty = \{ \alpha \mid 1 \leq \alpha \leq J, \lambda_{\alpha} \to \infty \}. \tag{94}
\]

When \(\alpha \in \Lambda_\infty\),

\[
\lim_{n \to \infty} \left\| g_{\alpha} \phi_\alpha^J \right\|_{L^m(\mathbb{R}^2)} = \lim_{n \to \infty} \left( \lambda_{\alpha} \right)^{-1+1/m} \left\| \hat{g}_{\alpha} \phi_\alpha^J \right\|_{L^m(\mathbb{R}^2)} \leq \lim_{n \to \infty} \left( \lambda_{\alpha} \right)^{-1+1/m} \left\| \phi_\alpha^J \right\|_{H^1(\mathbb{R}^2)} = 0. \tag{95}
\]

Hence, in order to prove (90), one only needs to prove

\[
\lim_{n \to \infty} \left\{ \left\| \sum_{\alpha \in \Lambda_1} g_{\alpha} \phi_\alpha^J \right\|_{L^m(\mathbb{R}^2)}^{2m} - \left\| \sum_{\alpha \in \Lambda_1} g_{\alpha} \phi_\alpha^J \right\|_{L^m(\mathbb{R}^2)}^{2m} \right\} = 0. \tag{96}
\]

If \(\alpha \in \Lambda_1\) and \(t_{\alpha} \to \infty\), for a function \(\overline{\phi}_\alpha^J \in H^{1/2} \cap L^{4/3}\), we have

\[
\left\| g_{\alpha} \phi_\alpha^J \right\|_{L^4(\mathbb{R}^2)} \leq \left\| g_{\alpha} \phi_\alpha^J \right\|_{H^{1/2}(\mathbb{R}^2)} + \left\| g_{\alpha} \phi_\alpha^J \right\|_{L^4(\mathbb{R}^2)} \leq \left\| \phi_\alpha^J \right\|_{H^{1/2}(\mathbb{R}^2)} \tag{97}
\]

By approximating \(\phi_\alpha^J\) by \(\overline{\phi}_\alpha^J \in C^\infty\) in \(H^{1/2}\) and sending \(n \to \infty\), we have \(\left\| g_{\alpha} \phi_\alpha^J \right\|_{L^4(\mathbb{R}^2)} \to 0\). Note that \(g_{\alpha} \overline{\phi}_\alpha^J \in H^1\); we obtain \(\left\| g_{\alpha} \phi_\alpha^J \right\|_{L^m(\mathbb{R}^2)} \to 0\) for all \(m \geq 2\).
If $\alpha \in \Lambda_1$ and $t_{\text{na}} \equiv 0$, we have orthogonal condition $|x_{n\alpha} - x_{n\alpha'}| \to \infty$ for any $\alpha \neq \alpha'$. Thus,

$$
\lim_{n \to \infty} \left\{ \left( \sum_{\alpha \in \Lambda_1, t_{\text{na}} = 0} g_{n\alpha} \phi^j_{\alpha} \right)^2 + \sum_{\alpha \in \Lambda_1, t_{\text{na}} = 0} \left\| g_{n\alpha} \phi^j_{\alpha} \right\|^2_{L^2(\mathbb{R}^2)} \right\} = 0.
$$

(98)

(96) holds and then (57) was proved. \square

5. The Proof of Theorem 3

Let $u$ be a solution of (5), $H(u_0) \leq 1$; by Strichartz estimate and (21),

$$
\|u\|_{L^4_t L^{12/5}_x(\mathbb{R} \times \mathbb{R}^2)} 
\leq C\|u_0\|_{L^2(\mathbb{R})} + C\|u\|^3_{L^6_x(\mathbb{R} \times \mathbb{R}^2)} 
+ C(E) \|u\|_{L^4_t L^{12/5}_x(\mathbb{R} \times \mathbb{R}^2)} \|u\|_{L^8_{x,t} L^{12/5}_x(\mathbb{R} \times \mathbb{R}^2)}.
$$

(99)

When $\|u_0\|_{L^2(\mathbb{R}^2)} \ll 1$, by standard continuity argument, we have

$$
\|u\|_{L^4_t L^{12/5}_x(\mathbb{R} \times \mathbb{R}^2)} \leq C\|u_0\|_{L^2(\mathbb{R})} < \infty.
$$

(100)

Hence, if $M(u) \ll 1$, then $\|u\|_{L^4_t L^{12/5}_x(\mathbb{R} \times \mathbb{R}^2)} < \infty$. In particular, we have scattering in both directions.

For any mass $m \geq 0$, we define

$$
A(m) := \sup \{ S(u) : u \text{ is the global solution of (5)}, \quad M(u) \leq m, \quad H(u) \leq 1 \}.
$$

(101)

Then $A : [0, +\infty) \to [0, +\infty)$ is a monotone increasing function of $m$. As $A$ is left-continuous and finite for small $m$, there must exist a unique critical mass $m_0 \in (0, +\infty)$ such that $A(m)$ is finite for all $m < m_0$ but infinite for all $m \geq m_0$.

To prove Theorem 3, one needs only to prove that the critical mass $m_0$ is infinite. We will prove that by contradiction.

Proposition 17. Suppose that the critical mass $m_0$ is finite. Let $u_n : \mathbb{R} \to \mathbb{C}$ for $n = 1, 2, \ldots$ be a sequence of solutions and let $t_n \in \mathbb{R}$ be a sequence of times such that $H(u_n) \leq 1$, lim sup$_{n \to \infty} M(u_n) = m_0$, and

$$
\lim_{n \to \infty} S_{\geq t_n}(u_n) = \lim_{n \to \infty} S_{\geq t_n}(u_n) = +\infty.
$$

(102)

Then there exists a sequence of $x_n = x_n(t_n) \in \mathbb{R}^2$ such that $u_n(t_n, x + x_n)$ has a subsequence which converges strongly into $L^2_x(\mathbb{R}^2)$. Especially, the Hamiltonian of the limiting function is not greater than $1$.

Proof. We can take $t_n = 0$ for all $n$ by translating $u_n$ in time. Thus,

$$
\lim_{n \to \infty} S_{> t_n}(u_n) = \lim_{n \to \infty} S_{< t_n}(u_n) = +\infty.
$$

(103)

By Lemma 16, up to a subsequence if necessary, we have

$$
u_n(0) = \sum_{\alpha \in \Lambda_1, \Lambda_{\infty}} g_{n\alpha} \phi^j_{\alpha} + R(n, J),
$$

(104)

where $\Lambda_1$ and $\Lambda_{\infty}$ were defined by (94). Suppose that

$$
g_{n\alpha} = h_{na} e^{i2\pi\alpha},
$$

(105)

where $t_{na} \in \mathbb{R}$ and $h_{na} \in G$. By (55),

$$
\sum_{\alpha \in \Lambda_1, \Lambda_{\infty}} M(\phi^j_{\alpha}) \leq \limsup_{n \to \infty} M(u_n(0)) \leq m_0.
$$

(106)

Hence,

$$
\sup_{\alpha \in \Lambda_1, \Lambda_{\infty}} M(\phi^j_{\alpha}) \leq m_0 - \sigma.
$$

(107)

Suppose

$$
\sup_{\alpha \in \Lambda_1, \Lambda_{\infty}} M(\phi^j_{\alpha}) \leq m_0 - \sigma.
$$

(108)

For some $\sigma > 0$, we will prove that this leads to a contradiction: defining the nonlinear profile $\phi^j_{\alpha} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ as follows.

(i) When $\alpha \in \Lambda_1$,

(1) if $t_{na} \equiv 0$, we define $\phi^j_{\alpha}$ to be the global solution of (5) with initial data $\phi^j_{\alpha}(0) = \phi^j_{\alpha}$;

(2) if $t_{na} \to +\infty$, we define $\phi^j_{\alpha}$ to be the global solution of (5) which scatters to $e^{i\mu\Delta} \phi^j_{\alpha}$ in $H^1$ when $t \to +\infty$;

(3) if $t_{na} \to -\infty$, we define $\phi^j_{\alpha}$ to be the global solution of (5) which scatters to $e^{i\mu\Delta} \phi^j_{\alpha}$ in $H^1$ when $t \to -\infty$.

(ii) When $\alpha \in \Lambda_{\infty}$,

(1) if $t_{na} \equiv 0$, we define $\phi^j_{\alpha}$ to be the global solution of $\dot{u}_n + \Delta u = |u|^2 u$ with initial data $\phi^j_{\alpha}(0) = \phi^j_{\alpha}$;

(2) if $t_{na} \to +\infty$, we define $\phi^j_{\alpha}$ to be the global solution of $\dot{u}_n + \Delta u = |u|^2 u$ which scatters to $e^{i\mu\Delta} \phi^j_{\alpha}$ in $H^1$ when $t \to +\infty$;

(3) if $t_{na} \to -\infty$, we define $\phi^j_{\alpha}$ to be the global solution of $\dot{u}_n + \Delta u = |u|^2 u$ which scatters to $e^{i\mu\Delta} \phi^j_{\alpha}$ in $H^1$ when $t \to -\infty$.
For each \( \alpha \in \Lambda_1 \), \( v'_\alpha \) is well defined by the definition of \( A(m) \), (100), and (108). For each \( \alpha \in \Lambda_{co} \), \( v'_\alpha \) is well defined by the scattering of cubic Schrödinger equation in \( L^2(\mathbb{R}^2) \) (see [7, 8]) and the same analysis as in Lemma 9.

Now, we define

\[
\tilde{u}_n = \sum_{\alpha \in \Lambda_1 \cup \Lambda_{co}} T_{h_n} \left[ v'_\alpha \cdot (t_n) \right] + e^{i\Delta} R(n, J) \tag{109}
\]

for \( n, J = 1, 2, \ldots \), and then we have the following two lemmas.

**Lemma 18**

\[
\lim_{n \to \infty} M(\tilde{u}_n(0) - u_n(0)) = 0,
\]

\[
\lim_{J \to \infty} \sup_{n \to \infty} \|\tilde{u}_n\|^2_{L^2(R^2)} \leq 1. \tag{110}
\]

**Proof.** Since

\[
\lim_{n \to \infty} M(\tilde{u}_n(0) - u_n(0)) \\
\leq \sum_{\alpha \in \Lambda_1 \cup \Lambda_{co}} \lim_{n \to \infty} \left\| T_{h_n} \left[ v'_\alpha \cdot (t_n) \right] - h_n \phi'_\alpha \right\|_1 \\
= \sum_{\alpha \in \Lambda_1 \cup \Lambda_{co}} \lim_{n \to \infty} M(h_n \left[ v'_\alpha (t_n) \right] - h_n e^{i\Delta} \phi'_\alpha) \\
= \sum_{\alpha \in \Lambda_1 \cup \Lambda_{co}} \lim_{n \to \infty} M(v'_\alpha (t_n) - e^{i\Delta} \phi'_\alpha) = 0,
\]

we get the first equality.

Note that \( T_{h_n} \) maps the solutions of \( iu_t + \Delta u = |u|^2 u \) from one to another; using (57) and energy conservation, we have

\[
\lim_{J \to \infty} \sup_{n \to \infty} \left\| \tilde{u}_n \right\|^2_{L^2(R^2)} \\
\leq \lim_{J \to \infty} \sup_{n \to \infty} \left\| \sum_{\alpha = 1}^J T_{h_n} \left[ v'_\alpha \cdot (t_n) \right] \right\|^2_{L^2(R^2)} + \|R(n, J)\|^2_{L^2(R^2)} \tag{112}
\]

\[
\leq \lim_{J \to \infty} \sup_{n \to \infty} \left\| \sum_{\alpha = 1}^J H \left( h_n \phi'_\alpha \right) + H \left( R(n, J) \right) \right\|
\leq \lim_{J \to \infty} \sup_{n \to \infty} \|H(u_n)\| \leq 1.
\]

**Lemma 19.** If

\[
\lim_{J \to \infty} \sup_{n \to \infty} \|\tilde{u}_n\|_{L^4(R \times R^2)} < \infty,
\]

\[
\|v'_\alpha\|_{L^4(R \times R^2)} < \infty \quad (\forall \alpha),
\]

then

\[
\lim_{J \to \infty} \sup_{n \to \infty} \left\| \int_0^J e^{i(t-t_0)\Delta} \left( (i\partial_t + \Delta) \tilde{u}_n - f(\tilde{u}_n) \right) \right\|_X = 0, \tag{114}
\]

where \( X = L^4_t \cap L^{2/(1-2\lambda)} L^1_x(\mathbb{R} \times \mathbb{R}^2) \).

**Proof.** Denote

\[
v'_{na} = T_{h_n} \left[ v'_\alpha \cdot (t_n) \right] \tag{115}
\]

By the definition of \( \tilde{u}_n \), we have

\[
\tilde{u}_n = \sum_{\alpha \in \Lambda_1} v'_{na} + e^{i\Delta} R(n, J), \tag{116}
\]

Thus, by triangle inequality, it suffices to show that

\[
\lim_{J \to \infty} \sup_{n \to \infty} \left\| \int_0^J e^{i(t-t_0)\Delta} \left( f \left( \tilde{u}_n - e^{i\Delta} R(n, J) \right) \right) \right\|_X = 0, \tag{117}
\]

\[
\lim_{J \to \infty} \sup_{n \to \infty} \left\| \int_0^J e^{i(t-t_0)\Delta} \left( f \left( \sum_{\alpha \in \Lambda_1} v'_{na} \right) \right) \right\|_X = 0, \tag{118}
\]

\[
\lim_{J \to \infty} \sup_{n \to \infty} \left\| \int_0^J e^{i(t-t_0)\Delta} \left( \sum_{\alpha \in \Lambda_1} f \left( v'_{na} \right) - \left| v'_{na} \right|^2 v'_{na} \right) \right\|_X = 0, \tag{119}
\]

By Lemma 18, let \( J \) and \( n \) be sufficiently large; we have

\[
\|\tilde{u}_n\|^2_{L^2(R^2)} \leq \frac{1}{1 - 4\varepsilon}, \quad \|\tilde{u}_n - e^{i\Delta} R(n, J)\|^2_{L^2(R^2)} \leq \frac{1}{1 - 4\varepsilon}. \tag{120}
\]

Since we have supposed \( 0 < \lambda < 4(1 - 4\varepsilon)^2 \pi \), Lemma 6 can also be used here. Then by the same estimates as (31),

\[
\left\| \int_0^J e^{i(t-t_0)\Delta} \left( f \left( \tilde{u}_n - e^{i\Delta} R(n, J) \right) - f(\tilde{u}_n) \right) \right\|_X \leq \left( \|\tilde{u}_n\|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)} + \|\Delta R(n, J)\|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)} \right) \times \|e^{i\Delta} R(n, J)\|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)} \tag{121}
\]

\[
+ \left( \|\tilde{u}_n\|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)} + \|\Delta R(n, J)\|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)} \right) \times \|e^{i\Delta} R(n, J)\|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)}.
\]

By (54),

\[
\lim_{J \to \infty} \sup_{n \to \infty} \left\| \int_0^J e^{i\Delta} R(n, J) \right\|_{L^2_t L^4_x(\mathbb{R} \times \mathbb{R}^2)} = 0. \tag{122}
\]
As
\[
\|e^{it\Delta} R(n, J)\|_{L_{t,x}^{1/2} L_{x}^{1}((0, \infty) \times \mathbb{R}^d)} \\
\leq \|e^{it\Delta} R(n, J)\|_{L_{t,x}^{1/2} L_{x}^{2}((0, \infty) \times \mathbb{R}^d)} \|e^{it\Delta} R(n, J)\|_{L_{t,x}^{1} L_{x}^{1/2}((0, \infty) \times \mathbb{R}^d)},
\]
\[
\|e^{it\Delta} R(n, J)\|_{L_{t,x}^{1/2} L_{x}^{1}((0, \infty) \times \mathbb{R}^d)} \leq \|R(n, J)\|_{L_{t,x}^{1}} < \infty,
\]
we have
\[
\lim_{f \to \infty} \lim_{n \to \infty} \|e^{it\Delta} R(n, J)\|_{L_{t,x}^{1/2} L_{x}^{1}((0, \infty) \times \mathbb{R}^d)} = 0.
\]
By \(\lim_{f \to \infty} \lim_{n \to \infty} \|G\|_{L_{t,x}^{1}((0, \infty) \times \mathbb{R}^d)} < \infty\), (117) was obtained.

Using Strichartz estimate,
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \left( f \left( \frac{\partial}{\partial \tau} \right) - \left| \frac{\partial}{\partial \tau} \right| \alpha \right) d\tau \right\|_X
\leq \sum_{m=2}^{\infty} \frac{\|\frac{\partial}{\partial \tau} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)}}{m!} \left( \|\frac{\partial}{\partial \tau} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^{2m-1}
\leq \sum_{m=2}^{\infty} \frac{1}{m!} \left( \|\frac{\partial}{\partial \tau} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^{2m-1}
\leq \sum_{m=2}^{\infty} \frac{1}{m!} \left( \alpha \right)^{2m-1} \left( \|\frac{\partial}{\partial \tau} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^{2m}.
\]

By Lemma 8,
\[
\left\| \frac{\partial}{\partial \tau} \right\|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \leq C \left( \|f\|_{L_{t,x}^{1}((0, \infty) \times \mathbb{R}^d)} \right)^{1/m} \left( \|\frac{\partial}{\partial \tau} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^{1/m}.
\]
Note that \(\|\frac{\partial}{\partial \tau} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} < \infty\). Equation (119) was obtained.
To prove Lemma 19, it is only left to prove (118). Note that
\[
\left| f \left( \sum_{\alpha=1}^{J} \frac{z_{\alpha}}{\alpha} \right) - \sum_{\alpha=1}^{J} f \left( z_{\alpha} \right) \right| \leq \sum_{\alpha \neq \alpha'} \left| \frac{z_{\alpha}}{\alpha} \right| \left| e^{\frac{i|\alpha|}{\alpha}} - 1 \right|.
\]
By (57), we have
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \left( f \left( \sum_{\alpha \in \Lambda} \frac{\nabla}{\alpha} \right) - \sum_{\alpha \in \Lambda} f \left( \frac{\nabla}{\alpha} \right) \right) d\tau \right\|_X
\leq \sum_{\alpha \neq \alpha'} \left( \|\frac{\nabla}{\alpha} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^2 \left( \|f\|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^2
\leq \sum_{\alpha \neq \alpha'} \left( \|\frac{\nabla}{\alpha} \|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^2 \left( \|f\|_{L_{t,x}^{2}((0, \infty) \times \mathbb{R}^d)} \right)^2,
\]
for some \(\sigma > 0\), by the definition of \(A(m)\) and (100), we have
\[
A(m) \leq Bm \quad \forall m \in [0, m_0 - \sigma],
\]
where \(B = B(\sigma) \in (0, +\infty)\). Then \(v_\alpha\) satisfies
\[
M \left( v_\alpha \right) = M \left( \phi_\alpha \right) \leq m_0 - \sigma,
\]
\[
S \left( v_\alpha \right) \leq A \left( M \left( \phi_\alpha \right) \right) \leq BM \left( \phi_\alpha \right).
\]
By (50), (106), and (132), we have
\[
\lim_{f \to \infty} \lim_{n \to \infty} S \left( \tilde{u}_n \right) \leq Bm_0.
\]
Using Lemmas 18 and 19, we have
\[
M \left( \tilde{u}_n \right) - u_n(0) \leq \delta, \quad S \left( \tilde{u}_n \right) \leq 2Bm_0,
\]
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \left( i\partial_\tau + \Delta \right) \tilde{u}_n - f \left( \tilde{u}_n \right) \right\|_{L_{t,x}^{1/2} L_{x}^{1}((0, \infty) \times \mathbb{R}^d)} \leq \delta,
\]
for \(\delta > 0\) sufficiently small, \(J = J(\delta)\), and \(n = n(f, \delta)\) sufficiently large. By Lemma 10, we obtain that \(S(u_n) \leq 3Bm_0\) which contradicts (103). Thus, (129) fails for all \(\sigma > 0\), and then
\[
\sup_{\alpha \in \Lambda_{\sigma}, \Lambda_{\sigma}} M \left( \phi_\alpha \right) = m_0.
\]
Comparing this with (106), we have
\[
u_n(0) = h_n e^{it\Delta} \phi + R_n
\]
with $t_n$ converging to $\pm \infty$ or $t_n \equiv 0$, $h_n \in G$, $M(\phi) = m_0$, and
\[ M(R_n) \to 0, \quad S(e^{\beta_n R_n}) \to 0 \quad \text{as} \quad n \to \infty. \quad (137) \]
Specially, the parameters $\lambda_n, \xi_n$ of $h_n$ must satisfy
\[ "\lambda_n \equiv 1 \quad \text{and} \quad \xi_n \equiv 0," \quad \text{or} \quad "\lambda_n \to \infty." \quad (138) \]
If $\lambda_n \to \infty$, similar to the former case, we can define the approximate solution
\[ \tilde{u}_n = T_{h_n} [v(\cdot + t_n)] + e^{\beta_n R_n}. \quad (139) \]
By the scattering of cubic Schrödinger equation, we have $S(v) < \infty$ and $\lim_{n \to \infty} S(\tilde{u}_n) < \infty$. By Lemmas 18, 19, and 10, for $n$ sufficiently large, we obtain $S(u_n) < \infty$ which contradicts (103).

If $\lambda_n \equiv 1$, $\xi_n \equiv 0$, and $t_n \to +\infty$, by Strichartz estimate and monotone convergence, we have
\[ \lim_{n \to \infty} S_{\geq 0} (e^{i\lambda_n t} e^{i\beta_n \phi}) = 0. \quad (140) \]
Thus,
\[ \lim_{n \to \infty} S_{<0} (e^{-i\lambda_n t} e^{-i\beta_n \phi}) = \lim_{n \to \infty} S_{\leq 0} (e^{i\lambda_n t} e^{i\beta_n \phi}) = 0. \quad (141) \]
Since $\lim_{n \to \infty} S(e^{i\lambda_n R_n}) = 0$, we can see from (136) that
\[ \lim_{n \to \infty} S_{\geq 0} (e^{i\lambda_n u_n(0)}) = 0. \quad (142) \]
Hence,
\[ \lim_{n \to \infty} \left\| \int_0^t e^{i(t-\tau)\Delta} f(e^{i\lambda_n u_n(0)} (\tau)) \, d\tau \right\|_{L^2((0,\infty) \times \mathbb{R}^2)} = 0. \quad (143) \]
By Lemma 10 (with $e^{i\lambda_n u_n(0)$ as the approximate solution and $u_n(0)$ as the initial data), we have
\[ \lim_{n \to \infty} S_{\geq 0} (u_n) = 0, \quad (144) \]
which contradicts one of the estimates in (103).

If $\lambda_n \equiv 1$, $\xi_n \equiv 0$, and $t_n \to -\infty$, the argument is the same and we can obtain a contradiction by using the other half of (103).

Now, the only case left is $\lambda_n \equiv 1$, $\xi_n \equiv 0$, and $t_n \equiv 0$. In this case, we have
\[ M(u_n(0) - h_n \phi) = M(R_n) \to 0 \quad \text{as} \quad n \to \infty. \quad (145) \]
Thus, $(h_n)^{-1} u_n(0) = e^{\beta_n} u_n(0, x + x_n)$ converges to $\phi$ in $L^2_x(\mathbb{R}^2)$. Since $H(\phi) = H(g_n \phi) \leq 1$, after passing to a subsequence if necessary and rotating $\phi$, the desired result follows. \qed

Let $\{u_n\}$ be the sequence given in Proposition 17 and satisfy $M(u_n) \leq m_0$ and suppose that $u_n(0, x + x_n)$ converges to $u_0$ strongly in $L^2_x(\mathbb{R}^2)$; then $M(u_n) \leq m_0$ and $H(u_n) \leq 1$.

Let $u$ be the global solution with initial data $u(0) = u_0$; by Lemma 10, we must have
\[ S_{<0} (u) = S_{\leq 0} (u) = +\infty. \quad (146) \]
By the definition of $m_0$, $M(u_0) \geq m_0$ and hence $M(u_0) = m_0$.

Since $u$ is locally in $L^1_{t,x}$, for all $t_n \in \mathbb{R}$, we have
\[ S_{\leq 0} (u) = S_{\geq 0} (u) = +\infty. \quad (147) \]
Using Proposition 17 for $\{u(t_n)\}$, we have that $u(t_n, x + x(t_n))$ converges into $L^1_x(\mathbb{R}^2)$. By Ascoli-Arzelà Theorem, we have the following.

**Proposition 18.** Suppose that the critical mass $m_0$ is finite. Then there exists a global solution $u$ with mass $m_0$, and for every $\eta > 0$ there exists $0 < C(\eta) < \infty$ such that
\[ \int_{|x-x(t)| \leq C(\eta)} |u(t, x)|^2 \, dx + \int_{|x-x(t)| > C(\eta)} |\bar{u}(t, \xi)|^2 \, d\xi \leq \eta \quad (148) \]
for all $t \in \mathbb{R}$, where the functions $x, \xi : \mathbb{R} \to \mathbb{R}^2$.

**Proposition 19.** The solution described in Proposition 18 does not exist.

Once we proved Proposition 19, we can say that $m_0 = \infty$ and thus Theorem 3 is true. In order to prove Proposition 19, we need the following two lemmas.

**Lemma 20** (see [6, Lemma 5.2]). Let $u$ be a global solution of (5). Then one has
\[ \int_{\mathbb{R} \times \mathbb{R}^2} \frac{(t)^2 G(u)}{(t)^3 + |x|^2} \, dx \, dt \leq C(E), \quad (149) \]
where $(t) = \sqrt{1 + |t|^2}$.

**Lemma 21** (see [6, Lemma 6.2]). Let $u$ be a global solution of (5). Let $B$ be a compact subset of $\mathbb{R}^2$. Then for any $R > 0$ and $T > 0$, one has
\[ \int_{\mathbb{R} \times \mathbb{R}^2} |u(T, x)|^2 \, dx \geq \int_B |u(0, x)|^2 \, dx - \frac{C(E) T}{R}, \quad (150) \]
where $B(R) := \{x \in \mathbb{R}^2 | \exists y \in B \text{ s.t. } |x - y| < R\}$.

**Proof of Proposition 19.** By Lemma 21, choosing $\eta$ sufficiently small,
\[ \int_{|x-x(0)| \leq C(\eta) + R|t|} |u(t, x)|^2 \, dx \geq \int_{|x-x(0)| \leq C(\eta)} |u(0, x)|^2 \, dx - \frac{C(E) T}{R} \quad (151) \]
\[ \geq m_0 - \eta - \frac{C(E) T}{R} \].
By Proposition 18,
\[ \int_{|x-x(t)| \leq C(\eta)} |u(t,x)|^2 dx \geq m_0 - \eta. \] (152)

For a fixed large number \( R \), we must have \( |x(t) - x(0)| \leq 2C(\eta) + R|t| \). By Lemma 20 and Hölder inequality,
\[ \int_{R} \int_{R} \frac{(t)^2 |u|^4}{(t)^3 + |x|^3} dx dt \geq \int_{R} \int_{|x-x(t)| \leq C(\eta)} \frac{|u|^4}{(t)^3 + |x|^3} dx dt \geq \int_{R} \int_{|x-x(t)| \leq C(\eta)} \frac{|u|^2}{(t)^3 + 1} dx dt \geq \int_{R} \frac{1}{(t)^3 + 1} dx dt = \infty. \] (153)

This is a contradiction. Proposition 19 was obtained.

Acknowledgment

This work is supported by China Scholarship Council and NNSF of China (no. 11271023).

References


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