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Research Article

Energy Scattering for Schrödinger Equation with Exponential Nonlinearity in Two Dimensions

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When the spatial dimensions $n = 2$, the initial data $u_0 \in H^1$, and the Hamiltonian $H(u_0) \leq 1$, we prove that the scattering operator is well defined in the whole energy space $H^1(\mathbb{R}^2)$ for nonlinear Schrödinger equation with exponential nonlinearity $(e^{\lambda|u|^2} - 1)u$, where $0 < \lambda < 4\pi$.

1. Introduction

We consider the Cauchy problem for the following nonlinear Schrödinger equation:

$$iu_t + \triangle u = f(u), \tag{1}$$

$$f(u) := \left(e^{\lambda|u|^2} - 1\right)u, \tag{2}$$

in two spatial dimensions with initial data $u_0 \in H^1$ and $0 < \lambda < 4\pi$. Solutions of the above problem satisfy the conservation of mass and Hamiltonian:

$$\begin{aligned} M(u;t) &:= \int_{\mathbb{R}^2} |u|^2 dx = M(u_0), \\ H(u;t) &:= \int_{\mathbb{R}^2} \left(|\nabla u|^2 + F(u)\right) dx = H(u_0), \end{aligned} \tag{3}$$

where

$$F(u) = \frac{1}{\lambda}\left(e^{\lambda|u|^2} - \lambda|u|^2 - 1\right). \tag{4}$$

Nakamura and Ozawa [1] showed the existence and uniqueness of the scattering operator of (1) with (2). Then, Wang [2] proved the smoothness of this scattering operator. However, both of these results are based on the assumption of small initial data u_0. In this paper, we remove this assumption and show that for arbitrary initial data $u_0 \in H^1(\mathbb{R}^2)$ and $H(u_0) \leq 1$, the scattering operator is always well defined.

Wang et al. [3] proved the energy scattering theory of (1) with $f(u) = (e^{\lambda|u|^2} - 1 - \lambda|u|^2 - (\lambda^2/2)|u|^4)u$, where $\lambda \in \mathbb{R}$ and the spatial dimension $n = 1$. Ibrahim et al. [4] showed the existence and asymptotic completeness of the wave operators for (1) with $f(u) = (e^{\lambda|u|^2} - 1 - \lambda|u|^2)u$ when the spatial dimensions $n = 2$, $\lambda = 4\pi$, and $H(u_0) \leq 1$. Under the same assumptions as [4], Colliander et al. [5] proved the global well-posedness of (1) with (2).

Theorem 1. *Assume that $u_0 \in H^1(\mathbb{R}^2)$, $H(u_0) \leq 1$, and $\lambda = 4\pi$. Then problem* (1) *with* (2) *has a unique global solution u in the class $C(\mathbb{R}, H^1(\mathbb{R}^2))$.*

Remark 2. In fact, by the proof in [5], the global well-posedness of (1) with (2) is also true for $0 < \lambda \leq 4\pi$.

In this paper, we further study the scattering of this problem. Note that $f(u) = (e^{\lambda|u|^2} - 1)u = \sum_{k=1}^{\infty}(\lambda^k/k!)|u|^{2k}u$. Nakanishi [6] proved the existence of the scattering operators in the whole energy space $H^1(\mathbb{R}^2)$ for (1) with $f(u) = |u|^p u$ when $p > 2$. Then, Killip et al. [7] and Dodson [8] proved the existence of the scattering operators in $L^2(\mathbb{R}^2)$ for (1) with $f(u) = |u|^2 u$. Inspired by these two works, we use the concentration compactness method, which was introduced by Kenig and Merle in [9], to prove the existence of the scattering operators for (1) with (2).

For convenience, we write (1) and (2) together; that is,

$$iu_t + \triangle u = f(u) := \left(e^{\lambda|u|^2} - 1\right)u, \quad u(0,x) = u_0, \tag{5}$$

where $u_0 \in H^1(\mathbb{R}^2)$ and $0 < \lambda < 4\pi$. Our main result is as follows.

Theorem 3. *Assume that the initial data $u_0 \in H^1(\mathbb{R}^2)$, $H(u_0) \le 1$, and $0 < \lambda < 4\pi$. Let u be a global solution of (5). Then*

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} < \infty. \tag{6}$$

In Section 2, Lemma 9 will show us that Theorem 3 implies the following scattering result.

Theorem 4. *Assume that the initial data $u_0 \in H^1(\mathbb{R}^2)$, $H(u_0) \le 1$, and $0 < \lambda < 4\pi$. Then the solution of (5) is scattering in the energy space $H^1(\mathbb{R}^2)$.*

We will prove Theorem 3 by contradiction in Section 5. In Section 2, we give some nonlinear estimates. In Section 3, we prove the stability of solutions. In Section 4, we give a new profile decomposition for H^1 sequence which will be used to prove concentration compactness.

Now, we introduce some notations:

$$\begin{aligned} G(u) &:= \overline{u} f(u) - F(u) = e^{\lambda|u|^2}|u|^2 - \frac{1}{\lambda}\left(e^{\lambda|u|^2} - 1\right) \\ &= \sum_{k=1}^{\infty} \frac{k\lambda^k |u|^{2k+2}}{(k+1)!}, \\ E &= E(u;t) := M(u;t) + H(u;t). \end{aligned} \tag{7}$$

We define

$$\begin{aligned} \|u\|_{H^s_q(\mathbb{R}^2)} &:= \left\|(I-\triangle)^{s/2}u\right\|_{L^q(\mathbb{R}^2)}, \\ \|u\|_{\dot{H}^s_q(\mathbb{R}^2)} &:= \left\|(-\triangle)^{s/2}u\right\|_{L^q(\mathbb{R}^2)}. \end{aligned} \tag{8}$$

For Banach space $X = H^s_q(\mathbb{R}^2)$, $\dot{H}^s_q(\mathbb{R}^2)$, or $L^q(\mathbb{R}^2)$, we denote

$$\|u\|_{L^p(\mathbb{R};X)} := \left(\int_{\mathbb{R}} \|u(t)\|_X^p dt\right)^{1/p}. \tag{9}$$

When $q = r$, $L^q_t L^r_x$ is abbreviated to $L^q_{t,x}$. When q or r is infinity or when the domain $\mathbb{R}\times\mathbb{R}^2$ is replaced by $I\times\mathbb{R}^2$, we make the usual modifications. Specially, we denote

$$S(u) := \|u\|^4_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)}. \tag{10}$$

For $t_0 \in \mathbb{R}$, we split $S(u) = S_{\le t_0}(u) + S_{\ge t_0}(u)$, where

$$\begin{aligned} S_{\le t_0}(u) &:= \int_{-\infty}^{t_0}\int_{\mathbb{R}^2} |u(t,x)|^4 dx\, dt, \\ S_{\ge t_0}(u) &:= \int_{t_0}^{+\infty}\int_{\mathbb{R}^2} |u(t,x)|^4 dx\, dt. \end{aligned} \tag{11}$$

For any two Banach spaces X and Y, $\|\cdot\|_{X\cap Y} := \max\{\|\cdot\|_X, \|\cdot\|_Y\}$. C denotes positive constant. If C depends upon some parameters, such as λ, we will indicate this with $C(\lambda)$.

Remark 5. Note that $0 < \lambda < 4\pi$ in Theorem 3; we only need to prove the result for $0 < \lambda < 4(1-4\varepsilon)^2\pi$, $\varepsilon \in (0, 1/8)$. Hence, we always suppose that $0 < \lambda < 4(1-4\varepsilon)^2\pi$ in the context.

Moreover, we always suppose that the initial data u_0 of (5) satisfies $u_0 \in H^1(\mathbb{R}^2)$ and $H(u_0) \le 1$.

2. Nonlinear Estimates

In order to estimate (2), we need the following Trudinger-type inequality.

Lemma 6 (see [10]). *Let $\lambda \in [0, 4\pi)$. Then for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$, one has*

$$\int_{\mathbb{R}^2} \left(e^{\lambda|u|^2} - 1\right) dx \le C(\lambda)\|u\|^2_{L^2(\mathbb{R}^2)}. \tag{12}$$

Note that for for all $\alpha \ge 1$,

$$\left(e^{\lambda|u|^2} - 1\right)^\alpha \le e^{\lambda\alpha|u|^2} - 1. \tag{13}$$

By Lemma 6 and Hölder inequality, for $\lambda \in (0, 4\pi)$ and for all $\beta \ge 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \left(e^{\lambda|u|^2} - 1\right)|u|^\beta dx &\le \left\|e^{\lambda|u|^2} - 1\right\|_{L^{1/(1-\varepsilon)}_x(\mathbb{R}^2)} \|u\|^\beta_{L^{\beta/\varepsilon}_x(\mathbb{R}^2)} \\ &\le \|u\|^2_{L^2(\mathbb{R}^2)} \|u\|^\beta_{H^1(\mathbb{R}^2)} \\ &\le C(\lambda,\beta)\|u\|^2_{L^2(\mathbb{R}^2)}, \end{aligned} \tag{14}$$

and thus

$$\begin{aligned} \int_{\mathbb{R}^2} \left(e^{\lambda|u|^2} - \lambda|u|^2 - 1\right) dx &\le \lambda \int_{\mathbb{R}^2} \left(e^{\lambda|u|^2} - 1\right)|u|^2 dx \\ &\le C(\lambda)\|u\|^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{15}$$

Lemma 7 (Strichartz estimates). *For $s = 0$ or 1,*

$$2 \le r, \ p < \infty, \quad \frac{1}{\gamma(p)} + \frac{1}{p} = \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{16}$$

(the pairs $(\gamma(p), p)$ were called admissible pairs) we have

$$\begin{aligned} \left\|e^{i(t-t_0)\triangle}u(t_0)\right\|_{L^{\gamma(p)}(\mathbb{R};H^s_p)} &\le C\|u(t_0)\|_{H^s(\mathbb{R}^2)}, \\ \left\|\int_{t_0}^t e^{i(t-\tau)\triangle} f(u(\tau))\, d\tau\right\|_{L^{\gamma(p)}(\mathbb{R};H^s_p)} &\le C\|f(u)\|_{L^{\gamma(r)'}(\mathbb{R};H^s_{r'})}. \end{aligned} \tag{17}$$

Lemma 8 (see [3, Proposition 2.3]). *Let $1 < r < p < \infty$ be fixed indices. Then for any $q \in [p, \infty)$,*

$$\|u\|_{L^q(\mathbb{R}^2)} \le C(p,r)\, q^{1/r' + p/rq} \|u\|^{p/q}_{L^p(\mathbb{R}^2)} \|u\|^{1-p/q}_{\dot{H}^{2/r}_r(\mathbb{R}^2)}. \tag{18}$$

As shown in [6, 11], to obtain the scattering result, it suffices to show that any finite energy solution has a finite global space-time norm. In fact, if Theorem 3 is true, we have the following theorem.

Lemma 9 (Theorem 3 implies Theorem 4). *Let u be a global solution of* (5), $H(u) \le 1$, *and* $\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} < \infty$. *Then, for all admissible pairs, we have*

$$\|u\|_{L^{\gamma(p)}(\mathbb{R};H^1_p)} < \infty. \tag{19}$$

Moreover, there exist $u_\pm \in H^1$ such that

$$\lim_{t\to\pm\infty}\left\|u - e^{it\triangle}u_\pm\right\|_{H^1(\mathbb{R}^2)} = 0. \tag{20}$$

Proof. Defining $X(I) = L^{2/(1-2\varepsilon)}(I;H^1_{1/\varepsilon})$, $Y(I) = L^4(I;H^1_4)$, by Strichartz estimates, (14) and (15),

$$\begin{aligned}
&\|u\|_{X(I)\cap Y(I)}\\
&\quad\le C\|u(S)\|_{H^1(\mathbb{R}^2)} + C\left\|\lambda|u|^2u\right\|_{L^{4/3}(I;H^1_{4/3})}\\
&\qquad + C\left\|\left(e^{\lambda|u|^2} - \lambda|u|^2 - 1\right)u\right\|_{L^{2/(1+2\varepsilon)}(I;H^1_{1/(1-\varepsilon)})}\\
&\quad\le C(E) + C\|u\|^2_{L^4_{t,x}(I\times\mathbb{R}^2)}\|u\|_{Y(I)}\\
&\qquad + C\|u\|^{8\varepsilon}_{L^4_{t,x}(I\times\mathbb{R}^2)}\|u\|_{X(I)}\\
&\qquad\times\left\|\left(e^{\lambda|u|^2} - \lambda|u|^2 - 1\right)u^{-8\varepsilon}\right\|_{L^\infty_tL^{1/(1-4\varepsilon)}_x(I\times\mathbb{R}^2)}\\
&\qquad + C\|u\|^{8\varepsilon}_{L^4_{t,x}(I\times\mathbb{R}^2)}\|u\|_{X(I)}\\
&\qquad\times\left\|\left(e^{\lambda|u|^2} - 1\right)|u|^{2-8\varepsilon}\right\|_{L^\infty_tL^{1/(1-4\varepsilon)}_x(I\times\mathbb{R}^2)}\\
&\quad\le C(E) + C(E)\left(\|u\|^2_{L^4_{t,x}(I\times\mathbb{R}^2)} + \|u\|^{8\varepsilon}_{L^4_{t,x}(I\times\mathbb{R}^2)}\right)\|u\|_{X(I)\cap Y(I)}.
\end{aligned} \tag{21}$$

Using the same way as in Bourgain [12], one can split $\mathbb{R}$ into finitely many pairwise disjoint intervals:

$$\mathbb{R} = \bigcup_{j=1}^{J} I_j, \qquad \|u\|_{L^4_{t,x}(I_j\times\mathbb{R}^2)} \le \eta, \qquad C(E)\left(\eta^2 + \eta^{8\varepsilon}\right) \le \frac{1}{2}. \tag{22}$$

By (21),

$$\|u\|_{L^{2/(1-2\varepsilon)}(I_j;H^1_{1/\varepsilon})\cap L^4(I_j;H^1_4)} \le C(E). \tag{23}$$

Since $\|u\|_{L^\infty(I_j;H^1)} \le C(E)$ and $\varepsilon \in (0, 1/8)$ can be chosen small arbitrarily, by interpolation,

$$\|u\|_{L^{\gamma(p)}(I_j;H^1_p)} \le C(E), \tag{24}$$

for all admissible pairs and $j = 1, 2, \ldots, J$. The desired result (19) follows.

Let

$$u_\pm = u_0 - i\int_0^{\pm\infty} e^{-i\tau\triangle} f(u(\tau))\,d\tau. \tag{25}$$

By (19) and (21),

$$\left\|u_\pm\right\|_{H^1(\mathbb{R}^2)} \lesssim 1 + \|u\|^3_{Y(0,\pm\infty)} + \|u\|^{1+8\varepsilon}_{X(0,\pm\infty)\cap Y(0,\pm\infty)} \lesssim 1. \tag{26}$$

Thus, $u_\pm$ were well defined and belong to H^1. Since

$$\begin{aligned}
\left\|u - e^{it\triangle}u_\pm\right\|_{H^1(\mathbb{R}^2)} &= \left\|\int_t^{\pm\infty} e^{i(t-\tau)\triangle} f(u(\tau))\,d\tau\right\|_{H^1(\mathbb{R}^2)}\\
&\lesssim \|u\|^3_{Y(t,\pm\infty)} + \|u\|^{1+8\varepsilon}_{X(t,\pm\infty)\cap Y(t,\pm\infty)},
\end{aligned} \tag{27}$$

$$\|u\|_{X(0,\pm\infty)\cap Y(0,\pm\infty)} \lesssim 1,$$

we must have

$$\begin{aligned}
&\lim_{t\to\pm\infty}\left\|u - e^{it\triangle}u_\pm\right\|_{H^1(\mathbb{R}^2)}\\
&\quad\lesssim \lim_{t\to\pm\infty}\left(\|u\|^3_{Y(t,\pm\infty)} + \|u\|^{1+8\varepsilon}_{X(t,\pm\infty)\cap Y(t,\pm\infty)}\right) = 0.
\end{aligned} \tag{28}$$

(20) was proved. □

3. Stability

Lemma 10 (stability). *For any $A > 0$ and $\sigma > 0$, there exists $\delta > 0$ with the following property: suppose that $u : [0,\infty)\times\mathbb{R}^2 \to \mathbb{C}$ satisfies $\|\nabla u\|_{L^2_x(\mathbb{R}^2)} \le 1$ for all $t \in [0,\infty)$, $\|u\|_{L^4_{t,x}([0,\infty)\times\mathbb{R}^2)} \le A$ and approximately solves* (5) *in the sense that*

$$\begin{aligned}
&\left\|\int_0^t e^{i(t-\tau)\triangle}\left(iu_t + \triangle u - f(u)\right)(\tau)\,d\tau\right\|_{L^4_{t,x}\cap L^{2/(1-2\varepsilon)}_tL^{1/\varepsilon}_x([0,\infty)\times\mathbb{R}^2)}\\
&\quad\le \delta.
\end{aligned} \tag{29}$$

Then for any initial data $v(0) \in H^1(\mathbb{R}^2)$ satisfying $H(v(0)) \le 1$ and $\|u(0) - v(0)\|_{L^2_x(\mathbb{R}^2)} \le \delta$, there is a unique global solution v to (5) *satisfying $\|u - v\|_{L^4_{t,x}([0,\infty)\times\mathbb{R}^2)} \le \sigma$.*

Proof. Denote $v = u + w$, then

$$iw_t + \triangle w = \left(f(u+w) - f(u)\right) - \left(iu_t + \triangle u - f(u)\right) \tag{30}$$

and $\|w(0)\|_{L^2(\mathbb{R}^2)} \le \delta$. Let $X = L^\infty_tL^2_x \cap L^4_{t,x} \cap L^{2/(1-2\varepsilon)}_tL^{1/\varepsilon}_x$. By the similar estimates as (21), we have

$$\begin{aligned}
\|w\|_X &\le \delta + \left\|\int_0^t e^{i(t-\tau)\triangle}\left(f(u+w) - f(u)\right)(\tau)\,d\tau\right\|_X + \delta\\
&\le 2\delta + C\left\||u+w|^2(u+w) - |u|^2u\right\|_{L^{4/3}_{t,x}}\\
&\qquad + C\left\|\left(e^{\lambda|u+w|^2} - \lambda|u+w|^2 - 1\right)(u+w)\right.\\
&\qquad\qquad \left. - \left(e^{\lambda|u|^2} - \lambda|u|^2 - 1\right)u\right\|_{L^{2/(1+2\varepsilon)}_tL^{1/(1-\varepsilon)}_x}
\end{aligned}$$

$$
\begin{aligned}
&\le 2\delta + C\left\| |u+w|^2 |w| + |u|^2 |w| \right\|_{L_{t,x}^{4/3}} \\
&\quad + C\left\| \left(e^{\lambda|u+w|^2} - \lambda|u+w|^2 - 1\right)|w| \right. \\
&\qquad \left. + \left(e^{\lambda|u|^2} - \lambda|u|^2 - 1\right)|w| \right\|_{L_t^{2/(1+2\varepsilon)} L_x^{1/(1-\varepsilon)}} \\
&\quad + C\left\| \left(e^{\lambda|u+w|^2} - 1\right)|u+w|^2 |w| \right. \\
&\qquad \left. + \left(e^{\lambda|u|^2} - 1\right)|u|^2 |w| \right\|_{L_t^{2/(1+2\varepsilon)} L_x^{1/(1-\varepsilon)}} \\
&\le 2\delta + C\left(\|u\|_{L_{t,x}^4}^2 + \|u+w\|_{L_{t,x}^4}^2\right)\|w\|_{L_{t,x}^4} \\
&\quad + C\left(\|u\|_{L_{t,x}^4}^{8\varepsilon} + \|u+w\|_{L_{t,x}^4}^{8\varepsilon}\right)\|w\|_{L_t^{2/(1-2\varepsilon)} L_x^{1/\varepsilon}} \\
&\le 2\delta + C\left(\|u\|_{L_{t,x}^4}^2 + \|u\|_{L_{t,x}^4}^{8\varepsilon}\right)\|w\|_X + \|w\|_X^{1+8\varepsilon} + \|w\|_X^3.
\end{aligned} \tag{31}
$$

Then we subdivide the time interval $[0,\infty)$ into finite subintervals $I_j = [t_j, t_{j+1}]$, $j = 1,\dots,J$, $J = J(A)$ such that

$$
C\left(\|u\|_{L_{t,x}^4(I_j\times\mathbb{R}^2)}^2 + \|u\|_{L_{t,x}^4(I_j\times\mathbb{R}^2)}^{8\varepsilon}\right) < \frac{1}{4} \tag{32}
$$

for each j. Let $\delta = \delta(A, J)$ be small such that

$$
C8^{2J}\delta \ll 1, \qquad 8^{2J}\delta \ll \sigma. \tag{33}
$$

Then by (31) on $I_1 \times \mathbb{R}^2$, we have $\|w\|_{X(I_1\times\mathbb{R}^2)} \le 8\delta$ and

$$
\begin{aligned}
&\|w(t_2)\|_{L_x^2(\mathbb{R}^2)} \\
&\quad \le \left\| e^{i(t_2-t_1)\triangle} w(t_1) \right\|_{L_x^2(\mathbb{R}^2)} \\
&\qquad + \left\| \int_{t_1}^{t_2} e^{i(t_2-\tau)\triangle} \left(f(u+w) - f(u)\right)(\tau)\, d\tau \right\|_{L_x^2(\mathbb{R}^2)} + \delta \\
&\quad \le 2\delta + \delta + C\left(\|u\|_{L_{t,x}^4(I_1\times\mathbb{R}^2)}^2 + \|u\|_{L_{t,x}^4(I_1\times\mathbb{R}^2)}^{8\varepsilon}\right)\|w\|_{X(I_1\times\mathbb{R}^2)} \\
&\qquad + \|w\|_{X(I_1\times\mathbb{R}^2)}^{1+8\varepsilon} + \|w\|_{X(I_1\times\mathbb{R}^2)}^3 \\
&\quad \le 2\delta + 4\cdot 8\delta \le 8^2\delta.
\end{aligned} \tag{34}
$$

Using the same analysis as above, we can get $\|w\|_{X(I_2\times\mathbb{R}^2)} \le 8^3\delta$. Iterating this for $I_2, I_3, \dots, I_J$, we obtain $\|w\|_{X([0,\infty)\times\mathbb{R}^2)} \le 8^{2J}\delta \le \sigma$; the desired result was obtained. □

4. Linear Profile Decomposition

In this section, we will give the linear profile decomposition for Schrödinger equation in $H^1(\mathbb{R}^2)$. First, we give some definitions and lemmas.

Definition 11 (symmetry group, [13]). For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^2$, frequency $\xi_0 \in \mathbb{R}^2$, and scaling parameter $\lambda > 0$, we define the unitary transformation $g_{\theta,\xi_0,x_0,\lambda} : L_x^2(\mathbb{R}^2) \to L_x^2(\mathbb{R}^2)$ by the formula

$$
g_{\theta,\xi_0,x_0,\lambda} f(x) := \frac{1}{\lambda} e^{i\theta} e^{ix\cdot\xi_0} f\left(\frac{x-x_0}{\lambda}\right). \tag{35}
$$

We let G be the collection of such transformations; this is a group with identity $g_{0,0,0,1}$, inverse $g_{\theta,\xi_0,x_0,\lambda}^{-1} = g_{-\theta-x_0\xi_0,-\lambda\xi_0,-x_0/\lambda,1/\lambda}$, and group law

$$
g_{\theta,\xi_0,x_0,\lambda} g_{\theta',\xi_0',x_0',\lambda'} = g_{\theta+\theta'-x_0\xi_0'/\lambda,\xi_0+\xi_0'/\lambda,x_0+\lambda x_0',\lambda\lambda'}. \tag{36}
$$

If $u : I \times \mathbb{R}^2 \to \mathbb{C}$ is a function, we define $T_{g_{\theta,\xi_0,x_0,\lambda}} u : \lambda^2 I \times \mathbb{R}^2 \to \mathbb{C}$, where $\lambda^2 I := \{\lambda^2 t : t \in I\}$ by the formula

$$
\left(T_{g_{\theta,\xi_0,x_0,\lambda}} u\right)(t,x) := \frac{1}{\lambda} e^{i\theta} e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u\left(\frac{t}{\lambda^2}, \frac{x-x_0-2\xi_0 t}{\lambda}\right), \tag{37}
$$

or equivalently

$$
\left(T_{g_{\theta,\xi_0,x_0,\lambda}} u\right)(t) = g_{\theta-t|\xi_0|^2,\xi_0,x_0+2\xi_0 t,\lambda}\left(u\left(\frac{t}{\lambda^2}\right)\right). \tag{38}
$$

If $g \in G$, we can easily prove that $M(T_g u) = M(u)$ and $S(T_g u) = S(u)$.

Definition 12 (enlarged group, [13]). For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^2$, frequency $\xi_0 \in \mathbb{R}^2$, scaling parameter $\lambda > 0$, and time t_0, we define the unitary transformation $g_{\theta,\xi_0,x_0,\lambda,t_0} : L_x^2(\mathbb{R}^2) \to L_x^2(\mathbb{R}^2)$ by the formula

$$
g_{\theta,\xi_0,x_0,\lambda,t_0} = g_{\theta,\xi_0,x_0,\lambda} e^{it_0\triangle}, \tag{39}
$$

or in other words

$$
g_{\theta,\xi_0,x_0,\lambda,t_0} f(x) := \frac{1}{\lambda} e^{i\theta} e^{ix\cdot\xi_0} \left(e^{it_0\triangle} f\right)\left(\frac{x-x_0}{\lambda}\right). \tag{40}
$$

Let G' be the collection of such transformations. We also let G' act on global space-time function $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ by defining

$$
\begin{aligned}
&T_{g_{\theta,\xi_0,x_0,\lambda,t_0}} u(t,x) \\
&\quad := \frac{1}{\lambda} e^{i\theta} e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} \left(e^{it_0\triangle} u\right)\left(\frac{t}{\lambda^2}, \frac{x-x_0-2\xi_0 t}{\lambda}\right),
\end{aligned} \tag{41}
$$

or equivalently

$$
\left(T_{g_{\theta,\xi_0,x_0,\lambda,t_0}} u\right)(t) = g_{\theta-t|\xi_0|^2,\xi_0,x_0+2\xi_0 t,\lambda,t_0}\left(u\left(\frac{t}{\lambda^2}\right)\right). \tag{42}
$$

Lemma 13 (linear profiles for L^2 sequence, [14]). *Let u_n be a bounded sequence in $L_x^2(\mathbb{R}^2)$. Then (after passing to a subsequence if necessary) there exists a family $\phi^{(j)}$, $j = 1, 2, \dots$ of functions in $L_x^2(\mathbb{R}^2)$ and group elements $g_n^{(j)} \in G'$ for $j, n = 1, 2, \dots$ such that one has the decomposition*

$$
u_n = \sum_{j=1}^{l} g_n^{(j)} \phi^{(j)} + w_n^{(l)} \tag{43}
$$

for all $l = 1, 2, \ldots$; here, $w_n^{(l)} \in L_x^2(\mathbb{R}^2)$ is such that its linear evolution has asymptotically vanishing scattering size:

$$\lim_{l\to\infty} \limsup_{n\to\infty} S\left(e^{it\triangle} w_n^{(l)}\right) = 0. \tag{44}$$

Moreover, for any $j \neq j'$,

$$\begin{aligned} &\frac{\lambda_n^{(j')}}{\lambda_n^{(j)}} + \frac{\lambda_n^{(j)}}{\lambda_n^{(j')}} + \lambda_n^{(j)}\lambda_n^{(j')}\left|\xi_n^{(j)} - \xi_n^{(j')}\right|^2 + \frac{\left|x_n^{(j)} - x_n^{(j')}\right|^2}{\lambda_n^{(j)}\lambda_n^{(j')}} \\ &\quad + \frac{\left|t_n^{(j)}\left(\lambda_n^{(j)}\right)^2 - t_n^{(j')}\left(\lambda_n^{(j')}\right)^2\right|}{\lambda_n^{(j)}\lambda_n^{(j')}} \longrightarrow \infty. \end{aligned} \tag{45}$$

Furthermore, for any $l \geq 1$, one has the mass decoupling property

$$\lim_{n\to\infty}\left[M(u_n) - \sum_{j=1}^{l} M\left(\phi^{(j)}\right) - M\left(w_n^{(l)}\right)\right] = 0. \tag{46}$$

For any $j \leq l$, we have

$$\left(g_n^{(j)}\right)^{-1} w_n^{(l)} \longrightarrow 0 \quad \text{weakly in } L_x^2\left(\mathbb{R}^2\right). \tag{47}$$

Remark 14. If the orthogonal condition (45) holds, then (see [14])

$$\begin{aligned} &\lim_{n\to\infty}\left\langle g_n^{(j)}\phi^{(j)}, g_n^{(j')}\phi^{(j')}\right\rangle_{L^2(\mathbb{R}^2)} = 0, \quad j \neq j', \\ &\lim_{n\to\infty}\left\langle g_n^{(j)}\phi^{(j)}, w_n^{(l)}\right\rangle_{L^2(\mathbb{R}^2)} = 0. \end{aligned} \tag{48}$$

Moreover, if $v^{(j)}, v^{(j')} \in L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)$, then (see [14, 15]), for any $0 < \theta < 1$,

$$\lim_{n\to\infty}\left\|\left|T_{g_n^{(j)}} v^{(j)}\right|^{1-\theta}\left|T_{g_n^{(j')}} v^{(j')}\right|^{\theta}\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)} = 0. \tag{49}$$

If $v^{(1)}, \ldots, v^{(l)} \in L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)$, then (see [16, Lemma 5.5])

$$\lim_{n\to\infty} S\left(\sum_{j=1}^{l} g_n^{(j)} v^{(j)}\right) \leq \sum_{j=1}^{l} S\left(v^{(j)}\right). \tag{50}$$

Remark 15. As each linear profile $\phi^{(j)}$ in Lemma 13 is constructed in the sense that

$$e^{it_n^{(j)}\triangle}\left\{e^{i\xi_n^{(j)}\cdot x}\lambda_n^{(j)} u_n\left(\lambda_n^{(j)} x\right)\right\}\left(x + x_n^{(j)}\right) \longrightarrow \phi^{(j)} \tag{51}$$

weakly in $L_x^2(\mathbb{R}^2)$ (see [14]), after passing to a subsequence in n, rearrangement, translation, and refining $\phi^{(j)}$ accordingly, we may assume that the parameters satisfy the following properties:

(i) $t_n^{(j)} \to \pm\infty$ as $n \to \infty$, or $t_n^{(j)} \equiv 0$ for all n, j;

(ii) $\lambda_n^{(j)} \to 0$ or ∞ as $n \to \infty$, or $\lambda_n^{(j)} \equiv 1$ for all n, j;

(iii) $|\xi_n^{(j)}| \to \infty$ as $n \to \infty$, or $\xi_n^{(j)} \equiv \xi^{(j)}$ with $|\xi^{(j)}| < \infty$;

(iv) when $\lambda_n^{(j)} \equiv 1$, $\xi_n^{(j)} \equiv \xi^{(j)}$ and $|\xi^{(j)}| < \infty$, we can let $\xi^{(j)} \equiv 0$.

Our main result in this section is the following lemma.

Lemma 16 (linear profiles for H^1 sequence). *Let u_n be a bounded sequence in $H^1(\mathbb{R}^2)$. Then up to a subsequence, for any $J \geq 1$, there exists a sequence ϕ_α^J in $H^1(\mathbb{R}^2)$ and a sequence of group elements $g_{n\alpha} = g_{\theta_{n\alpha},\xi_{n\alpha},x_{n\alpha},\lambda_{n\alpha},t_{n\alpha}} \in G'$ such that*

$$u_n = \sum_{\alpha=1}^{J} g_{n\alpha}\phi_\alpha^J + R(n, J). \tag{52}$$

Here, for each α, $\lambda_{n\alpha}$ and $\xi_{n\alpha}$ must satisfy

$$\text{“}\lambda_{n\alpha} \equiv 1 \text{ and } \xi_{n\alpha} \equiv 0\text{,”} \quad \text{or} \quad \text{“}\lambda_{n\alpha} \longrightarrow \infty\text{.”} \tag{53}$$

$R(n, J) \in H^1(\mathbb{R}^2)$ is such that

$$\lim_{J\to\infty} \limsup_{n\to\infty} S\left(e^{it\triangle} R(n, J)\right) = 0. \tag{54}$$

Moreover, for any $\alpha \neq \alpha'$, one has the same orthogonal conditions as (45). For any $J \geq 1$, one has the following decoupling properties:

$$\lim_{n\to\infty}\left\{\|u_n\|_{L^2(\mathbb{R}^2)}^2 - \sum_{\alpha=1}^{J}\left\|\phi_\alpha^J\right\|_{L^2(\mathbb{R}^2)}^2 - \|R(n, J)\|_{L^2(\mathbb{R}^2)}^2\right\} = 0, \tag{55}$$

$$\begin{aligned} &\lim_{n\to\infty}\left\{\|u_n\|_{\dot{H}^1(\mathbb{R}^2)}^2 - \sum_{\alpha=1}^{J}\left\|g_{n\alpha}\phi_\alpha^J\right\|_{\dot{H}^1(\mathbb{R}^2)}^2 - \|R(n, J)\|_{\dot{H}^1(\mathbb{R}^2)}^2\right\} \\ &\quad = 0, \end{aligned} \tag{56}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty}\left\{H(u_n) - \sum_{\alpha=1}^{J} H\left(g_{n\alpha}\phi_\alpha^J\right) - H(R(n, J))\right\} = 0. \tag{57}$$

Proof. Let

$$\square_k = \mathscr{F}^{-1}\chi_k\mathscr{F}, \qquad \chi_k = \begin{cases} 1 & 2^{k-1} < |\xi| \leq 2^k, \\ 0 & \text{else.} \end{cases} \tag{58}$$

Then, we have

$$\begin{aligned} u_n &= \sum_{k=-\infty}^{+\infty} \square_k u_n := \sum_{|k|\leq N} \square_k u_n + R_N, \\ \|u_n\|_{L^2(\mathbb{R}^2)}^2 &= \sum_{|k|\leq N} \|\square_k u_n\|_{L^2(\mathbb{R}^2)}^2 + \|R_N\|_{L^2(\mathbb{R}^2)}^2, \\ \|u_n\|_{\dot{H}^1(\mathbb{R}^2)}^2 &= \sum_{|k|\leq N} \|\square_k u_n\|_{\dot{H}^1(\mathbb{R}^2)}^2 + \|R_N\|_{\dot{H}^1(\mathbb{R}^2)}^2, \\ &\lim_{N\to\infty} \limsup_{n\to\infty} \|R_N\|_{L^2(\mathbb{R}^2)} = 0. \end{aligned} \tag{59}$$

By Lemma 13, after passing to a subsequence if necessary, we can obtain

$$\square_k u_n = \sum_{j=1}^{l_k} g_{nk}^{(j)} \phi_k^{(j)} + w_{nk}^{(l_k)} \tag{60}$$

with the stated properties (i)–(iv) in Remark 15 and (43)–(47). Denote

$$\begin{aligned}
\Lambda_{1,0} &= \left\{(k,j) \mid |k| \le N, 1 \le j \le l_k, \lambda_{nk}^{(j)} \equiv 1, \xi_{nk}^{(j)} \equiv 0\right\},\\
\Lambda_{1,\infty} &= \left\{(k,j) \mid |k| \le N, 1 \le j \le l_k, \lambda_{nk}^{(j)} \equiv 1, \left|\xi_{nk}^{(j)}\right| \longrightarrow \infty\right\},\\
\Lambda_{0} &= \left\{(k,j) \mid |k| \le N, 1 \le j \le l_k, \lambda_{nk}^{(j)} \longrightarrow 0\right\},\\
\Lambda_{\infty,0} &= \left\{(k,j) \mid |k| \le N, 1 \le j \le l_k, \lambda_{nk}^{(j)} \longrightarrow \infty, \xi_{nk}^{(j)} \equiv \xi_k^{(j)},\right.\\
&\qquad \left.\left|\xi_k^{(j)}\right| < 2^{k-1}\right\},\\
\Lambda_{\infty,1} &= \left\{(k,j) \mid |k| \le N, 1 \le j \le l_k, \lambda_{nk}^{(j)} \longrightarrow \infty, \xi_{nk}^{(j)} \equiv \xi_k^{(j)},\right.\\
&\qquad \left.\left|\xi_k^{(j)}\right| \in \left[2^{k-1}, 2^k\right]\right\},\\
\Lambda_{\infty,\infty} &= \left\{(k,j) \mid |k| \le N, 1 \le j \le l_k, \lambda_{nk}^{(j)} \longrightarrow \infty, \left|\xi_{nk}^{(j)}\right| \longrightarrow \infty\right.\\
&\qquad \left. \text{or } \xi_{nk}^{(j)} \equiv \xi_k^{(j)}, \left|\xi_k^{(j)}\right| > 2^k\right\}.
\end{aligned} \tag{61}$$

Step 1. We prove that

$$u_n = \sum_{(k,j)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}} g_{nk}^{(j)} \phi_k^{(j)} + R \tag{62}$$

with $\phi_k^{(j)} \in H^1$ and for each fixed N,

$$\begin{aligned}
&\lim_{n\to\infty} \left\{ \|u_n\|_{L^2(\mathbb{R}^2)}^2 - \sum_{(k,j)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}} \left\|\phi_k^{(j)}\right\|_{L^2(\mathbb{R}^2)}^2 \right.\\
&\qquad \left. - \|R\|_{L^2(\mathbb{R}^2)}^2 \right\} = 0,\\
&\lim_{n\to\infty} \left\{ \|u_n\|_{\dot{H}^1(\mathbb{R}^2)}^2 - \sum_{(k,j)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}} \left\|g_{nk}^{(j)} \phi_k^{(j)}\right\|_{\dot{H}^1(\mathbb{R}^2)}^2 \right.\\
&\qquad \left. - \|R\|_{\dot{H}^1(\mathbb{R}^2)}^2 \right\} = 0,
\end{aligned} \tag{63}$$

$$\lim_{N\to\infty} \lim_{l_k\to\infty} \limsup_{n\to\infty} S\left(e^{it\triangle} R\right) = 0, \tag{64}$$

where

$$R = R_N + R_w, \quad R_w = \sum_{|k|\le N} w_{nk}^{(l_k)}. \tag{65}$$

By (44) and $\lim_{N\to\infty} \limsup_{n\to\infty} \|R_N\|_{L^2(\mathbb{R}^2)} = 0$, (64) holds obviously. For (62), we prove it by induction. For every k, suppose that

$$\square_k u_n = g_{nk}^{(1)} \phi_k^{(1)} + w_{nk}^{(1)}. \tag{66}$$

Case 1. If $(k,1) \in \Lambda_{1,\infty} \cup \Lambda_0 \cup \Lambda_{\infty,0} \cup \Lambda_{\infty,\infty}$, we have $\phi_k^{(1)} = 0$.

In fact, by (66),

$$\phi_k^{(1)} = \left(g_{nk}^{(1)}\right)^{-1} \square_k u_n - \left(g_{nk}^{(1)}\right)^{-1} w_{nk}^{(1)}. \tag{67}$$

Thus,

$$\begin{aligned}
\left\|\phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 &= \left\langle \left(g_{nk}^{(1)}\right)^{-1} \square_k u_n - \left(g_{nk}^{(1)}\right)^{-1} w_{nk}^{(1)}, \phi_k^{(1)} \right\rangle_{L^2(\mathbb{R}^2)}\\
&= \left\langle u_n, \square_k g_{nk}^{(1)} \phi_k^{(1)} \right\rangle_{L^2(\mathbb{R}^2)}\\
&\quad - \left\langle \left(g_{nk}^{(1)}\right)^{-1} w_{nk}^{(1)}, \phi_k^{(1)} \right\rangle_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{68}$$

Using (47),

$$\left\langle \left(g_{nk}^{(1)}\right)^{-1} w_{nk}^{(1)}, \phi_k^{(1)} \right\rangle_{L^2(\mathbb{R}^2)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{69}$$

By direct calculation,

$$\begin{aligned}
&\square_k g_{nk}^{(1)} \phi_k^{(1)}\\
&\quad = \mathscr{F}^{-1} \chi_k(\xi) \lambda_{nk}^{(1)} e^{i\theta_{nk}^{(1)}} e^{-ix_{nk}^{(1)}(\xi+\xi_{nk}^{(1)})} e^{-it_{nk}^{(1)}(\lambda_{nk}^{(1)})^2|\xi+\xi_{nk}^{(1)}|^2}\\
&\qquad \times \widehat{\phi_k^{(1)}}\left(\lambda_{nk}^{(1)}\left(\xi + \xi_{nk}^{(1)}\right)\right)\\
&\quad = \frac{1}{\lambda_{nk}^{(1)}} e^{i\theta_{nk}^{(1)}} e^{ix\cdot\xi_{nk}^{(1)}} \left(\mathscr{F}^{-1} \chi_k\left(\frac{\xi}{\lambda_{nk}^{(1)}} + \xi_{nk}^{(1)}\right) e^{-it_{nk}^{(1)}|\xi|^2} \widehat{\phi_k^{(1)}}\right)\\
&\qquad \times \left(\frac{x - x_{nk}^{(1)}}{\lambda_{nk}^{(1)}}\right).
\end{aligned} \tag{70}$$

Let $n \to \infty$. When $(k,1) \in \Lambda_{1,\infty}$,

$$\left\|\square_k g_{nk}^{(1)} \phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \le \int_{2^{k-1}\le|\xi+\xi_{nk}^{(1)}|\le 2^k} \left|\mathscr{F}\phi_k^{(1)}\right|^2 d\xi \longrightarrow 0. \tag{71}$$

When $(k,1) \in \Lambda_0$,

$$\left\|\square_k g_{nk}^{(1)} \phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \le \int_{\lambda_{nk}^{(1)} 2^{k-1}\le|\xi+\lambda_{nk}^{(1)}\xi_{nk}^{(1)}|\le\lambda_{nk}^{(1)} 2^k} \left|\mathscr{F}\phi_k^{(1)}\right|^2 d\xi \longrightarrow 0. \tag{72}$$

When $(k,1) \in \Lambda_{\infty,0}$,

$$\left\|\square_k g_{nk}^{(1)} \phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \le \int_{|\xi|\ge\lambda_{nk}^{(1)}(2^{k-1}-|\xi_k^{(1)}|)} \left|\mathscr{F}\phi_k^{(1)}\right|^2 d\xi \longrightarrow 0. \tag{73}$$

When $(k,1)\in\Lambda_{\infty,\infty}$,

$$\left\|\square_k g_{nk}^{(1)}\phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \le \int_{|\xi|\ge\lambda_{nk}^{(1)}(|\xi_{nk}^{(1)}|-2^k)} \left|\mathcal{F}\phi_k^{(1)}\right|^2 d\xi \longrightarrow 0. \quad (74)$$

By (68)–(74), $\|\phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 = 0$ and thus $\phi_k^{(1)} = 0$.

Case 2. If $(k,1)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}$, we can prove

$$\left\|g_{nk}^{(1)}\phi_k^{(1)} - \square_k g_{nk}^{(1)}\phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)} \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \quad (75)$$

By absorbing the error into $w_{nk}^{(1)}$, we can suppose $g_{nk}^{(1)}\phi_k^{(1)} = \square_k g_{nk}^{(1)}\phi_k^{(1)}$. Since $\square_k g_{nk}^{(1)}\phi_k^{(1)} \in H^1$ for each fixed n, we must have $\phi_k^{(1)}\in H^1$.

Now, we begin to prove (75). Let $\chi_{A_k^{(1)}}$ be the characteristic function of the set $A_k^{(1)}$ and $P_{A_k^{(1)}} = \mathcal{F}^{-1}\chi_{A_k^{(1)}}\mathcal{F}$, and then

$$\begin{aligned} g_{nk}^{(1)}\left(P_{A_k^{(1)}}\phi_k^{(1)} + P_{A_k^{(1)}}\left(g_{nk}^{(1)}\right)^{-1}w_{nk}^{(1)}\right) &= P_{\widetilde{A}_k^{(1)}}\left(g_{nk}^{(1)}\phi_k^{(1)} + w_{nk}^{(1)}\right) \\ &= P_{\widetilde{A}_k^{(1)}}\square_k u_n, \end{aligned} \quad (76)$$

where

$$P_{\widetilde{A}_k^{(1)}} = \mathcal{F}^{-1}\chi_{A_k^{(1)}}\left(\lambda_{nk}^{(1)}\left(\xi-\xi_{nk}^{(1)}\right)\right)\mathcal{F}. \quad (77)$$

Note that

$$\begin{aligned} &\left\langle P_{A_k^{(1)}}\phi_k^{(1)}, P_{A_k^{(1)}}\left(g_{nk}^{(1)}\right)^{-1}w_{nk}^{(1)}\right\rangle_{L^2(\mathbb{R}^2)} \\ &\quad = \left\langle P_{A_k^{(1)}}\phi_k^{(1)}, \left(g_{nk}^{(1)}\right)^{-1}w_{nk}^{(1)}\right\rangle_{L^2(\mathbb{R}^2)} \longrightarrow 0. \end{aligned} \quad (78)$$

We have

$$\begin{aligned} \lim_{n\to\infty}\Bigg(&\left\|P_{\widetilde{A}_k^{(1)}}\square_k u_n\right\|_{L^2(\mathbb{R}^2)}^2 - \left\|P_{A_k^{(1)}}\phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \\ &-\left\|P_{A_k^{(1)}}\left(g_{nk}^{(1)}\right)^{-1}w_{nk}^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2\Bigg) = 0. \end{aligned} \quad (79)$$

When $(k,1)\in\Lambda_{1,0}$, we have $P_{A_k^{(1)}} = P_{\widetilde{A}_k^{(1)}}$. Choosing $A_k^{(1)} = \{\xi \mid |\xi|\le 2^{k-1} \text{ or } |\xi| > 2^k\}$, then by (79), $P_{A_k^{(1)}}\phi_k^{(1)} = 0$, the desired result follows.

When $(k,1)\in\Lambda_{\infty,1}$ and $|\xi_k^{(1)}|\in(2^{k-1},2^k)$, we have

$$\begin{aligned} &\left\|g_{nk}^{(1)}\phi_k^{(1)} - \square_k g_{nk}^{(1)}\phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad\le \int_{|\xi+\lambda_{nk}^{(1)}\xi_k^{(1)}|\le\lambda_{nk}^{(1)}2^{k-1}} \left|\mathcal{F}\phi_k^{(1)}\right|^2 d\xi \\ &\qquad + \int_{|\xi+\lambda_{nk}^{(1)}\xi_k^{(1)}|\ge\lambda_{nk}^{(1)}2^{k}} \left|\mathcal{F}\phi_k^{(1)}\right|^2 d\xi \\ &\quad\le \int_{|\xi|\ge\lambda_{nk}^{(1)}(|\xi_k^{(1)}|-2^{k-1})} \left|\mathcal{F}\phi_k^{(1)}\right|^2 d\xi \\ &\qquad + \int_{|\xi|\ge\lambda_{nk}^{(1)}(2^k-|\xi_k^{(1)}|)} \left|\mathcal{F}\phi_k^{(1)}\right|^2 d\xi \longrightarrow 0 \\ &\qquad\qquad \text{as } n\longrightarrow\infty. \end{aligned} \quad (80)$$

When $(k,1)\in\Lambda_{\infty,1}$ and $|\xi_k^{(1)}| = 2^k$, we denote $\xi = (\xi_1,\xi_2)$ and $\xi_k^{(1)} = (\xi_{k1}^{(1)},\xi_{k2}^{(1)})$. The line $\xi_2 = -(\xi_{k1}^{(1)}/\xi_{k2}^{(1)})\xi_1$ (when $\xi_{k2}^{(j)} = 0$, we use the line $\xi_1 = 0$ instead) separates the frequency space $\mathbb{R}^2$ into two half-planes. We let $A_k^{(1)}$ to be the half-plane which contains the point $\xi_k^{(1)}$, and then

$$P_{\widetilde{A}_k^{(1)}}\square_k u_n = \mathcal{F}^{-1}\chi_{A_k^{(1)}}\left(\lambda_{nk}^{(1)}\left(\xi-\xi_k^{(1)}\right)\right)\chi_k\mathcal{F}u_n = 0. \quad (81)$$

By (79), we have $P_{A_k^{(1)}}\phi_k^{(1)} = 0$. Note that

$$\begin{aligned} &\lim_{n\to\infty}\left\|g_{nk}^{(1)}\left(I-P_{A_k^{(1)}}\right)\phi_k^{(1)} - \square_k g_{nk}^{(1)}\phi_k^{(j)}\right\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad = \lim_{n\to\infty}\left\|\left[\left(1-\chi_{A_k^{(1)}}\right)(\cdot) - \chi_k\left(\frac{\cdot}{\lambda_{nk}^{(1)}} + \xi_k^{(1)}\right)\right]\mathcal{F}\phi_k^{(1)}\right\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad \le \lim_{n\to\infty}\int_{\xi\in\mathbb{R}^2\setminus(A_k^{(1)}\cup\{2^{k-1}\le|\xi/\lambda_{nk}^{(1)}+\xi_k^{(1)}|\le 2^k\})} \left|\mathcal{F}\phi_k^{(1)}\right|^2 d\xi \\ &\quad = 0. \end{aligned} \quad (82)$$

(75) holds.

When $(k,1)\in\Lambda_{\infty,1}$ and $|\xi_k^{(1)}| = 2^{k-1}$, let $A_k^{(1)}$ be the half-plane which does NOT contain the point $\xi_k^{(1)}$; we can prove (75) similarly as above.

By the proof above, we get

$$\square_k u_n = g_{nk}^{(1)}\phi_k^{(1)} + w_{nk}^{(1)}, \quad g_{nk}^{(1)}\phi_k^{(1)} = \square_k g_{nk}^{(1)}\phi_k^{(1)}, \quad \phi_k^{(1)}\in H^1 \quad (83)$$

and $(k,1)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}$. Denote $u_n^{(1)} = u_n - g_{nk}^{(1)}\phi_k^{(1)}$ and suppose

$$\square_k u_n^{(1)} = g_{nk}^{(2)}\phi_k^{(2)} + w_{nk}^{(2)}. \quad (84)$$

Repeating the proof above, we can get $g_{nk}^{(2)}\phi_k^{(2)} = \square_k g_{nk}^{(2)}\phi_k^{(2)}$, $\phi_k^{(2)}\in H^1$, and $(k,2)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}\ldots$; by induction, we obtain (62).

By the orthogonal condition (45), following the proof in [14], we can obtain that for fix k and for all $j\ne j'$,

$$\begin{aligned} &\lim_{n\to\infty}\left\langle g_{nk}^{(j)}\phi_k^{(j)}, g_{nk}^{(j')}\phi_k^{(j')}\right\rangle_{\dot{H}^1(\mathbb{R}^2)} \\ &\quad = \lim_{n\to\infty}\left\langle \square_k g_{nk}^{(j)}\phi_k^{(j)}, \square_k g_{nk}^{(j')}\phi_k^{(j')}\right\rangle_{\dot{H}^1(\mathbb{R}^2)} = 0, \\ &\lim_{n\to\infty}\left\langle g_{nk}^{(j)}\phi_k^{(j)}, w_{nk}^{(l_k)}\right\rangle_{\dot{H}^1(\mathbb{R}^2)} \\ &\quad = \lim_{n\to\infty}\left\langle \square_k g_{nk}^{(j)}\phi_k^{(j)}, \square_k w_{nk}^{(l_k)}\right\rangle_{\dot{H}^1(\mathbb{R}^2)} = 0, \end{aligned} \quad (85)$$

(63) were proved.

Step 2. For arbitrary $(k_1, j_1), (k_2, j_2) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}$, we define $(k_1, j_1) \sim (k_2, j_2)$ if the orthogonal condition (45) is NOT true for any subsequence; that is,

$$\limsup_{n\to\infty}\left(\frac{\lambda_{nk_1}^{(j_1)}}{\lambda_{nk_2}^{(j_2)}}+\frac{\lambda_{nk_2}^{(j_2)}}{\lambda_{nk_1}^{(j_1)}}+\lambda_{nk_2}^{(j_2)}\lambda_{nk_1}^{(j_1)}\left|\xi_{nk_2}^{(j_2)}-\xi_{nk_1}^{(j_1)}\right|^2 + \frac{\left|x_{nk_2}^{(j_2)}-x_{nk_1}^{(j_1)}\right|^2}{\lambda_{nk_2}^{(j_2)}\lambda_{nk_1}^{(j_1)}}+\frac{\left|t_{nk_2}^{(j_2)}\left(\lambda_{nk_2}^{(j_2)}\right)^2-t_{nk_1}^{(j_1)}\left(\lambda_{nk_1}^{(j_1)}\right)^2\right|}{\lambda_{nk_2}^{(j_2)}\lambda_{nk_1}^{(j_1)}}\right) < \infty. \tag{86}$$

By the definition above, if $(k_1, j_1) \sim (k_2, j_2)$, we have

$$\lambda_{nk_1}^{(j_1)} \sim \lambda_{nk_2}^{(j_2)}, \qquad \lambda_{nk_1}^{(j_1)}\xi_{nk_1}^{(j_1)} \sim \lambda_{nk_2}^{(j_2)}\xi_{nk_2}^{(j_2)}, \qquad \frac{x_{nk_1}^{(j_1)}}{\lambda_{nk_1}^{(j_1)}} \sim \frac{x_{nk_2}^{(j_2)}}{\lambda_{nk_2}^{(j_2)}}, \qquad t_{nk_1}^{(j_1)} \sim t_{nk_2}^{(j_2)}. \tag{87}$$

Note that

$$g_{\theta,\xi_0,x_0,\lambda,t_0} f(x) := \frac{1}{\lambda} e^{i\theta} e^{i(x/\lambda)\cdot\lambda\xi_0}\left(e^{it_0\triangle} f\right)\left(\frac{x}{\lambda}-\frac{x_0}{\lambda}\right). \tag{88}$$

By Remark 15, we can put these two profiles together as one profile. Then, by denoting $(\Lambda_{1,0} \cup \Lambda_{\infty,1})/\sim$ as $\{1, 2, \ldots, J\}$, we can obtain the sequence ϕ_α^J, $\alpha = 1, 2, \ldots, J$; and (52)–(56) were proved.

Specially, since $\phi_k^{(j)} \in H^1$ for each $(k, j) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}$, we have

$$\sum_{(k,j)\in\Lambda_{1,0}\cup\Lambda_{\infty,1}} \phi_k^{(j)} \in H^1 \tag{89}$$

for fixed N and l_k, and hence $\phi_\alpha^J \in H^1(\mathbb{R}^2)$ for any fixed J and $1 \le \alpha \le J$.

Step 3. We prove (57) now. By (56), we only need to prove that for all $m \in \mathbb{N}$, $m \ge 2$,

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\{\|u_n\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \sum_{\alpha=1}^{J}\left\|g_{n\alpha}\phi_\alpha^J\right\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \|R(n,J)\|_{L^{2m}(\mathbb{R}^2)}^{2m}\right\} = 0. \tag{90}$$

As

$$\|R(n,J)\|_{L^{2m}(\mathbb{R}^2)} \lesssim \|R(n,J)\|_{L^4(\mathbb{R}^2)}^{1/(m-1)}\|R(n,J)\|_{H^1(\mathbb{R}^2)}^{(m-2)/(m-1)} \tag{91}$$

and for $1/4 < \theta < 1/2$,

$$\begin{aligned}
&\|R(n,J)\|_{L^4(\mathbb{R}^2)} \\
&\quad\le \left\|e^{it\triangle}R(n,J)\right\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^2)} \\
&\quad\lesssim \left\|e^{it\triangle}R(n,J)\right\|_{H_4^\theta L_x^4(\mathbb{R}\times\mathbb{R}^2)} \\
&\quad= \left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{1-2\theta}\left\|e^{it\triangle}R(n,J)\right\|_{H_4^{1/2}L_x^4(\mathbb{R}\times\mathbb{R}^2)}^{2\theta} \\
&\quad= \left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{1-2\theta}\left\|e^{it\triangle}R(n,J)\right\|_{L_t^4H_4^1(\mathbb{R}\times\mathbb{R}^2)}^{2\theta} \\
&\quad\lesssim \left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{1-2\theta}.
\end{aligned} \tag{92}$$

By (54), we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|R(n,J)\|_{L^{2m}(\mathbb{R}^2)}^{2m} = 0. \tag{93}$$

We separate the set $1 \le \alpha \le J$ into two subsets:

$$\Lambda_1 = \{\alpha \mid 1 \le \alpha \le J, \lambda_{n\alpha} \equiv 1, \xi_{n\alpha} \equiv 0\}, \qquad \Lambda_\infty = \{\alpha \mid 1 \le \alpha \le J, \lambda_{n\alpha} \longrightarrow \infty\}. \tag{94}$$

When $\alpha \in \Lambda_\infty$,

$$\begin{aligned}
\lim_{n\to\infty}\left\|g_{n\alpha}\phi_\alpha^J\right\|_{L^{2m}(\mathbb{R}^2)} &= \lim_{n\to\infty}(\lambda_{n\alpha})^{-1+1/m}\left\|e^{it_{n\alpha}\triangle}\phi_\alpha^J\right\|_{L^{2m}(\mathbb{R}^2)} \\
&\le \lim_{n\to\infty}(\lambda_{n\alpha})^{-1+1/m}\left\|\phi_\alpha^J\right\|_{H^1(\mathbb{R}^2)} = 0.
\end{aligned} \tag{95}$$

Hence, in order to prove (90), one only needs to prove

$$\lim_{n\to\infty}\left\{\left\|\sum_{\alpha\in\Lambda_1} g_{n\alpha}\phi_\alpha^J\right\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \sum_{\alpha\in\Lambda_1}\left\|g_{n\alpha}\phi_\alpha^J\right\|_{L^{2m}(\mathbb{R}^2)}^{2m}\right\} = 0. \tag{96}$$

If $\alpha \in \Lambda_1$ and $t_{n\alpha} \to \infty$, for a function $\tilde{\phi}_\alpha^J \in \dot{H}^{1/2} \cap L^{4/3}$, we have

$$\begin{aligned}
\left\|g_{n\alpha}\phi_\alpha^J\right\|_{L^4(\mathbb{R}^2)} &\le \left\|g_{n\alpha}\phi_\alpha^J - g_{n\alpha}\tilde{\phi}_\alpha^J\right\|_{L^4(\mathbb{R}^2)} + \left\|g_{n\alpha}\tilde{\phi}_\alpha^J\right\|_{L^4(\mathbb{R}^2)} \\
&\le \left\|\phi_\alpha^J - \tilde{\phi}_\alpha^J\right\|_{\dot{H}^{1/2}(\mathbb{R}^2)} + \left|t_{n\alpha}\right|^{-1/2}\left\|\tilde{\phi}_\alpha^J\right\|_{L^{4/3}(\mathbb{R}^2)}.
\end{aligned} \tag{97}$$

By approximating ϕ_α^J by $\tilde{\phi}_\alpha^J \in C_c^\infty$ in $\dot{H}^{1/2}$ and sending $n \to \infty$, we have $\|g_{n\alpha}\phi_\alpha^J\|_{L^4(\mathbb{R}^2)} \to 0$. Note that $g_{n\alpha}\phi_\alpha^J \in H^1$; we obtain $\|g_{n\alpha}\phi_\alpha^J\|_{L^{2m}(\mathbb{R}^2)} \to 0$ for all $m \ge 2$.

If $\alpha \in \Lambda_1$ and $t_{n\alpha} \equiv 0$, we have orthogonal condition $|x_{n\alpha} - x_{n\alpha'}| \to \infty$ for any $\alpha \neq \alpha'$. Thus,

$$\lim_{n\to\infty} \left\{ \left\| \sum_{\alpha\in\Lambda_1, t_{n\alpha}\equiv 0} g_{n\alpha}\phi_\alpha^J \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \sum_{\alpha\in\Lambda_1, t_{n\alpha}\equiv 0} \left\| g_{n\alpha}\phi_\alpha^J \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} \right\}$$
$$= \lim_{n\to\infty} \left\{ \left\| \sum_{\alpha\in\Lambda_1, t_{n\alpha}\equiv 0} e^{i\theta_{n\alpha}}\phi_\alpha^J(\cdot - x_{n\alpha}) \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \sum_{\alpha\in\Lambda_1, t_{n\alpha}\equiv 0} \left\| \phi_\alpha^J(\cdot - x_{n\alpha}) \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} \right\} = 0. \tag{98}$$

(96) holds and then (57) was proved. □

5. The Proof of Theorem 3

Let u be a solution of (5), $H(u_0) \leq 1$; by Strichartz estimate and (21),

$$\begin{aligned} &\|u\|_{L^4_{t,x}\cap L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)} \\ &\quad \leq C\|u_0\|_{L^2(\mathbb{R}^2)} + C\|u\|^3_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \\ &\qquad + C(E)\|u\|_{L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}\|u\|^{8\varepsilon}_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \\ &\quad \leq C\|u_0\|_{L^2(\mathbb{R}^2)} + C\|u\|^3_{L^4_{t,x}\cap L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)} \\ &\qquad + C(E)\|u\|^{1+8\varepsilon}_{L^4_{t,x}\cap L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}. \end{aligned} \tag{99}$$

When $\|u_0\|_{L^2(\mathbb{R}^2)} \ll 1$, by standard continuity argument, we have

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \leq C\|u_0\|_{L^2(\mathbb{R}^2)} < \infty. \tag{100}$$

Hence, if $M(u) \ll 1$, then $\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} < \infty$. In particular, we have scattering in both directions.

For any mass $m \geq 0$, we define

$$A(m) := \sup\{S(u) : u \text{ is the global solution of } (5), \; M(u) \leq m, \; H(u) \leq 1\}. \tag{101}$$

Then $A : [0, +\infty) \to [0, +\infty]$ is a monotone increasing function of m. As A is left-continuous and finite for small m, there must exist a unique critical mass $m_0 \in (0+\infty]$ such that $A(m)$ is finite for all $m < m_0$ but infinite for all $m \geq m_0$.

To prove Theorem 3, one only needs to prove that the critical mass m_0 is infinite. We will prove that by contradiction.

Proposition 17. *Suppose that the critical mass m_0 is finite. Let $u_n : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ for $n = 1, 2, \ldots$ be a sequence of solutions and let $t_n \in \mathbb{R}$ be a sequence of times such that $H(u_n) \leq 1$, $\limsup_{n\to\infty} M(u_n) = m_0$, and*

$$\lim_{n\to\infty} S_{\geq t_n}(u_n) = \lim_{n\to\infty} S_{\leq t_n}(u_n) = +\infty. \tag{102}$$

Then there exists a sequence of $x_n = x_n(t_n) \in \mathbb{R}^2$ such that $u_n(t_n, x + x_n)$ has a subsequence which converges strongly into $L^2_x(\mathbb{R}^2)$. Especially, the Hamiltonian of the limiting function is not greater than 1.

Proof. We can take $t_n = 0$ for all n by translating u_n in time. Thus,

$$\lim_{n\to\infty} S_{\geq 0}(u_n) = \lim_{n\to\infty} S_{\leq 0}(u_n) = +\infty. \tag{103}$$

By Lemma 16, up to a subsequence if necessary, we have

$$u_n(0) = \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty} g_{n\alpha}\phi_\alpha^J + R(n, J), \tag{104}$$

where Λ_1 and Λ_∞ were defined by (94). Suppose that

$$g_{n\alpha} = h_{n\alpha}e^{it_{n\alpha}\triangle}, \tag{105}$$

where $t_{n\alpha} \in \mathbb{R}$ and $h_{n\alpha} \in G$. By (55),

$$\sum_{\alpha\in\Lambda_1\cup\Lambda_\infty} M(\phi_\alpha^J) \leq \limsup_{n\to\infty} M(u_n(0)) \leq m_0. \tag{106}$$

Hence,

$$\sup_{\alpha\in\Lambda_1\cup\Lambda_\infty} M(\phi_\alpha^J) \leq m_0. \tag{107}$$

Suppose

$$\sup_{\alpha\in\Lambda_1\cup\Lambda_\infty} M(\phi_\alpha^J) \leq m_0 - \sigma. \tag{108}$$

For some $\sigma > 0$, we will prove that this leads to a contradiction: defining the nonlinear profile $v_\alpha^J : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ as follows.

(i) When $\alpha \in \Lambda_1$,

(1) if $t_{n\alpha} \equiv 0$, we define v_α^J to be the global solution of (5) with initial data $v_\alpha^J(0) = \phi_\alpha^J$;

(2) if $t_{n\alpha} \to +\infty$, we define v_α^J to be the global solution of (5) which scatters to $e^{it\triangle}\phi_\alpha^J$ in H^1 when $t \to +\infty$;

(3) if $t_{n\alpha} \to -\infty$, we define v_α^J to be the global solution of (5) which scatters to $e^{it\triangle}\phi_\alpha^J$ in H^1 when $t \to -\infty$.

(ii) When $\alpha \in \Lambda_\infty$,

(1) if $t_{n\alpha} \equiv 0$, we define v_α^J to be the global solution of $iu_t + \triangle u = |u|^2u$ with initial data $v_\alpha^J(0) = \phi_\alpha^J$;

(2) if $t_{n\alpha} \to +\infty$, we define v_α^J to be the global solution of $iu_t + \triangle u = |u|^2u$ which scatters to $e^{it\triangle}\phi_\alpha^J$ in H^1 when $t \to +\infty$;

(3) if $t_{n\alpha} \to -\infty$, we define v_α^J to be the global solution of $iu_t + \triangle u = |u|^2u$ which scatters to $e^{it\triangle}\phi_\alpha^J$ in H^1 when $t \to -\infty$.

For each $\alpha \in \Lambda_1$, v_α^J is well defined by the definition of $A(m)$, (100), and (108). For each $\alpha \in \Lambda_\infty$, v_α^J is well defined by the scattering of cubic Schrödinger equation in $L^2(\mathbb{R}^2)$ (see [7, 8]) and the same analysis as in Lemma 9.

Now, we define

$$\widetilde{u}_n = \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty} T_{h_{n\alpha}}\left[v_\alpha^J(\cdot + t_{n\alpha})\right] + e^{it\triangle}R(n,J) \tag{109}$$

for $n, J = 1, 2, \ldots$, and then we have the following two lemmas.

Lemma 18

$$\begin{gathered}\lim_{n\to\infty} M\left(\widetilde{u}_n(0) - u_n(0)\right) = 0,\\ \lim_{J\to\infty}\limsup_{n\to\infty}\|\widetilde{u}_n\|_{\dot{H}^1(\mathbb{R}^2)}^2 \le 1.\end{gathered} \tag{110}$$

Proof. Since

$$\begin{aligned}&\lim_{n\to\infty} M\left(\widetilde{u}_n(0) - u_n(0)\right)\\ &\quad\le \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty}\lim_{n\to\infty} M\left(T_{h_{n\alpha}}\left[v_\alpha^J(\cdot + t_{n\alpha})\right](0) - g_{n\alpha}\phi_\alpha^J\right)\\ &\quad= \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty}\lim_{n\to\infty} M\left(h_{n\alpha}\left[v_\alpha^J(t_{n\alpha})\right] - h_{n\alpha}e^{it_{n\alpha}\triangle}\phi_\alpha^J\right)\\ &\quad= \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty}\lim_{n\to\infty} M\left(v_\alpha^J(t_{n\alpha}) - e^{it_{n\alpha}\triangle}\phi_\alpha^J\right) = 0,\end{aligned} \tag{111}$$

we get the first equality.

Note that $T_{h_{n\alpha}}$ maps the solutions of $iu_t + \triangle u = |u|^2 u$ from one to another; using (57) and energy conservation, we have

$$\begin{aligned}&\lim_{J\to\infty}\limsup_{n\to\infty}\|\widetilde{u}_n\|_{\dot{H}^1(\mathbb{R}^2)}^2\\ &\quad\le \lim_{J\to\infty}\limsup_{n\to\infty}\left\{\sum_{\alpha=1}^{J}\left\|T_{h_{n\alpha}}\left[v_\alpha^J(\cdot + t_{n\alpha})\right]\right\|_{\dot{H}^1(\mathbb{R}^2)}^2\right.\\ &\qquad\left. + \|R(n,J)\|_{\dot{H}^1(\mathbb{R}^2)}^2\right\}\\ &\quad\le \lim_{J\to\infty}\limsup_{n\to\infty}\left\{\sum_{\alpha=1}^{J} H\left(g_{n\alpha}\phi_\alpha^J\right) + H(R(n,J))\right\}\\ &\quad\le \lim_{J\to\infty}\limsup_{n\to\infty} H(u_n) \le 1.\end{aligned} \tag{112}$$

Lemma 19. *If*

$$\begin{gathered}\lim_{J\to\infty}\limsup_{n\to\infty}\|\widetilde{u}_n\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)} < \infty,\\ \left\|v_\alpha^J\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)} < \infty \quad (\forall\alpha),\end{gathered} \tag{113}$$

then

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\int_0^t e^{i(t-\tau)\triangle}\left((i\partial_t + \triangle)\widetilde{u}_n - f(\widetilde{u}_n)\right)(\tau)\,d\tau\right\|_X = 0, \tag{114}$$

where $X = L_{t,x}^4 \cap L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)$.

Proof. Denote

$$v_{n\alpha}^J = T_{h_{n\alpha}}\left[v_\alpha^J(\cdot + t_{n\alpha})\right]. \tag{115}$$

By the definition of $\widetilde{u}_n$, we have

$$\begin{gathered}\widetilde{u}_n = \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty} v_{n\alpha}^J + e^{it\triangle}R(n,J),\\ (i\partial_t + \triangle)\widetilde{u}_n = \sum_{\alpha\in\Lambda_1} f\left(v_{n\alpha}^J\right) + \sum_{\alpha\in\Lambda_\infty}\left|v_{n\alpha}^J\right|^2 v_{n\alpha}^J.\end{gathered} \tag{116}$$

Thus, by triangle inequality, it suffices to show that

$$\begin{aligned}&\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\int_0^t e^{i(t-\tau)\triangle}\left(f\left(\widetilde{u}_n - e^{it\triangle}R(n,J)\right)\right.\right.\\ &\qquad\left.\left. - f(\widetilde{u}_n)\right)(\tau)\,d\tau\right\|_X = 0,\end{aligned} \tag{117}$$

$$\begin{aligned}&\lim_{n\to\infty}\left\|\int_0^t e^{i(t-\tau)\triangle}\left(f\left(\sum_{\alpha\in\Lambda_1\cup\Lambda_\infty} v_{n\alpha}^J\right)\right.\right.\\ &\qquad\left.\left. - \sum_{\alpha\in\Lambda_1\cup\Lambda_\infty} f\left(v_{n\alpha}^J\right)\right)(\tau)\,d\tau\right\|_X = 0,\end{aligned} \tag{118}$$

$$\begin{aligned}&\lim_{n\to\infty}\left\|\int_0^t e^{i(t-\tau)\triangle}\sum_{\alpha\in\Lambda_\infty}\left(f\left(v_{n\alpha}^J\right) - \left|v_{n\alpha}^J\right|^2 v_{n\alpha}^J\right)(\tau)\,d\tau\right\|_X\\ &\quad= 0.\end{aligned} \tag{119}$$

By Lemma 18, let J and n be sufficiently large; we have

$$\|\widetilde{u}_n\|_{\dot{H}^1(\mathbb{R}^2)}^2 \le \frac{1}{1-4\varepsilon}, \qquad \left\|\widetilde{u}_n - e^{it\triangle}R(n,J)\right\|_{\dot{H}^1(\mathbb{R}^2)}^2 \le \frac{1}{1-4\varepsilon}. \tag{120}$$

Since we have supposed $0 < \lambda < 4(1-4\varepsilon)^2\pi$, Lemma 6 can also be used here. Then by the same estimates as (31),

$$\begin{aligned}&\left\|\int_0^t e^{i(t-\tau)\triangle}\left(f\left(\widetilde{u}_n - e^{it\triangle}R(n,J)\right) - f(\widetilde{u}_n)\right)(\tau)\,d\tau\right\|_X\\ &\quad\lesssim \left(\|\widetilde{u}_n\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^2 + \left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^2\right)\\ &\qquad\times\left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}\\ &\qquad+\left(\|\widetilde{u}_n\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{8\varepsilon} + \left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{8\varepsilon}\right)\\ &\qquad\times\left\|e^{it\triangle}R(n,J)\right\|_{L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}.\end{aligned} \tag{121}$$

By (54),

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)} = 0. \tag{122}$$

As

$$\begin{aligned}
&\left\|e^{it\triangle}R(n,J)\right\|_{L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}\\
&\quad\lesssim \left\|e^{it\triangle}R(n,J)\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{2\varepsilon/(1-2\varepsilon)}\left\|e^{it\triangle}R(n,J)\right\|_{L_t^{2/(1-\varepsilon)}L_x^{2/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}^{(1-4\varepsilon)/(1-2\varepsilon)},\\
&\left\|e^{it\triangle}R(n,J)\right\|_{L_t^{2/(1-\varepsilon)}L_x^{2/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}\lesssim \|R(n,J)\|_{L_x^2(\mathbb{R}^2)}<\infty,
\end{aligned}\tag{123}$$

we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|e^{it\triangle}R(n,J)\right\|_{L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}=0.\tag{124}$$

By $\lim_{J\to\infty}\limsup_{n\to\infty}\|\tilde{u}_n\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}<\infty$, (117) was obtained.

Using Strichartz estimate,

$$\begin{aligned}
&\left\|\int_0^t e^{i(t-\tau)\triangle}\left(f\left(v_{n\alpha}^J\right)-\left|v_{n\alpha}^J\right|^2v_{n\alpha}^J\right)(\tau)\,d\tau\right\|_X\\
&\quad\le\left\|\sum_{m=2}^{\infty}\frac{\left|v_{n\alpha}^J\right|^{2m}v_{n\alpha}^J}{m!}\right\|_{L_{t,x}^{4/3}(\mathbb{R}\times\mathbb{R}^2)}\\
&\quad\le\sum_{m=2}^{\infty}\frac{1}{m!}\left\|v_{n\alpha}^J\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}\left\|v_{n\alpha}^J\right\|_{L_{t,x}^{4m}(\mathbb{R}\times\mathbb{R}^2)}^{2m}\\
&\quad\le\sum_{m=2}^{\infty}\frac{1}{m!}\left(\lambda_{n\alpha}\right)^{-2(m-1)}\left\|v_\alpha^J\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}\left\|v_\alpha^J\right\|_{L_{t,x}^{4m}(\mathbb{R}\times\mathbb{R}^2)}^{2m}.
\end{aligned}\tag{125}$$

By Lemma 8,

$$\left\|v_\alpha^J\right\|_{L_{t,x}^{4m}(\mathbb{R}\times\mathbb{R}^2)}\le Cm^{1/2+1/2m}\left\|v_\alpha^J\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{1/m}\left\|v_\alpha^J\right\|_{L^\infty(\mathbb{R},\dot{H}^1)}^{1-1/m}.\tag{126}$$

Note that $\|v_\alpha^J\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}<\infty$. Equation (119) was obtained.

To prove Lemma 19, it is only left to prove (118). Note that

$$\left|f\left(\sum_{\alpha=1}^J z_\alpha\right)-\sum_{\alpha=1}^J f(z_\alpha)\right|\lesssim\sum_{\alpha\ne\alpha'}\left|z_\alpha\right|\left|e^{\lambda|z_{\alpha'}|^2}-1\right|.\tag{127}$$

By (57), we have

$$\begin{aligned}
&\left\|\int_0^t e^{i(t-\tau)\triangle}\left(f\left(\sum_{\alpha\in\Lambda_1\cup\Lambda_\infty}v_{n\alpha}^J\right)-\sum_{\alpha\in\Lambda_1\cup\Lambda_\infty}f\left(v_{n\alpha}^J\right)\right)(\tau)\,d\tau\right\|_X\\
&\quad\lesssim\sum_{\alpha\ne\alpha'}\left(\left\|v_{n\alpha}^J\left|v_{n\alpha'}^J\right|^2\right\|_{L_{t,x}^{4/3}(\mathbb{R}\times\mathbb{R}^2)}\right.\\
&\qquad\left.+\left\|v_{n\alpha}^J\left(e^{\lambda|v_{n\alpha'}^J|^2}-\lambda\left|v_{n\alpha'}^J\right|^2-1\right)\right\|_{L_t^{2/(1+2\varepsilon)}L_x^{1/(1-\varepsilon)}(\mathbb{R}\times\mathbb{R}^2)}\right)\\
&\quad\lesssim\sum_{\alpha\ne\alpha'}\left\|\left|v_{n\alpha}^J\right|^{1/2}\left|v_{n\alpha'}^J\right|^{1/2}\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^2\left\|v_{n\alpha'}^J\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}\\
&\qquad+\sum_{\alpha\ne\alpha'}\left\|\left|v_{n\alpha}^J\right|^{\varepsilon/(1+\varepsilon)}\left|v_{n\alpha'}^J\right|^{1/(1+\varepsilon)}\right\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^2)}^{4\varepsilon(1+\varepsilon)/(1-2\varepsilon)}\\
&\qquad\times\left\|v_{n\alpha}^J\right\|_{L_t^{2(1-2\varepsilon-4\varepsilon^2)/(1-2\varepsilon-6\varepsilon^2)}L_x^{(1-2\varepsilon-4\varepsilon^2)/\varepsilon^2}(\mathbb{R}\times\mathbb{R}^2)}^{(1-2\varepsilon-4\varepsilon^2)/(1-2\varepsilon)}\\
&\qquad\times\left\|\left|v_{n\alpha'}^J\right|^{-4\varepsilon/(1-2\varepsilon)}\right.\\
&\qquad\left.\times\left(e^{\lambda|v_{n\alpha'}^J|^2}-\lambda\left|v_{n\alpha'}^J\right|^2-1\right)\right\|_{L_t^\infty L_x^{(1-2\varepsilon)/(1-4\varepsilon)}(\mathbb{R}\times\mathbb{R}^2)}.
\end{aligned}\tag{128}$$

By (49), we immediately obtain (118).

If

$$\sup_{\alpha\in\Lambda_1\cup\Lambda_\infty}M(\phi_\alpha)\le m_0-\sigma\tag{129}$$

for some $\sigma>0$, by the definition of $A(m)$ and (100), we have

$$A(m)\le Bm\quad\forall m\in[0,m_0-\sigma],\tag{130}$$

where $B=B(\sigma)\in(0,+\infty)$. Then v_α^J satisfies

$$M\left(v_\alpha^J\right)=M\left(\phi_\alpha^J\right)\le m_0-\sigma,\tag{131}$$

$$S\left(v_\alpha^J\right)\le A\left(M\left(\phi_\alpha^J\right)\right)\le BM\left(\phi_\alpha^J\right).\tag{132}$$

By (50), (106), and (132), we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}S(\tilde{u}_n)\le Bm_0.\tag{133}$$

Using Lemmas 18 and 19, we have

$$\begin{aligned}
&M(\tilde{u}_n(0)-u_n(0))\le\delta,\qquad S(\tilde{u}_n)\le 2Bm_0,\\
&\left\|\int_0^t e^{i(t-\tau)\triangle}\left((i\partial_t+\triangle)\tilde{u}_n-f(\tilde{u}_n)\right)(\tau)\,d\tau\right\|_{L_{t,x}^4\cap L_t^{2/(1-2\varepsilon)}L_x^{1/\varepsilon}(\mathbb{R}\times\mathbb{R}^2)}\\
&\quad\le\delta,
\end{aligned}\tag{134}$$

for $\delta>0$ sufficiently small, $J=J(\delta)$, and $n=n(J,\delta)$ sufficiently large. By Lemma 10, we obtain that $S(u_n)\le 3Bm_0$ which contradicts (103). Thus, (129) fails for all $\sigma>0$, and then

$$\sup_{\alpha\in\Lambda_1\cup\Lambda_\infty}M(\phi_\alpha)=m_0.\tag{135}$$

Comparing this with (106), we have

$$u_n(0)=h_ne^{it_n\triangle}\phi+R_n\tag{136}$$

with t_n converging to $\pm\infty$ or $t_n \equiv 0$, $h_n \in G$, $M(\phi) = m_0$, and

$$M(R_n) \longrightarrow 0, \quad S\left(e^{it\triangle}R_n\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{137}$$

Specially, the parameters λ_n, ξ_n of h_n must satisfy

$$\text{“}\lambda_n \equiv 1 \text{ and } \xi_n \equiv 0\text{,”} \quad \text{or} \quad \text{“}\lambda_n \longrightarrow \infty\text{.”} \tag{138}$$

If $\lambda_n \to \infty$, similar to the former case, we can define the approximate solution

$$\widetilde{u}_n = T_{h_n}\left[v\left(\cdot + t_n\right)\right] + e^{it\triangle}R_n. \tag{139}$$

By the scattering of cubic Schrödinger equation, we have $S(v) < \infty$ and $\lim_{n\to\infty} S(\widetilde{u}_n) < \infty$. By Lemmas 18, 19, and 10, for n sufficiently large, we obtain $S(u_n) < \infty$ which contradicts (103).

If $\lambda_n \equiv 1$, $\xi_n \equiv 0$, and $t_n \to +\infty$, by Strichartz estimate and monotone convergence, we have

$$\lim_{n\to\infty} S_{\geq 0}\left(e^{it\triangle}e^{it_n\triangle}\phi\right) = 0. \tag{140}$$

Thus,

$$\lim_{n\to\infty} S_{\geq 0}\left(e^{it\triangle}h_n e^{it_n\triangle}\phi\right) = \lim_{n\to\infty} S_{\geq 0}\left(e^{it\triangle}e^{it_n\triangle}\phi\right) = 0. \tag{141}$$

Since $\lim_{n\to\infty} S(e^{it\triangle}R_n) = 0$, we can see from (136) that

$$\lim_{n\to\infty} S_{\geq 0}\left(e^{it\triangle}u_n(0)\right) = 0. \tag{142}$$

Hence,

$$\lim_{n\to\infty}\left\|\int_0^t e^{i(t-\tau)\triangle} f\left(e^{it\triangle}u_n(0)\right)(\tau)\,d\tau\right\|_{L^4_{t,x}\cap L^{2/(1-2\varepsilon)}_t L^{1/\varepsilon}_x([0,\infty)\times\mathbb{R}^2)} = 0. \tag{143}$$

By Lemma 10 (with $e^{it\triangle}u_n(0)$ as the approximate solution and $u_n(0)$ as the initial data), we have

$$\lim_{n\to\infty} S_{\geq 0}(u_n) = 0, \tag{144}$$

which contradicts one of the estimates in (103).

If $\lambda_n \equiv 1$, $\xi_n \equiv 0$, and $t_n \to -\infty$, the argument is the same and we can obtain a contradiction by using the other half of (103).

Now, the only case left is $\lambda_n \equiv 1$, $\xi_n \equiv 0$, and $t_n \equiv 0$. In this case, we have

$$M(u_n(0) - h_n\phi) = M(R_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{145}$$

Thus, $(h_n)^{-1}u_n(0) = e^{i\theta_n}u_n(0, x+x_n)$ converges to ϕ in $L^2_x(\mathbb{R}^2)$. Since $H(\phi) = H(g_n\phi) \leq 1$, after passing to a subsequence if necessary and rotating ϕ, the desired result follows. □

Let $\{u_n\}$ be the sequence given in Proposition 17 and satisfy $M(u_n) \leq m_0$ and suppose that $u_n(0, x + x_n)$ converges to u_0 strongly in $L^2_x(\mathbb{R}^2)$; then $M(u_0) \leq m_0$ and $H(u_0) \leq 1$. Let u be the global solution with initial data $u(0) = u_0$; by Lemma 10, we must have

$$S_{\geq 0}(u) = S_{\leq 0}(u) = +\infty. \tag{146}$$

By the definition of m_0, $M(u_0) \geq m_0$ and hence $M(u_0) = m_0$.

Since u is locally in $L^4_{t,x}$, for all $t_n \in \mathbb{R}$, we have

$$S_{\geq t_n}(u) = S_{\leq t_n}(u) = +\infty. \tag{147}$$

Using Proposition 17 for $\{u(t_n)\}$, we have that $u(t_n, x + x(t_n))$ converges into $L^2_x(\mathbb{R}^2)$. By Ascoli-Arzela Theorem, we have the following.

Proposition 18. *Suppose that the critical mass m_0 is finite. Then there exists a global solution u with mass m_0, and for every $\eta > 0$ there exists $0 < C(\eta) < \infty$ such that*

$$\int_{|x-x(t)|\geq C(\eta)} |u(t,x)|^2 dx + \int_{|\xi-\xi(t)|\geq C(\eta)} \left|\widehat{u}(t,\xi)\right|^2 d\xi \leq \eta \tag{148}$$

for all $t \in \mathbb{R}$, where the functions $x, \xi : \mathbb{R} \to \mathbb{R}^2$.

Proposition 19. *The solution described in Proposition 18 does not exist.*

Once we proved Proposition 19, we can say that $m_0 = \infty$ and thus Theorem 3 is true. In order to prove Proposition 19, we need the following two lemmas.

Lemma 20 (see [6, Lemma 5.2]). *Let u be a global solution of (5). Then one has*

$$\iint_{\mathbb{R}\times\mathbb{R}^2} \frac{\langle t\rangle^2 G(u)}{\langle t\rangle^3 + |x|^3} dx\,dt \leq C(E), \tag{149}$$

where $\langle t\rangle = \sqrt{1 + |t|^2}$.

Lemma 21 (see [6, Lemma 6.2]). *Let u be a global solution of (5). Let B be a compact subset of $\mathbb{R}^2$. Then for any $R > 0$ and $T > 0$, one has*

$$\int_{B(R)} |u(T,x)|^2 dx \geq \int_B |u(0,x)|^2 dx - \frac{C(E)T}{R}, \tag{150}$$

where $B(R) := \{x \in \mathbb{R}^2 \mid \exists y \in B \text{ s.t. } |x - y| < R\}$.

Proof of Proposition 19. By Lemma 21, choosing η sufficiently small,

$$\begin{aligned}\int_{|x-x(0)|\leq C(\eta)+R|t|} |u(t,x)|^2 dx &\geq \int_{|x-x(0)|\leq C(\eta)} |u(0,x)|^2 dx - \frac{C(E)}{R} \\ &\geq m_0 - \eta - \frac{C(E)}{R}.\end{aligned} \tag{151}$$

By Proposition 18,

$$\int_{|x-x(t)|\le C(\eta)} |u(t,x)|^2 dx \ge m_0 - \eta. \tag{152}$$

For a fixed large number R, we must have $|x(t) - x(0)| \le 2C(\eta) + R|t|$. By Lemma 20 and Hölder inequality,

$$\begin{aligned}
\infty &> \iint_{\mathbb{R}\times\mathbb{R}^2} \frac{\langle t\rangle^2 |u|^4}{\langle t\rangle^3 + |x|^3} dx\, dt \\
&\ge \int_{\mathbb{R}} \int_{|x-x(t)|\le C(\eta)} \frac{\langle t\rangle^2 |u|^4}{\langle t\rangle^3 + |x|^3} dx\, dt \\
&\ge \int_{\mathbb{R}} \int_{|x-x(t)|\le C(\eta)} \frac{|u|^4}{\langle t\rangle + 1} dx\, dt \\
&\ge \int_{\mathbb{R}} \int_{|x-x(t)|\le C(\eta)} \frac{|u|^2}{\langle t\rangle + 1} dx\, dt \\
&\ge (m_0 - \eta) \int_{\mathbb{R}} \frac{1}{\langle t\rangle + 1} dx\, dt = \infty.
\end{aligned} \tag{153}$$

This is a contradiction. Proposition 19 was obtained. □

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