Research Article

Sobolev Embeddings for Generalized Riesz Potentials of Functions in Morrey Spaces $L^{(1,\varphi)}(G)$ over Nondoubling Measure Spaces

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Our aim in this paper is to deal with the Sobolev embeddings for generalized Riesz potentials of functions in Morrey spaces $L^{(1,\varphi)}(G)$ over nondoubling measure spaces.

1. Introduction

In this paper, we show that many endpoint results about the Adams theorem still hold in the nondoubling setting and that the integral kernel can be generalized to a large extent. In [1], in the setting of the Lebesgue measure, for $0 < \alpha < n$, recall that Adams considered and proved the boundedness of the fractional integral operator $I_\alpha$ given by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$ (1)

The operator $I_\alpha$ is also called the fractional integral operator or the Riesz potential. We denote by $B(z,r)$ the ball $\{x \in \mathbb{R}^n : |x-z| < r\}$ with center $z$ and of radius $r > 0$, and by $|B(z,r)|$ its Lebesgue measure, that is, $|B(z,r)| = \omega_n r^n$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Let $G$ be a bounded open subset of $\mathbb{R}^n$. We denote its diameter by $d_G$;

$$d_G = \sup \{|x-y| : x, y \in G\}.$$ (2)

For $u \in L^1(G)$, we define the integral mean over $B(z,r)$ by

$$u_{B(z,r)} = \frac{1}{|B(z,r)|} \int_{B(z,r)} u(x) \, dx = \int_{G \cap B(z,r)} u(x) \, dx.$$ (3)

Let $1 \leq p < \infty$. If $\varphi$ is a positive function on the interval $(0, \infty)$ satisfying the doubling condition (see (23)), then we define the Morrey space $L^{(p,\varphi)}(G)$ to be the family of all $f \in L^p_{\text{loc}}(G)$ for which there is a positive constant $C$ such that

$$\int_{B(z,r)} \|f(x)\|^p \, dx \leq C \varphi(r)$$ (4)

whenever $z \in G$, $0 < r \leq d_G$.

The norm of $f \in L^{(p,\varphi)}(G)$ is defined by the infimum of the constants $C$ satisfying the inequality above. When $\varphi(r) \equiv r^{-\lambda}$ ($r > 0$), $L^{(p,\varphi)}(G)$ is denoted by $L^{p,\lambda}(G)$.

A direct consequence of this notation is that

$$L^{(p,\lambda)}(G) \supset L^{n/\lambda}(G)$$ (5)

for $0 < \lambda \leq n$ and $p \in [1, n/\lambda)$.

Some prefer to use the notation

$$\|f\|_{M_{p,\alpha}} = \sup_{r > 0} w(r) \|f\|_{L^p(B(z,r))}$$ (6)

with

$$w(r) = (\omega_n r^n)^{-1/p} \varphi(r)^{-1/p}$$ (7)

references [2–5].
Much about the case \( p > 1 \) is known. Recall that the Adams theorem about the boundedness of fractional integral operators [1, Theorem 3.1] asserts that
\[
\|I_\alpha f\|_{L^{p,\lambda}} \leq C\|f\|_{L^{p,\lambda}},
\]
provided the parameters \( p, q, \lambda \) satisfy
\[
0 < \lambda \leq n, \quad 1 < p < q < \infty, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\lambda}.
\]

See also research papers [2–4, 6–16] and a survey [5].

Meanwhile, only a few results are known for the case \( p = 1 \). Trudinger [17, Theorem 1] proved that if \( f \in L_{1,1}(G) = L^1(G) \) then \( \exp(a I_1 f) \in L^1(G) \) for some constant \( a > 0 \); this implies that the operator \( I_1 \) is bounded from \( L_{1,1}(G) \) to \( \exp(L^1(G)) \). See also Serrin [18] for an alternative proof.

Recently, the boundedness of Riesz potentials from \( L^{1,\phi}(G) \) to Orlicz-Morrey spaces was shown in [19]. This result extends [20, 21]. One of the reasons why the case when \( p = 1 \) is difficult is the failure of the boundedness of the Hardy-Littlewood maximal operator \( M \). In connection with this failure, we do not have Littlewood-Paley characterization. Due to these two difficulties, the case when \( p = 1 \) is hard to analyze. However, from the point of PDEs, we are faced with analyzing the quantity
\[
\lim_{r \downarrow 0} \left( \sup_{x \in \mathbb{R}^n} \int_{B(x,r)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy \right)^{\frac{1}{2}}
\]
in connection of the Kato condition, where \( V \) is the potential operator of the operator \( -\Delta + V \). See [22, Section 2], for example. Consequently, despite the difficulty arising from harmonic analysis, the case when \( p = 1 \) occurs naturally. As another evidence that the case when \( p = 1 \) is of importance, we recall that the space \( L_{1,1}(\mathbb{R}^n) \) appears naturally in the following sharp maximal inequalities [23, Theorem 4.7], [24, Theorem 1.3], and [25, Theorem 1.2]: let \( 1 < p < \infty \) and \( \lambda \in (0, n] \). Then, there exists a constant \( C > 0 \) such that
\[
C^{-1} \left( \|M^1 f\|_{L^{p,\lambda}} + \|f\|_{L^{1,1}} \right) \leq \|f\|_{L^{p,\lambda}} \leq C \left( \|M^1 f\|_{L^{p,\lambda}} + \|f\|_{L^{1,1}} \right)
\]
for any measurable function \( f \), where
\[
M^1 f(x) = \sup_{y \in \mathbb{R}^n, r > 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} f(y) dy
\]
\[
\times \int_{B(y,r)} \frac{1}{B(y,r)} \int_{B(y,r)} f(w) dw dz
\]
is the sharp maximal operator due to Fefferman and Stein [26]. A disadvantage of using the Littlewood-Paley theory is that we lose the integrability of functions a little when we consider the inequality
\[
\sup_{j \in \mathbb{Z}} \|S_j f\|_{L^{1,\lambda}} \leq C\|f\|_{L^{1,\lambda}},
\]
where \( \{S_j\}_{j=-\infty}^{\infty} \) is a Littlewood-Paley patch. By choosing a smooth function \( \varphi \in C_c^\infty(\mathbb{R}^n) \) such that \( \chi_{B(0,4)}(B(0,2)) \leq \varphi \leq \chi_{B(0,8)}(B(0,1)) \), recall that we can define the \( j \)-th Littlewood-Paley patch by
\[
S_j f(x) := \mathcal{F}^{-1} \left[ \varphi \left( 2^{-j} \cdot \right) \mathcal{F} f(x) \right]
\]
for \( f \in \mathcal{S}'(\mathbb{R}^n) \). Note that (13) is a direct consequence of the translation invariance of the space \( L_{1,1}(\mathbb{R}^n) \). But this loss caused by (13) is quite big. Note that
\[
\|f\|_{L_{1,1}} \leq C \sup_{j \in \mathbb{Z}} \|S_j f\|_{L_{1,1}}
\]
fails. See the appendix for a proof. When \( p > 1 \), an approach using the Littlewood-Paley patch is taken effectively [27]. Indeed,
\[
C^{-1} \|f\|_{L_{p,\lambda}} \leq \left( \sum_{j=-\infty}^{\infty} \|S_j f\|_{L^{p,\lambda}}^2 \right)^{1/2} \leq C\|f\|_{L_{p,\lambda}}
\]
for all \( f \in L_{p,\lambda}(\mathbb{R}^n) \). However, for the case when \( p = 1 \), due to the fact that the estimate (13) is essential when we consider the Littlewood-Paley patch, we prefer to avoid the Littlewood-Paley patch. See [28–43] for a huge amount of culmination of this approach. Instead of using the Littlewood-Paley patch, we still have a good approach for the case when \( p = 1 \). Just make a closer look at the integral kernel. Our method being simple enough, there is no need to stick to the geometric structure of \( \mathbb{R}^n \).

Our result relies completely only upon the positivity of the integral kernel. So, here and below, we work on a separable metric space \( X \) equipped with a nonnegative Radon measure \( \mu \), where we do not postulate any other condition on \( \mu \). By \( B(x, r) \), we denote the open ball centered at \( x \) of radius \( r > 0 \). While, given a point \( p_1 \) and \( p_2 \) in \( \mathbb{R}^n \), we write \( [p_1, p_2] \) for the distance of the points \( p_1 \) and \( p_2 \), and we write \( d(x, y) \) for the distance of the points \( x \) and \( y \) in \( X \). We assume that \( \mu([x]) = 0 \) and that \( 0 < \mu(B(x, r)) < \infty \) for \( x \in X \) and \( r > 0 \) for simplicity. In the present paper, we do not postulate on \( \mu \) the "so-called" doubling condition. Recall that a Radon measure \( \mu \) is said to be doubling, if there exists a constant \( C > 0 \) such that
\[
\mu(B(x,2r)) \leq C \mu(B(x,r))
\]
for all \( x \in \text{supp}(\mu) (= X) \) and \( r > 0 \). Otherwise \( \mu \) is said to be nondoubling. In connection with the 5r-covering lemma, the doubling condition had been a key condition in harmonic analysis.

Our aim in this paper is to show that, for the case \( p = 1 \), the operator \( I_\alpha \) and its generalization \( I_\lambda \) are bounded from Morrey spaces \( L^{1,\phi}(\mathbb{R}^n) \) to Orlicz-Morrey spaces, or, to generalized Hölder spaces, whose definitions will be given in the next section, in the nondoubling setting. Our result extends the results in [17–21]. The definition of \( I_\lambda \) is the following: let \( \rho \) be a function from \( (0, \infty) \) to itself and satisfy
\[
\int_0^\tau \frac{\rho(t)}{t} dt < +\infty
\]
for all sufficiently small $r > 0$. We do not have to postulate the doubling condition on $\rho$. See Remark 3 for an example which fails the doubling condition. We define

$$I_\rho f(x) = \int_G \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} f(y)\,d\mu(y), \quad (19)$$

where $f \in L^1(G)$. Instead of using

$$I_\rho^k f(x) = \int_G \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} f(y)\,d\mu(y), \quad (20)$$

we discuss $I_\rho$ defined above. This modification will be necessary in Lemma 9 for example. An example in [44, Section 2] shows that $I_\rho^k$ is less likely to be bounded in general, although there does not exist a proof. We refer to [45] for an attempt of defining fractional integral operators by using the underlying measure $\mu$.

Note that (18) is necessary in order that the image by $I_\rho$ of $\chi_{B(x,r)}$, the indicator functions of the balls, belongs to $L^{p\#}\mu(G)$ at least when $\mu$ is the Lebesgue measure. Indeed, if

$$\int_0^r \frac{\rho(t)}{t} \,dt = \infty \quad (21)$$

for any sufficiently small $r > 0$. Then, for $y \in B(x,r/2)$ such that $B(x,r) \subset G$, we have

$$I_\rho \chi_{B(x,r)}(y) = \int_{B(x,r)} \frac{\rho(|y-z|)}{|B(y,4|y-z|)|} \,dz$$

$$\geq \int_{B(y,r/2)} \frac{\rho(|y-z|)}{|B(y,4|y-z|)|} \,dz \quad (22)$$

$$= C \int_0^{r/2} \frac{\rho(t)}{t} \,dt$$

$$= \infty$$

by using the spherical coordinate.

We organize the remaining part of the present paper as follows. In Section 2, we set up some notations. Section 3 is devoted to stating our main results fully based on the notations in Section 2. Some auxiliary lemmas are collected in Section 4. Finally, theorems in the present paper are proven in Section 5.

2. Notation and Terminologies

Let $\mathcal{G}$ be the set of all continuous functions from $(0, \infty)$ to itself with the doubling condition, that is, there exists a constant $c_\varphi \geq 1$ such that

$$\frac{1}{c_\varphi} \leq \frac{\varphi(r)}{\varphi(s)} \leq c_\varphi \quad \text{for } r, s > 0 \text{ with } \frac{1}{2} \leq \frac{r}{s} \leq 2. \quad (23)$$

We call the smallest number $c_\varphi$ satisfying (23) the doubling constant of $\varphi$. Note that in view of [46, page 445] and [47, (1.2)], the doubling condition on $\varphi$ is a natural one. For $\varphi \in \mathcal{G}$, we define the Morrey space $L^{(1,\varphi)}(G)$ as follows:

$$L^{(1,\varphi)}(G) := \{ f \in L^1_{loc}(G) : \| f \|_{L^{(1,\varphi)}(G)} < \infty \} \quad (24)$$

with the norm

$$\| f \|_{L^{(1,\varphi)}(G)} = \sup_{x \in G, 0 < r \leq d_x} \frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f(x)|\,d\mu(x). \quad (25)$$

Then, a routine argument shows that $L^{(1,\varphi)}(G)$ is a Banach space. Due to the fact that $\mathbb{R}^n$ is a geometrically doubling space, we can prove that

$$C^{-1} \| f \|_{L^{(1,\varphi)}(G)} \leq \sup_{x \in G, 0 < r \leq d_x} \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} |f(x)|\,d\mu(x) \leq C \| f \|_{L^{(1,\varphi)}(G)}$$

for all $k > 1$. See [48, Proposition 1.1] for a technique used to prove this inequality. Note here that if $\varphi_1, \varphi_2 \in \mathcal{G}$ and $\varphi_1/\varphi_2$ is bounded above on $(0, d_G)$, then

$$L^{(1,\varphi_1)}(G) \subset L^{(1,\varphi_2)}(G), \quad (27)$$

in particular, if there exists a constant $C \geq 1$ such that $C^{-1} \varphi_1(r) \leq \varphi_2(r) \leq C \varphi_1(r)$ for all $r > 0$, then

$$L^{(1,\varphi_1)}(G) = L^{(1,\varphi_2)}(G) \quad (28)$$

with equivalent norms. A ball testing shows the following.

**Proposition 1.** The function $\varphi_1/\varphi_2$ is bounded above on $(0, d_G)$ if $L^{(1,\varphi_1)}(G) \subset L^{(1,\varphi_2)}(G)$ when $\mu = dx$.

Here and below, we write $A \leq B$ to indicate that there exists a constant $C$ independent of Morrey functions such that $A \leq CB$. The symbol $A \sim B$ stands for $A \leq B \leq A$.

**Proof.** According to [49, Proposition A], for any ball $B(x_0, r)$ contained in $G$, we have

$$\| \chi_{B(x_0,r)} \|_{L^{(1,\varphi_1)}(G)} \sim \frac{1}{\varphi_1(r)}, \quad \| \chi_{B(x_0,r)} \|_{L^{(1,\varphi_2)}(G)} \sim \frac{1}{\varphi_2(r)}. \quad (29)$$

If $L^{(1,\varphi_1)}(G) \subset L^{(1,\varphi_2)}(G)$, in the sense of sets, then by the closed graph theorem and the doubling condition on $\varphi_1$ and $\varphi_2$, we conclude

$$\| f \|_{L^{(1,\varphi_2)}(G)} \leq C \| f \|_{L^{(1,\varphi_1)}(G)} \quad (30)$$

If we combine (29) and (30), then we obtain that $\varphi_1/\varphi_2$ is bounded above on $(0, d_G)$. □
Let us consider the family \( \mathcal{Y} \) of all continuous, increasing, convex, and bijective functions from \([0, \infty)\) to itself. For \( \Phi \in \mathcal{Y} \), the Orlicz space \( L^\Phi (G) \) is defined by
\[
L^\Phi (G) := \{ f \in L^1_{\text{loc}} (G) : \| f \|_{L^\Phi (G)} < \infty \},
\]
where
\[
\| f \|_{L^\Phi (G)} := \inf \left\{ \lambda > 0 : \int_G \Phi \left( \frac{|f(x)|}{\lambda} \right) d \mu(x) \leq 1 \right\}.
\]
If \( \Phi_1, \Phi_2 \in \mathcal{Y} \) are equivalent in the sense that there exists a constant \( C \geq 1 \) with
\[
\Phi_1 \left( C^{-1} r \right) \leq \Phi_2 (r) \leq \Phi_1 (Cr)
\]
for all \( r > 0 \), then we see easily that
\[
L^\Phi_1 (G) = L^\Phi_2 (G)
\]
with equivalent norms. If
\[
\Phi (r) = \exp (r^p), \quad \exp (r^p),
\]
for large \( r > 0 \), then \( L^\Phi (G) \) will be denoted by
\[
\exp (L^p) (G), \quad \exp (L^p) (G),
\]
respectively.

For \( \Phi \in \mathcal{Y} \) and \( \varphi \in \mathcal{G} \), the Orlicz-Morrey space \( L^{\Phi, \varphi} (G) \) is defined by
\[
L^{\Phi, \varphi} (G) := \{ f \in L^1_{\text{loc}} (G) : \| f \|_{L^{\Phi, \varphi} (G)} < \infty \},
\]
where
\[
\| f \|_{L^{\Phi, \varphi} (G)} := \sup_{z \in G, 0 < r \leq d_G} \left\{ \int_{B(z, r)} \frac{1}{\mu (B(z, 2r))} \int_{B(z, r)} \Phi \left( \frac{|f(x)|}{\lambda} \right) d \mu(x) \leq \varphi (r) \right\}.
\]

(see [50, 51]). Then, again it is routine to prove that \( \| \cdot \|_{L^{\Phi, \varphi} (G)} \) is a norm and that \( L^{\Phi, \varphi} (G) \) is a Banach space. Note that the space \( L^\Phi \) is a special case of Orlicz-Morrey spaces when \( \mu = dx \).

For \( \varphi \in \mathcal{G} \) such that \( \varphi \) is bounded, the generalized Hölder space is defined by
\[
\Lambda^\varphi (G) := \{ f : \| f \|_{\Lambda^\varphi (G)} < \infty \},
\]
where
\[
\| f \|_{\Lambda^\varphi (G)} = \sup_{x, y \in G, x \neq y} \frac{|f(x) - f(y)|}{\varphi (2d(x, y))}.
\]

Then, \( \| f \|_{\Lambda^\varphi (G)} \) is a norm modulo constants and thereby \( \Lambda^\varphi (G) \) is a Banach space. Since \( \varphi \) is bounded, every \( f \in \Lambda^\varphi (G) \) is bounded. If \( \varphi (r) \to 0 \) as \( r \downarrow 0 \), then every \( f \in \Lambda^\varphi (G) \) is continuous. For details, we refer to [52].

### 3. Main Results

In this section, we state our main theorems, whose proofs are given in Section 5.

Throughout this paper, let \( G \) be a bounded open set in \( X \) and denote by \( c_p \), the doubling constant of \( \varphi \in \mathcal{G} \).

Let us begin with the following result, which is the one of Gunawan type [9].

**Theorem 2.** Let \( \rho : (0, \infty) \to (0, \infty) \) be a measurable function such that there exist \( k_1, k_2, C_0 \) such that
\[
0 \leq 16k_1 \leq 1 \leq k_2 < \infty,
\]
\[
\sup_{r/2 \leq s \leq r} \rho (s) \leq C_r \int_{k_r}^{k_r s} \rho (t) \frac{ds}{t} \quad (r > 0).
\]

Let \( \varphi \in \mathcal{G} \), and define
\[
\varphi (r) := \left( \int_0^{4k_r} \rho (t) \frac{dt}{t} \right) \varphi (r) + \int_{4k_r}^{d_G} \rho (t) \varphi (t) \frac{dt}{t}
\]
for \( 0 < r \leq d_G \). Then, there exists a constant \( C > 0 \) such that
\[
\frac{1}{\mu (B(z, 4r))} \int_{B(z, r)} \left| f(x) \right|^p d \mu (x) \leq C \varphi (r) \| f \|_{L^{\varphi} (G)}
\]
for \( z \in G, 0 < r \leq d_G \) and \( f \in L^{\varphi} (G) \), where \( C > 0 \) is a constant depending only on \( C_0, c_p, k_1, k_2 \).

**Remark 3.**
1. Here it is not significant for us to choose 16; it counts that any number will do as long as it is small enough.
2. The number 4 in the right-hand side seems to be essential. According to [44, Section 2], it can happen that the norms
\[
\sup_{z \in G, 0 < r \leq d_G} \left( \int_{B(z, r)} \left| f(w) \right|^p d \mu(w) \right)^{1/p}
\]}

are not equivalent for \( 1 \leq p < \infty \).

3. In view of [53, Lemma 2.5], we see that \((1 - \Delta)^{-\alpha/2}\) falls under the scope of Theorem 2. Indeed, Nagayasu and Wadade showed that the kernel \( \rho \) which corresponds to \((1 - \Delta)^{-\alpha/2}\) satisfies
\[
\rho (r) \sim r^\alpha \quad (0 < r < 1), \quad \rho (r) \leq e^{-\gamma} \quad (r \geq 1).
\]
This means that we have (41) with \( k_1 = 1/16 \) and \( k_2 = 1 \). Note that \( \rho \in \mathcal{G} \) implies (41). See also [54, Remark 2.2].

**Remark 4.** Theorem 2 is proved in [19] when \( G = \mathbb{R}^n \) and \( \rho \in \mathcal{G} \). See also [21].

We now state a result for Orlicz-Morrey spaces.

**Theorem 5.** Let \( \rho, \tilde{\rho} : (0, \infty) \to (0, \infty) \) be measurable functions such that there exist \( k_1, k_2, C_\rho \) such that \( 0 < 16k_1 \leq 1 \leq k_2 < \infty \) and that

\[
\sup_{r/2 \leq s \leq r} \rho(s) \leq C_\rho \int_{k_1r}^{k_2r} \frac{\rho(s)}{s} ds,
\]

and

\[
\sup_{r/2 \leq s \leq r} \tilde{\rho}(s) \leq C_\tilde{\rho} \int_{k_1r}^{k_2r} \frac{\tilde{\rho}(s)}{s} ds \quad (r > 0).
\]

Let \( \varphi \in \mathcal{G} \). Assume

\[
\int_0^1 \frac{\rho(t) \varphi(t)}{t} dt = \infty
\]

and that \( \tilde{\varphi}/\varphi \) is continuous and decreasing. Define

\[
\psi_1(r) := \int_{2kr}^{4k_1r} \frac{\rho(t) \varphi(t)}{t} dt,
\]

\[
\kappa(r) := \frac{\psi_1(r) \tilde{\rho}(4k_2r)}{\rho(4k_2r)},
\]

\[
\psi(r) := \left( \int_0^{2kr} \frac{\tilde{\rho}(t)}{t} dt \right) \varphi(r) + \int_{2kr}^{4k_1r} \frac{\tilde{\rho}(t) \varphi(t)}{t} dt
\]

for \( 0 < r \leq d_G \). If \( \Phi \in \mathcal{Y} \) satisfies

\[
C_G = \sup \left\{ \left( \psi \circ \kappa^{-1} \right)(s) : \kappa(d_G) \leq s < \infty \right\} < \infty,
\]

then there exists a constant \( A > 0 \) such that

\[
\frac{1}{\mu(B(z, 4r))} \int_{B(z, r)} \phi \left( \frac{|f|}{\mu f \mathbb{L}^{1,\rho}(G)} \right) d\mu(x) \leq \psi(r)
\]

for \( z \in G, 0 < r \leq d_G \) and \( f \in \mathbb{L}^{1,\rho}(G) \), where \( A > 0 \) is a constant depending only on \( C_\rho, C_\tilde{\rho}, \varphi, k_1, k_2, \) and \( C_G \).

**Remark 6.** Note that \( \kappa \) is bijective from \( (0, d_G] \) to \( [\kappa(d_G), \infty) \) by the assumptions in the theorem. Indeed, by the definition of \( \kappa \) above, \( \kappa \) is a decreasing function. In addition, \( \lim_{t \to 0} \kappa(t) = \infty \), showing that \( \kappa : (0, d_G] \to [\kappa(d_G), \infty) \) is bijective.

Finally, we shall show a result of Gunawan type about continuity.

**Theorem 7.** Let \( \rho : (0, \infty) \to (0, \infty) \) be a measurable function such that there exist \( k_1, k_2, C_\rho \) such that

\[
0 < 16k_1 \leq 1 \leq k_2 < \infty,
\]

\[
\sup_{r/2 \leq s \leq r} \rho(s) \leq C_\rho \int_{k_1r}^{k_2r} \frac{\rho(s)}{s} ds \quad (r > 0).
\]

Let \( \varphi \in \mathcal{G} \). Assume the following condition on \( \rho \). There are \( 0 < \theta \leq 1 \) and \( C^\rho > 0 \) such that

\[
\frac{\rho(d(x, y))}{\mu(B(x, 4d(x, y)))} - \frac{\rho(d(z, y))}{\mu(B(z, 4d(z, y)))} \leq C^\rho \left( \frac{d(x, z)}{d(x, y)} \right)^\theta \frac{\rho(d(x, y))}{\mu(B(x, 4d(x, y)))}
\]

whenever \( d(x, z) \leq d(x, y)/2 \). Assume in addition the Dini condition

\[
\int_0^1 \frac{\rho(t) \varphi(t)}{t} dt < \infty.
\]

If

\[
\psi(r) = \int_0^{3k_1r} \frac{\rho(t) \varphi(t)}{t} dt + r^\theta \int_{2kr}^{4k_1r} \frac{\rho(t) \varphi(t)}{t^{1+\theta}} dt
\]

for \( 0 < r \leq d_G \), then \( I_\rho \) is bounded from \( \mathbb{L}^{1,\varphi}(G) \) to \( \Lambda^\varphi(G) \). More precisely,

\[
\left\| I_\rho f \right\|_{\Lambda^\varphi(G)} \leq C \| f \|_{\mathbb{L}^{1,\varphi}(G)},
\]

where \( C > 0 \) is a constant depending only on \( C_\rho, C^\rho, \rho, k_1, k_2, \) and \( \theta \).

Note that if \( \int_0^1 (\rho(t) \varphi(t)/t) dt < \infty \) and \( 0 < \theta \leq 1 \), then

\[
r \in (0, d_G] \mapsto r^\theta \int_{2kr}^{4k_1r} \frac{\rho(t) \varphi(t)}{t^{1+\theta}} dt \in (0, \infty)
\]

is bounded.

4. **Preliminary Lemmas**

**Lemma 8.** Let \( \rho : (0, \infty) \to (0, \infty) \) be a measurable function such that there exist \( k_1, k_2, C_\rho \) such that

\[
0 < 16k_1 \leq 1 \leq k_2 < \infty,
\]

\[
\sup_{r/2 \leq s \leq r} \rho(s) \leq C_\rho \int_{k_1r}^{k_2r} \frac{\rho(s)}{s} ds \quad (r > 0).
\]

Let \( \varphi \in \mathcal{G} \). Then

\[
\int_{B(x, r)} \frac{\rho(d(x, y))}{\mu(B(x, 4d(x, y)))} |f(y)| d\mu(y)
\]

\[
\leq C \left( \int_0^{2k_1r} \frac{\rho(t) \varphi(t)}{t} dt \right) \| f \|_{\mathbb{L}^{1,\varphi}(G)},
\]

where \( C > 0 \) is a constant depending only on \( C_\rho, \varphi, k_1, \) and \( k_2 \).
Moreover, if $k \geq 0$, then
\[
\int_{B(x,2r)} \mu(B(x,4d(x,y))) \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \leq C \left( \sum_{l=0}^{k} \frac{\rho(t) \varphi(t)}{t^{1+k}} \right) \|f\|_{L^{1,\psi}(G)},
\]
where $C > 0$ is a constant depending only on $C_{\rho}, c_{\psi}, k_{1}, k_{2}$, and $k$.

Proof. If $y \in B(x,2r) \setminus B(x,2^{-l}r)$ and $j \in \mathbb{Z}$, then a geometric observation shows
\[
\int_{B(x,2^{-l}r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \leq \frac{1}{\mu(B(x,2^{j+1}r))(2^{j+1}r)^{k-1}} \sup_{2^{j+1}r \leq s \leq 2^{j+1}r} \rho(s).
\]
which proves (58).

We choose $j_{0} \in \mathbb{Z}$, so that $d_{G} \leq 2^{j_{0}}r - 2^{j_{0}+1}r$. Then, we have
\[
\int_{B(x,2^{j_{0}+1}r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \leq \sup_{2^{j_{0}+1}r \leq s \leq 2^{j_{0}+1}r} \rho(s).
\]

Set $d := [1 + \log_{2}(k_{2}/k_{1})]$. Then, by virtue of the doubling condition on $\varphi$, we have
\[
\varphi \left( \frac{2^{j_{0}+1}r}{2^{j_{0}+1}r} \right) \int_{2^{j_{0}+1}r}^{2^{j_{0}+1}r} \frac{\rho(s)}{s} \, ds \leq 2^{k} \varphi \left( \frac{2^{j_{0}+1}r}{2^{j_{0}+1}r} \right) \int_{2^{j_{0}+1}r}^{2^{j_{0}+1}r} \frac{\rho(s)}{s} \, ds \leq 2^{2k} \varphi \left( \frac{2^{j_{0}+1}r}{2^{j_{0}+1}r} \right) \int_{2^{j_{0}+1}r}^{2^{j_{0}+1}r} \frac{\rho(s)}{s} \, ds
\]
and that
\[
\sup_{r/2 \leq s \leq r} \frac{\rho(s)}{s} \leq C_{\rho} \frac{k_{2}r}{r} \int_{k_{2}r}^{kr} \frac{\rho(s)}{s} \, ds \quad (r > 0).
\]
Then, for all \( f \in L^{1,\varphi}(G) \),
\[
\frac{1}{\mu(B(z,2r))} \times \int_{B(z,r)} \left( \int_{B(z,r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \right) \, d\mu(x) \\
\leq C\varphi(r) \left( \int_0^{2k} \frac{\rho(t)}{t} \, dt \right) \|f\|_{L^{1,\varphi}(G)}.
\]
(67)

Proof. By Fubini’s theorem and the dyadic decomposition of the ball, we have
\[
\int_{B(z,r)} \left( \int_{B(z,r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \right) \, d\mu(x) \\
= \int_{B(z,r)} |f(y)| \times \left( \int_{B(z,r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} \, d\mu(x) \right) \, d\mu(y) \\
\leq \int_{B(z,r)} |f(y)| \times \left( \sum_{j=0}^{\infty} \int_{B(y,2^{-j}r)} \frac{\sup_{2^{-j}r \leq s \leq 2^{-j}r} \rho(s)}{\mu(B(y,2^{-j+1}r))} \, d\mu(x) \right) \, d\mu(y).
\]
(68)

Since \( \rho \) satisfies (66), we have
\[
\int_{B(z,r)} \left( \int_{B(z,r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \right) \, d\mu(x) \\
\leq C\rho \int_{B(z,r)} |f(y)| \times \left( \sum_{j=0}^{\infty} \int_{B(y,2^{-j+r}r)} \frac{\rho(s)}{\mu(B(y,2^{-j+1}r))} \, d\mu(x) \right) \, d\mu(y) \\
\leq C\rho \left( \int_0^{2k} \frac{\rho(t)}{t} \, dt \right) \int_{B(z,r)} |f(y)| \, d\mu(y) \\
\leq C\rho \varphi(r) \mu(B(z,2r)) \left( \int_0^{2k} \frac{\rho(t)}{t} \, dt \right) \|f\|_{L^{1,\varphi}(G)},
\]
(69)
as required. \qed

5. Proofs of the Theorems

We are now ready to prove our theorems.

Proof of Theorem 2. Let \( z \in G \) and \( r \in (0, d_G] \). By the positivity of the kernel, we may assume that \( f \geq 0 \). We write
\[
\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} I_p f(x) \, d\mu(x) \\
\leq \frac{1}{\mu(B(z,4r))} \times \int_{B(z,2r)} \left( \int_{B(x,4d(x,y))} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} \, d\mu(y) \right) \, d\mu(x) \\
+ \frac{1}{\mu(B(z,4r))} \times \int_{B(z,2r)} \left( \int_{B(x,4d(x,y))} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} \, d\mu(y) \right) \, d\mu(x)
\]
(70)

Since \( \rho \) satisfies (66), we have
\[
\int_{B(z,r)} \left( \int_{B(z,r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)| \, d\mu(y) \right) \, d\mu(x) \\
\leq C\rho \int_{B(z,r)} |f(y)| \times \left( \sum_{j=0}^{\infty} \int_{B(y,2^{-j}r)} \frac{\rho(s)}{\mu(B(y,2^{-j+1}r))} \, d\mu(x) \right) \, d\mu(y) \\
\leq C\rho \left( \int_0^{4k} \frac{\rho(t)}{t} \, dt \right) \int_{B(z,r)} |f(y)| \, d\mu(y) \\
\leq C\rho \varphi(r) \mu(B(z,2r)) \left( \int_0^{4k} \frac{\rho(t)}{t} \, dt \right) \|f\|_{L^{1,\varphi}(G)}.
\]
(71)

Meanwhile, by Lemma 8 we have
\[
I_2 \leq C_2 \left( \int_{2k}^{4k} \frac{\rho(t) \varphi(t)}{t} \, dt \right) \|f\|_{L^{1,\varphi}(G)}
\]
(72)

as required.
Hence, it follows from (71) and (72) that
\[
\frac{1}{\mu(B(z, 4r))} \int_{B(z,r)} I_p f(x) \, d\mu(x) \leq C \psi(r) \|f\|_{L^{1,\psi}(G)},
\] (73)
where \( C > 0 \) depends only on \( C_p, c_p, k_1, \) and \( k_2 \). \( \square \)

**Proof of Theorem 5.** By Theorem 2, we have
\[
\frac{1}{\mu(B(z, 4r))} \int_{B(z,r)} |I_p f(x)| \, d\mu(x) \leq C_1 \psi(r) \|f\|_{L^{1,\psi}(G)}.
\] (74)
for \( z \in G \) and \( 0 < r \leq d_G \). Let \( g := |f|/\|f\|_{L^{1,\psi}(G)} \). For \( x \in G \) and \( 0 < \delta \leq d_G \), since \( \tilde{\rho}/\rho \) is decreasing, we have by Lemma 8
\[
I_p g(x) = \int_{B(x,\delta)} \frac{\rho (d(x, y))}{\mu(B(x, 4d(x, y)))} g(y) \, d\mu(y)
+ \int_{B(x,\delta) \setminus B(x,\delta)} \frac{\rho (d(x, y))}{\mu(B(x, 4d(x, y)))} g(y) \, d\mu(y)
\leq \frac{\rho (\delta)}{\tilde{\rho} (\delta)} \int_{B(x,\delta)} \frac{\tilde{\rho} (d(x, y))}{\mu(B(x, 4d(x, y)))} g(y) \, d\mu(y)
+ C_2 \frac{\rho (4k_2 \delta)}{\rho (4k_1 \delta)} I_p g(x) + C_2 \psi_1 (\delta)
\leq \frac{\rho (\delta)}{\tilde{\rho} (\delta)} I_p g(x) + C_2 \psi_1 (\delta).
\] (75)
Hence, in view of the definition of \( \kappa \), we have
\[
I_p g(x) \leq \frac{\psi_1 (\delta)}{\kappa (\delta)} I_p g(x) + C_2 \psi_1 (\delta).
\] (76)
Now let
\[
\delta := \begin{cases} 
\kappa^{-1} (I_p g(x)) & \text{when } I_p g(x) \geq \kappa (d_G), \\
-d_G & \text{when } I_p g(x) < \kappa (d_G).
\end{cases}
\] (77)
Observe that
\[
\psi_1 (\delta) = \begin{cases} 
\psi_1 (\kappa^{-1} (I_p g(x))) & \text{when } I_p g(x) \geq \kappa (d_G), \\
\psi_1 (d_G) & \text{when } I_p g(x) < \kappa (d_G),
\end{cases}
\] (78)
by definition.
We claim that
\[
\frac{\psi_1 (\delta)}{\kappa (\delta)} I_p g(x) \leq \begin{cases} 
\psi_1 (\kappa^{-1} (I_p g(x))) & \text{when } I_p g(x) \geq \kappa (d_G), \\
\psi_1 (d_G) & \text{when } I_p g(x) < \kappa (d_G).
\end{cases}
\] (79)
Indeed, when \( I_p g(x) < \kappa (d_G) \), we have \( \delta = d_G \). Hence,
\[
\psi_1 (\delta) \frac{I_p g(x)}{\kappa (\delta)} = \psi_1 (d_G) \times \frac{1}{\kappa (d_G)} I_p g(x) \leq \psi_1 (d_G).
\] (80)
When \( I_p g(x) \geq \kappa (d_G) \), we have \( \delta = \kappa^{-1} (I_p g(x)) \). Hence,
\[
\psi_1 (\delta) \frac{I_p g(x)}{\kappa (\delta)} = \frac{\psi_1 (\kappa^{-1} (I_p g(x)))}{I_p g(x)} I_p g(x)
= \psi_1 (\kappa^{-1} (I_p g(x))).
\] (81)
Consequently our claim (79) is justified.
It follows from (76) and (79) that
\[
I_p g(x) \leq (1 + C_2) \max \{\psi_1 (\kappa^{-1} (I_p g(x))), \psi_1 (d_G)\}.
\] (82)
By (49), we obtain
\[
(\psi_1 \circ \kappa^{-1})(s) \leq C_G \Phi^{-1}(s) \quad \text{for } \kappa (d_G) \leq s < \infty.
\] (83)
Hence, taking \( A := C_G (C_1 + 1)(1 + C_2) \), we establish
\[
\frac{|I_p f(x)|}{A} \leq \frac{I_p g(x)}{A} \leq \frac{\max \{\psi_1 (\kappa^{-1} (I_p g(x))), \psi_1 (d_G)\}}{C_G (C_1 + 1)}
= \frac{\max \{\psi_1 (\kappa^{-1} (I_p g(x))), \psi_1 (\kappa^{-1} (\kappa (d_G)))\}}{C_G (C_1 + 1)}
\leq \frac{\max \{\Phi^{-1} (I_p g(x)), \Phi^{-1}(\kappa (d_G))\}}{C_G (C_1 + 1)}.
\] (84)
Since \( \tilde{\rho}/\rho \) is decreasing and
\[
(\tilde{\rho} (4k_2 d_G) / \rho (4k_2 d_G)) \psi_1 (d_G) = \kappa (d_G),
\] (85)
we see that
\[
\psi (r) \geq \int_{2k_2 d_G}^{4k_2 d_G} \frac{\tilde{\rho} (t) \varphi (t)}{t} \, dt
\geq \frac{\tilde{\rho} (4k_2 d_G)}{\rho (4k_2 d_G)} \int_{2k_2 d_G}^{4k_2 d_G} \frac{\rho (t) \varphi (t)}{t} \, dt
= \kappa (d_G).
\] (86)
for all \(0 < r \leq d_G\). Hence, with the aid of (74), we have

\[
\frac{1}{\mu(B(z, 4r))} \int_{B(z, r)} \Phi \left( \frac{|I_p f(x)|}{A \| f \|_{L^{1+\eta}(G)}} \right) d\mu(x) \\
\leq \frac{1}{C_1 + 1} \times \frac{1}{\mu(B(z, 4r))} \int_{B(z, r)} \max \{ I_p g(x), \kappa(d_G) \} d\mu(x) \\
\leq \frac{1}{C_1 + 1} \times \left( \frac{1}{\mu(B(z, 4r))} \int_{B(z, r)} I_p g(x) d\mu(x) \\
+ \frac{1}{\mu(B(z, 4r))} \int_{B(z, r)} \kappa(d_G) d\mu(x) \right) \\
\leq \frac{1}{C_1 + 1} \left( C_1 \psi(r) + \psi(r) \right) = \psi(r),
\]

which proves (50).

\[\square\]

**Proof of Theorem 7.** Write

\[
I_p f(x) - I_p f(z) = \int_{B(x, 2d(x,z))} \rho(d(x,y)) \mu(B(x, 4d(x,y))) f(y) d\mu(y) \\
- \int_{B(x,2d(x,z))} \rho(d(z,y)) \mu(B(x, 4d(z,y))) f(y) d\mu(y) \\
+ \int_{G \setminus B(x,2d(x,z))} \left( \frac{\rho(d(x,y))}{\mu(B(x, 4d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z, 4d(z,y)))} \right) f(y) d\mu(y).
\]

By (58), we have

\[
\int_{B(x,2d(x,z))} \rho(d(x,y)) \mu(B(x, 4d(x,y))) f(y) d\mu(y) \\
\leq C_1 \psi(2d(x,z)) \| f \|_{L^{1+\eta}(G)},
\]

\[
\int_{B(x,2d(x,z))} \rho(d(z,y)) \mu(B(z, 4d(z,y))) f(y) d\mu(y) \\
\leq \int_{B(z,4d(x,z))} \rho(d(z,y)) \mu(B(z, 4d(z,y))) f(y) d\mu(y) \\
\leq \psi(2d(x,z)) \| f \|_{L^{1+\eta}(G)}
\]

for \(x, z \in G\). On the other hand, we have by (52) and (59)

\[
\int_{G \setminus B(x,2d(x,z))} \left( \frac{\rho(d(x,y))}{\mu(B(x, 4d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z, 4d(z,y)))} \right) f(y) d\mu(y) \\
\times |f(y)| d\mu(y) \\
\leq C'_p d(z,x)^{\theta} \left( \int_{4k,4d(x,z)} \frac{\rho(t) \varphi(t)}{t^{1+\theta}} dt \right) \| f \|_{L^{1+\eta}(G)} \\
\leq C_p \psi(2d(x,z)) \| f \|_{L^{1+\eta}(G)},
\]

which proves (50).

\[\square\]

**Appendix**

**A. Disproof of (15)**

Inequality (15) can be disproved in terms of Besov spaces and Triebel-Lizorkin spaces. Let \(\psi \in C^0(R^d)\) satisfy

\[
\chi_{B(0,1)} \leq \psi \leq \chi_{B(0,3/2)}.
\]

Define \(T'_0 f := \mathcal{F}^{-1} \{ |\psi \cdot \mathcal{F} f| \} \) for \(f \in \mathcal{S}'(\mathbb{R}^n)\). For parameters \(p \in (0, \infty)\) and \(q \in (0, \infty)\), the Besov norm \(\| \cdot \|_{\dot{B}^p_{p,q}}\) and the Triebel-Lizorkin norm \(\| \cdot \|_{\dot{F}^p_{p,q}}\) are defined by

\[
\| f \|_{\dot{B}^p_{p,q}} := \| T'_0 f \|_{L^p} + \left( \sum_{j=1}^{\infty} \| S_j f \|_{L^p}^{q} \right)^{1/q},
\]

\[
\| f \|_{\dot{F}^p_{p,q}} := \| T'_0 f \|_{L^p} + \left( \sum_{j=1}^{\infty} \| 2^{j\alpha} S_j f \|_{L^p}^{q} \right)^{1/q},
\]

respectively, and for \(p \in (0, \infty)\) and for \(f \in \mathcal{S}'(\mathbb{R}^n)\), the Besov norm \(\| \cdot \|_{\dot{B}^p_{p,\infty}}\) and the Triebel-Lizorkin norm \(\| \cdot \|_{\dot{F}^p_{p,\infty}}\) are defined by

\[
\| f \|_{\dot{B}^p_{p,\infty}} := \| T'_0 f \|_{L^p} + \sup_{j \in \mathbb{N}} 2^{j\alpha} \| S_j f \|_{L^p},
\]

\[
\| f \|_{\dot{F}^p_{p,\infty}} := \| T'_0 f \|_{L^p} + \sup_{j \in \mathbb{N}} 2^{j\alpha} \| S_j f \|_{L^p}.
\]

Meanwhile, by denoting \(\mathcal{P}(\mathbb{R}^n)\) the set of all polynomials, for parameters \(p \in (1, \infty)\) and \(q \in (1, \infty)\) and for a distribution \(f \in \mathcal{S}'(\mathbb{R}^n)\), the homogeneous Besov norm \(\| \cdot \|_{\dot{B}^p_{p,q}}\) and the homogeneous Triebel-Lizorkin norm \(\| \cdot \|_{\dot{F}^p_{p,q}}\) are defined
by
\[
\|f\|_{p,q}^* := \left( \sum_{j=-\infty}^{\infty} (2^j \|S_j f\|_{L^p})^q \right)^{1/q},
\] (A.5)
and
\[
\|f\|_{p,q} : = \left( \sum_{j=-\infty}^{\infty} (2^j \|S_j f\|_{L^p})^q \right)^{1/q}.
\] (A.6)
respectively. Also, for \( p \in (1, \infty) \) and \( f \in \mathcal{S}'(\mathbb{R}^n) \), one defines
\[
\|f\|_{p,s}^* := \sup_{j \in \mathbb{N}} 2^j \|S_j f\|_{L^p},
\] (A.7)
and
\[
\|f\|_{p,s} := \sup_{j \in \mathbb{N}} 2^j \|S_j f\|_{L^p}.
\] (A.8)
respectively.

It follows from (A.7) and (A.8) that
\[
\|f\|_{p,s}^* \geq \|f\|_{p,s} \quad (f \in \mathcal{S}'(\mathbb{R}^n)).
\] (A.9)

Let \( 0 < p, q \leq \infty \), and \( s \in \mathbb{R} \). The inhomogeneous Besov space \( \dot{B}_{p,q}^s(\mathbb{R}^n) \) (resp. the homogeneous Besov space \( B_{p,q}^s(\mathbb{R}^n) \)) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) (resp. \( f \in \mathcal{S}(\mathbb{R}^n) \)) for which the norm \( \|f\|_{\dot{B}_{p,q}^s} \) (resp. \( \|f\|_{B_{p,q}^s} \)) is finite, when \( 0 < p, q \leq \infty \). Likewise, for \( 0 < p < \infty, 0 < q \leq \infty \) and \( s \in \mathbb{R} \), the inhomogeneous Triebel-Lizorkin space \( \dot{F}_{p,q}^s(\mathbb{R}^n) \) (resp. the homogeneous Triebel-Lizorkin space \( F_{p,q}^s(\mathbb{R}^n) \)) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) (resp. \( f \in \mathcal{S}(\mathbb{R}^n) \)) for which the norm \( \|f\|_{\dot{F}_{p,q}^s} \) (resp. \( \|f\|_{F_{p,q}^s} \)) is finite. To simplify the matters, even when we consider representatives in the function spaces \( \dot{B}_{p,q}^s(\mathbb{R}^n) \) and \( B_{p,q}^s(\mathbb{R}^n) \), we forget that they are in equivalence classes, and we regard the function spaces \( \dot{B}_{p,q}^s(\mathbb{R}^n) \) and \( B_{p,q}^s(\mathbb{R}^n) \) as subspaces of \( \mathcal{S}'(\mathbb{R}^n) \).

Keeping this in mind, let us disprove (15). We have
\[
\sup_{j \in \mathbb{Z}} \|S_j f\|_{L^1} \leq c \sup_{j \in \mathbb{Z}} \|S_j f\|_{L^\eta} = c \|f\|_{p,s}^* \leq c \|f\|_{p,s}.
\] (A.10)
from (5), (A.7), and (A.9).

However, according to [55, Theorem 11.2, (i), (11.2)], there exists \( f \in \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n) \) such that it is not represented by \( L_{1,\lambda}^1(\mathbb{R}^n) \)-functions:
\[
f \in \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n) \setminus L_{1,\lambda}^1(\mathbb{R}^n).
\] (A.11)

If we consider \( \mathcal{F}^{-1}(1 - \psi) \cdot \mathcal{F} f \), where \( \psi \in C_c^{\infty}(\mathbb{R}^n) \) is from (A.1), we can arrange that \( f \in \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n) \) can be chosen so that supp(\( \mathcal{F} f \)) \( \cap \{ B(0,4) \} = \emptyset \). Indeed,
\[
\mathcal{F}^{-1}(1 - \psi) \cdot \mathcal{F} f \in C_c^{\infty}(\mathbb{R}^n).
\] (A.12)

We suppose that the Fourier support of \( f \) is away from \( B(0,4) \). Let us admit that
\[
\|\mathcal{F}^{-1}(1 - \psi) \cdot \mathcal{F} f\|_{p,s}^* \leq C \|f\|_{p,s} \quad (f \in \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n)).
\] (A.13)
under the understanding \( \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \). Note also that \( L_{1,\lambda}(\mathbb{R}^n) \) is a subset of \( L_{1,\lambda}^1(\mathbb{R}^n) \), hence our observation can be summarized as follows:
\[
L_{1,\lambda}(\mathbb{R}^n) \subset L_{1,\lambda}^1(\mathbb{R}^n), \quad f \in \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n) \setminus L_{1,\lambda}^1(\mathbb{R}^n).
\] (A.14)
It then follows immediately that (15) fails since (15) implies
\[
L_{1,\lambda}(\mathbb{R}^n) \supset \dot{F}_{n,\lambda,\infty}^0(\mathbb{R}^n).
\] (A.15)

Inclusions (A.14), (A.15) contradict obviously.

It remains to prove (A.13). Note that the frequency support of \( f \) does not intersect with \( B(0,4) \). Observe also that \( (2^{-j}) \) has the frequency support in \( B(0,4) \). Thus, we have
\[
S_j f = \mathcal{F}^{-1}[\psi(2^{-j}) \cdot \mathcal{F} f] = 0 \quad (j \leq -1),
\]
and hence
\[
\|f\|_{p,s}^* = \sup_{j \in \mathbb{Z}} \|S_j f\|_{L^\eta} = \sup_{j \in \mathbb{Z} \setminus \{0\}} \|S_j f\|_{L^\eta} \leq \|f\|_{p,s} + \|S_0 f\|_{L^\eta}.
\] (A.16)

Define
\[
Wf := \mathcal{F}^{-1}[\psi \cdot \mathcal{F} f], \quad \psi := \frac{\varphi}{\psi + \varphi(2^{-1})}.
\] (A.17)

In view of the size of frequency support, we conclude \( S_0 f = VT_0 f + VS_1 f \). Now we invoke the following Planchrel-Polya-Nikolskij lemma.

**Lemma A.1** (Planchrel-Polya-Nikolskij [56, page 16]). Let \( 0 < \eta \leq 1 \). Assume that \( f \in \mathcal{S}'(\mathbb{R}^n) \) has frequency support in \( Q(0,R) \). Then, denoting by \( M \) the Hardy-Littlewood maximal operator, we have
\[
\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{(1 + R |y|)^{n/\eta}} \leq CM \left( |f| \right)^{1/\eta},
\] (A.18)
where \( C \) is independent of \( R > 0 \).

According to Lemma A.1 with \( \eta = 1/2 \) and \( R = 16 \), we conclude that
\[
|S_0 f(x)| \leq C \left( \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \psi(y) \cdot |T_0 f(x-y)| dy 
+ \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \psi(y) \cdot |S_1 f(x-y)| dy \right)
\]
\[
\leq C \left( M \left( |T_0 f|^{1/2} \right)^2 \right) + M \left( |S_1 f|^{1/2} \right)^2
\]
\[
\times \left( \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \psi(y) \cdot (1 + |y|)^{-2} dy \right)
\]
\[
= C \left( M \left( |T_0 f|^{1/2} \right)^2 \right) + M \left( |S_1 f|^{1/2} \right)^2,
\] (A.19)
where for the last inequality, we invoked
\[
|\mathcal{F}^{-1} \psi(y)| \leq \frac{C}{(1 + |y|)^{3r+1}} \quad (y \in \mathbb{R}^n).
\] (A.20)
By the fact that \( \|F\|_L^{n/\lambda} = \|\sqrt{|F|}\|_L^{2n/\lambda} \) and the \( L^{2n/\lambda}(\mathbb{R}^n) \)-boundedness of the Hardy-Littlewood maximal operator, we conclude

\[
\left\| S_0 f \right\|_{L^{n/\lambda}} \leq C \left( \left\| M \left[ T_0 f \right]^{1/2} \right\|_{L^{2n/\lambda}}^2 + \left\| M \left[ S_1 f \right]^{1/2} \right\|_{L^{2n/\lambda}}^2 \right)
\]

\[
\leq C \left( \left\| T_0 f \right\|_{L^{n/\lambda}}^{1/2} + \left\| S_1 f \right\|_{L^{n/\lambda}}^{1/2} \right)^2
\]

\[
= C \left( \left\| T_0 f \right\|_{L^{n/\lambda}} + \left\| S_1 f \right\|_{L^{n/\lambda}} \right)^2.
\]

(A.21)

Combining (A.16) and (A.21), we obtain the desired result.

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References


