Research Article

Stability of Pexider Equations on Semigroup with No Neutral Element

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Let S be a commutative semigroup with no neutral element, Y a Banach space, and $\mathbb{C}$ the set of complex numbers. In this paper we prove the Hyers-Ulam stability for Pexider equation $\|f(x+y) - g(x) - h(y)\| \leq \epsilon$ for all $x, y \in S$, where $f, g, h : S \rightarrow Y$. Using Jung's theorem we obtain a better bound than that usually obtained. Also, generalizing the result of Baker (1980) we prove the superstability for Pexider-exponential equation $|f(t+s) - g(t)h(s)| \leq \epsilon$ for all $t, s \in S$, where $f, g, h : S \rightarrow \mathbb{C}$. As a direct consequence of the result we also obtain the general solutions of the Pexider-exponential equation $f(t+s) = g(t)h(s)$ for all $t, s \in S$, a closed form of which is not yet known.

1. Introduction

Throughout this paper, we denote by S a commutative semigroup with no neutral element, Y a Banach space, and $\mathbb{C}$ the set of complex numbers and $\epsilon \geq 0$. A function $A : S \rightarrow Y$ is called an *additive function* provided that $A(x+y) = A(x) + A(y)$ for all $x, y \in S$, and $m : S \rightarrow \mathbb{C}$ is called an *exponential function* provided that $m(x+y) = m(x)m(y)$ for all $x, y \in S$. The Hyers-Ulam stability problems of functional equation have been originated by Ulam 1960 [1]. One of the first assertions to be obtained is the following result, essentially due to Hyers [2] that gives an answer for the question of Ulam.

Theorem 1. Let $f : S \rightarrow Y$ satisfy the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \quad (1)$$

for all $x, y \in S$. Then there exists a unique additive function $A : S \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \epsilon \quad (2)$$

for all $x \in S$.

If S has a neutral element, then as an easy consequence of Theorem 1 we have the following stability theorem for Pexider equation [3].

Theorem 2. Suppose that S has a neutral element and $f, g, h : S \rightarrow Y$ satisfy

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \quad (3)$$

for all $x, y \in S$. Then there exist a unique additive function $A : S \rightarrow Y$ and $\alpha, \beta \in Y$ such that

$$\|f(x) - A(x) - \alpha - \beta\| \leq 3\epsilon,$$

$$\|g(x) - A(x) - \alpha\| \leq 4\epsilon, \quad (4)$$

$$\|h(x) - A(x) - \beta\| \leq 4\epsilon$$

for all $x \in S$.

As a Hyers-Ulam stability theorem for the exponential functional equation, Baker proved that if $f : S \rightarrow \mathbb{C}$ satisfies the exponential functional inequality

$$|f(t+s) - f(t)f(s)| \leq \epsilon \quad (5)$$

for all $t, s \in S$, then either f is a bounded function satisfying $|f(t)| \leq (1/2)(1 + \sqrt{1 + 4\epsilon})$ for all $t \in S$ or f is an exponential function (see [4, 5]).

If S has a neutral element, we obtain the following stability theorem for Pexider-exponential equation [3].

Theorem 3. Suppose that S has a neutral element and $f, g, h : S \rightarrow \mathbb{C}$ satisfy the functional inequality

$$|f(t+s) - g(t)h(s)| \leq \epsilon \quad (6)$$

for all $t, s \in S$. Then either there exist positive constants C_1, C_2, C_3 such that

$$|g(t)| \leq C_1, \quad |h(t)| \leq C_2, \quad |f(t)| \leq C_3, \quad \forall t \in S, \quad (7)$$

or else there exist an unbounded exponential function m and nonzero constants $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$g(t) = \lambda_1 m(t), \quad \forall t \in S, \\ h(t) = \lambda_2 m(t), \quad \forall t \in S, \quad (8)$$

$$|f(t) - \lambda_1 \lambda_2 m(t)| \leq \epsilon, \quad \forall t \in S.$$

There are numerous results on the stability theorem for Pexider equations [6–12]. It is also very likely that there are some results on the stability of the inequalities (3) and (6) when S has no neutral element. However, in the scope of the author, no results are found. In this paper, we prove the stability theorem for the functional inequalities (3) and (6) when S has no neutral element.

Also, for inequality (3), using Jung's theorem which shows the ratio of the diameter of a bounded set in the Euclidean space and the radius of smallest circle enclosing the set, we show how to improve the bound when the Euclidean space is the target space of functions in given functional inequalities.

2. Stability of Pexider Equation

Throughout this section we assume that S is a commutative semigroup with no neutral element. We prove the Hyers-Ulam stability of (3). We denote $S + S = \{t + s : t, s \in S\}$.

Lemma 4. Assume that $g : S \rightarrow Y$ satisfies the functional inequality

$$\|g(x+y) + g(z) - g(x) - g(y+z)\| \leq \epsilon \quad (9)$$

for all $x, y, z \in S$. Then there exist a unique additive function $A : S \rightarrow Y$ and $\alpha \in Y$ such that

$$\|g(x) - A(x) - \alpha\| \leq 2\epsilon \quad (10)$$

for all $x \in S$.

Proof. Replacing (x, y, z) by (y, z, z) in (9) we have

$$\|g(y+z) + g(z) - g(y) - g(2z)\| \leq \epsilon. \quad (11)$$

From (9) and (11) using the triangle inequality we have

$$\|g(x+y) - g(x) - g(y) + 2g(z) - g(2z)\| \leq 2\epsilon. \quad (12)$$

Fix $z = z_0$ and let $g_0(x) = g(x) - \alpha$ with $\alpha = 2g(z_0) - g(2z_0)$. Then we have

$$\|g_0(x+y) - g_0(x) - g_0(y)\| \leq 2\epsilon \quad (13)$$

for all $x, y \in S$. By Theorem 1, there exists a unique additive function $A : S \rightarrow Y$ such that

$$\|g(x) - A(x) - \alpha\| \leq 2\epsilon \quad (14)$$

for all $x \in S$. This completes the proof. $\square$

Theorem 5. Assume that $f, g, h : S \rightarrow Y$ satisfy

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \quad (15)$$

for all $x, y \in S$. Then there exist a unique additive function $A : S \rightarrow Y$ and $\alpha, \beta \in Y$ such that

$$\begin{aligned} \|g(x) - A(x) - \alpha\| &\leq 8\epsilon, \\ \|h(x) - A(x) - \beta\| &\leq 8\epsilon, \\ \|f(t) - A(t) - \alpha - \beta\| &\leq 17\epsilon \end{aligned} \quad (16)$$

for all $x \in S, t \in S + S$.

Proof. Using (15) and the triangle inequality we have

$$\begin{aligned} &\|g(x+y) + g(z) - g(x) - g(y+z)\| \\ &\leq \|g(x+y) + h(u) - f(x+y+u)\| \\ &\quad + \|f(x+y+u) - g(x) - h(y+u)\| \\ &\quad + \|g(z) + h(y+u) - f(z+y+u)\| \\ &\quad + \|f(y+z+u) - g(y+z) - h(u)\| \leq 4\epsilon \end{aligned} \quad (17)$$

for all $x, y, z, u \in S$. By Lemma 4, there exist a unique additive function $A : S \rightarrow Y$ and $\alpha \in Y$ such that

$$\|g(x) - A(x) - \alpha\| \leq 8\epsilon \quad (18)$$

for all $x \in S$. Changing the role of g and h in (15) we have

$$\|h(x) - B(x) - \beta\| \leq 8\epsilon \quad (19)$$

for some additive function $B : S \rightarrow Y$ and $\beta \in Y$. Replacing (x, y) by (y, x) in (15) and using the triangle inequality with (15) and the result we have

$$\|g(x) - h(x)\| \leq 2\epsilon + \|g(y) - h(y)\|. \quad (20)$$

Fix $y = y_0$ in (20). Then from (18), (19), and (20) using the triangle inequality we have

$$\|A(x) - B(x)\| \leq 18\epsilon + \|\alpha - \beta\| + \|g(y_0) - h(y_0)\| \quad (21)$$

for all $x \in S$, which implies $A = B$. From (15), (18), and (19) using the triangle inequality we have

$$\|f(t) - A(t) - \alpha - \beta\| \leq 17\epsilon \quad (22)$$

for all $t \in S + S$. This completes the proof. $\square$

As a particular case of inequality (15), we have the following.

Theorem 6. Assume that $f, g : S \rightarrow Y$ satisfy

$$\|f(x+y) - g(x) - g(y)\| \leq \epsilon \quad (23)$$

for all $x, y \in S$. Then there exist a unique additive function $A : S \rightarrow Y$ and $\alpha \in Y$ such that

$$\begin{aligned} \|g(x) - A(x) - \alpha\| &\leq 4\epsilon, \\ \|f(t) - A(t) - 2\alpha\| &\leq 9\epsilon \end{aligned} \quad (24)$$

for all $x \in S, t \in S + S$.

Proof. Using (23) and the triangle inequality we have

$$\begin{aligned} &\|g(x+y) + g(z) - g(x) - g(y+z)\| \\ &\leq \|g(x+y) + g(z) - f(x+y+z)\| \\ &\quad + \|f(x+y+z) - g(x) - g(y+z)\| \\ &\leq 2\epsilon \end{aligned} \quad (25)$$

for all $x, y, z \in S$. By Lemma 4, there exist a unique additive function $A : S \rightarrow Y$ and $\alpha \in Y$ such that

$$\|g(x) - A(x) - \alpha\| \leq 4\epsilon \quad (26)$$

for all $x \in S$. From (23) and (26) using the triangle inequality we have

$$\|f(t) - A(t) - 2\alpha\| \leq 9\epsilon \quad (27)$$

for all $t \in S + S$. This completes the proof. $\square$

As a direct consequence of the above result we have the following stability theorem for Jensen functional equation when S is 2-divisible.

Corollary 7. Assume that $g : S \rightarrow Y$ satisfy

$$\left\| 2g\left(\frac{x+y}{2}\right) - g(x) - g(y) \right\| \leq \epsilon \quad (28)$$

for all $x, y \in S$. Then there exist a unique additive function $A : S \rightarrow Y$ and $\alpha \in Y$ such that

$$\|g(x) - A(x) - \alpha\| \leq 4\epsilon \quad (29)$$

for all $x \in S$.

Theorem 8. Assume that $f, h : S \rightarrow Y$ satisfy

$$\|f(x+y) - f(x) - h(y)\| \leq \epsilon. \quad (30)$$

Then there exist a unique additive function $A : S \rightarrow Y$ and $\alpha \in Y$ such that

$$\begin{aligned} \|f(x) - A(x) - \alpha\| &\leq 3\epsilon, \\ \|h(x) - A(x)\| &\leq 3\epsilon \end{aligned} \quad (31)$$

for all $x \in S$.

Proof. Replacing (x, y) by (y, x) in (30), using the triangle inequality with (30) and the result, putting $y = y_0$, and letting $\alpha = f(y_0) - h(y_0)$ we have

$$\|f(x) - h(x) - \alpha\| \leq 2\epsilon \quad (32)$$

for all $x \in S$. From (30) and (32) using the triangle inequality we have

$$\|f(x+y) - f(x) - f(y) + \alpha\| \leq 3\epsilon \quad (33)$$

for all $x, y \in S$. By Theorem 1, there exists a unique additive function $A : S \rightarrow Y$ such that

$$\|f(x) - A(x) - \alpha\| \leq 3\epsilon \quad (34)$$

for all $x \in S$. On the other hand, using (30) and the triangle inequality we have

$$\begin{aligned} &\|h(x+y) - h(x) - h(y)\| \\ &= \|f(x+y+z) - f(y+z) - h(x)\| \\ &= \|-f(x+y+z) + f(z) + h(x+y)\| \\ &= \|f(y+z) - f(z) - h(y)\| \leq 3\epsilon \end{aligned} \quad (35)$$

for all $x, y, z \in S$. By Theorem 1, there exists a unique additive function $B : S \rightarrow Y$ such that

$$\|h(x) - B(x)\| \leq 3\epsilon \quad (36)$$

for all $x \in S$. From (32), (34), and (36) using the triangle inequality we have

$$\|A(x) - B(x)\| \leq 8\epsilon \quad (37)$$

for all $x \in S$, which implies $A = B$. This completes the proof. $\square$

Using Jung's theorem (see [13] for more details) we slightly improve the bound in Theorem 8 when Y is n -dimensional Euclidean space.

Lemma 9. Let K be a bounded subset of $\mathbb{R}^n$, the n -dimensional Euclidean space, and let

$$\text{diam}(K) = \sup_{p, q \in K} \|p - q\| \quad (38)$$

be the diameter of K . Then there exists a closed ball $D_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$ with radius

$$r \leq \text{diam}(K) \sqrt{\frac{n}{2n+2}} \quad (39)$$

such that $K \subset D_r(x_0)$.

Theorem 10. Let $f, h : S \rightarrow \mathbb{R}^n$ satisfy

$$\|f(x+y) - f(x) - h(y)\| \leq \epsilon \quad (40)$$

for all $x, y \in S$. Then there exist a unique additive function $A : S \rightarrow \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$ such that

$$\|f(x) - A(x) - \alpha\| \leq \left(1 + \sqrt{\frac{2n}{n+1}}\right)\epsilon, \quad (41)$$

$$\|h(x) - A(x)\| \leq 3\epsilon$$

for all $x \in S$.

Proof. Replacing (x, y) by (y, x) in (40) and using the triangle inequality with (40) and the result we have

$$\|f(x) - h(x) - f(y) + h(y)\| \leq 2\epsilon \quad (42)$$

for all $x, y \in S$. Let $q(x) = f(x) - h(x)$. Then inequality (42) says that $\text{diam}(q(S)) \leq 2\epsilon$. Thus, by Lemma 9, there exists a closed ball of radius $\leq (\sqrt{2n/(n+1)})\epsilon$ containing $q(S)$. Let α be the center of the ball. Then we have

$$\|f(x) - h(x) - \alpha\| \leq \left(\sqrt{\frac{2n}{n+1}}\right)\epsilon \quad (43)$$

for all $x \in S$. From (40) and (43) using the triangle inequality we have

$$\|f(x+y) - f(x) - f(y) + \alpha\| \leq \left(1 + \sqrt{\frac{2n}{n+1}}\right)\epsilon \quad (44)$$

for all $x, y \in S$. By Theorem 1, there exists a unique additive function $A : S \rightarrow Y$ such that

$$\|f(x) - A(x) - \alpha\| \leq \left(1 + \sqrt{\frac{2n}{n+1}}\right)\epsilon \quad (45)$$

for all $x \in S$. This completes the proof. $\square$

As a direct consequence of the above result we have the following.

Corollary 11. Let $f, h : S \rightarrow \mathbb{C}$ satisfy

$$|f(x+y) - f(x) - h(y)| \leq \epsilon \quad (46)$$

for all $x, y \in S$. Then there exist a unique additive function $A : S \rightarrow \mathbb{C}$ and $\alpha \in \mathbb{C}$ such that

$$|f(x) - A(x) - \alpha| \leq \left(1 + \frac{2}{\sqrt{3}}\right)\epsilon, \quad (47)$$

$$|h(x) - A(x)| \leq 3\epsilon$$

for all $x \in S$.

Remark 12. The author wants to know if the bounds in the above theorems can be replaced by smaller ones. Also, the author guesses that Jung's theorem can be applied to obtain smaller bounds when dealing with some other functional inequalities.

3. Stability of Pexider-Exponential Equation

Throughout this section we assume that S is a semigroup with no neutral element. Let $f, g, h : S \rightarrow \mathbb{C}$. We consider the stability of the functional inequality

$$|f(t+s) - g(t)h(s)| \leq \epsilon \quad (48)$$

for all $t, s \in S$. Hereafter, we exclude the trivial cases $g(t) \equiv 0$ or $h(t) \equiv 0$.

We need the following lemma.

Lemma 13. Let $g : S \rightarrow \mathbb{C}$. Then g satisfies the functional equation

$$g(t+s)g(r) = g(t)g(s+r), \quad \forall t, s, r \in S \quad (49)$$

if and only if either there exist nonzero constant $\lambda \in \mathbb{C}$ and an exponential function $m : S \rightarrow \mathbb{C}$ such that

$$g(t) = \lambda m(t), \quad \forall t \in S \quad (50)$$

or else

$$g(t) = 0, \quad \forall t \in S + S, \quad (51)$$

$$g(t) : \text{arbitrary}, \quad \forall t \notin S + S.$$

In particular, if S is 2-divisible or has a neutral element, then every solution of (49) is given by (50).

Proof. Multiplying $g(u)$ in (49) we have

$$g(t+s)g(r)g(u) = g(t)g(s+r)g(u) \\ = g(t)g(s)g(r+u). \quad (52)$$

If (51) fails, then there exist $r_0, u_0 \in S$ such that $g(r_0 + u_0) \neq 0$. Putting $r = r_0, u = u_0, t = s = r_0 + u_0$ in (52) we have

$$g(r_0 + u_0 + r_0 + u_0)g(r_0)g(u_0) = [g(r_0 + u_0)]^3 \neq 0. \quad (53)$$

Thus, we have

$$g(r_0) \neq 0, \quad g(s_0) \neq 0. \quad (54)$$

Putting $r = r_0$ and $u = u_0$ in (52) and dividing the result by $g(r_0)^2 g(u_0)^2 / g(r_0 + u_0)$ we have

$$\frac{g(t+s)}{\lambda} = \frac{g(t)}{\lambda} \cdot \frac{g(s)}{\lambda}, \quad (55)$$

where $\lambda := g(r_0)g(u_0)/g(r_0 + u_0)$, which gives (50). Obviously, both (50) and (51) are solutions of (49). If in particular, S is 2-divisible or has a neutral element, then we have $S = S + S$. Thus, case (51) does not occur since $g \not\equiv 0$. This completes the proof. $\square$

Theorem 14. Assume that $f, g, h : S \rightarrow \mathbb{C}$ satisfy inequality (48). Then (f, g, h) satisfies one of the following.

(i) There exist positive constants C_1, C_2, C_3 such that

$$\begin{aligned} |g(t)| &\leq C_1, \quad \forall t \in S, \\ |h(t)| &\leq C_2, \quad \forall t \in S, \\ |f(t)| &\leq C_3, \quad \forall t \in S + S, \\ f(t) &: \text{arbitrary}, \quad \forall t \in S \setminus (S + S). \end{aligned} \quad (56)$$

(ii) There exist an unbounded exponential function m and nonzero constants $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\begin{aligned} g(t) &= \lambda_1 m(t), \quad \forall t \in S, \\ h(t) &= \lambda_2 m(t), \quad \forall t \in S, \\ |f(t) - \lambda_1 \lambda_2 m(t)| &\leq \epsilon, \quad \forall t \in S + S, \\ f(t) &: \text{arbitrary}, \quad \forall t \in S \setminus (S + S). \end{aligned} \quad (57)$$

(iii) There exists $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ such that

$$\begin{aligned} g(t) &= 0, \quad \forall t \in S + S, \\ |g(t_1)g(s_1) - g(t_2)g(s_2)| &\leq \frac{2\epsilon}{|\lambda|}, \\ \forall t_1 + s_1 &= t_2 + s_2, \\ h(t) &= \lambda g(t), \quad \forall t \in S \\ f(t) &: \text{arbitrary}, \quad \forall t \in S \setminus (S + S). \end{aligned} \quad (58)$$

Proof. Replacing t by s and s by t in (48), respectively, and using triangle inequality we have

$$|g(t)h(s) - g(s)h(t)| \leq 2\epsilon \quad (59)$$

for all $t, s \in S$. Since we exclude the trivial cases when $g(t) \equiv 0$ or $h(t) \equiv 0$, it follows from inequality (59) that there exist constants $c_1, c_2, d_1, d_2 \geq 0$ such that

$$\begin{aligned} |g(t)| &\leq c_1 |h(t)| + d_1, \\ |h(t)| &\leq c_2 |g(t)| + d_2 \end{aligned} \quad (60)$$

for all $t \in S$. It follows from (60) that $g(t)$ is bounded if and only if $h(t)$ is bounded. Assume that h is bounded. Then by (48) and (60) we get (56). Assume that h is unbounded. Dividing both sides of (59) by $|h(s)|$ we have

$$|g(t) - \lambda_s h(t)| \leq \frac{2\epsilon}{|h(s)|} \quad (61)$$

for all $s \in J := \{s : h(s) \neq 0\}$, where $\lambda_s = g(s)/h(s)$. Since g is unbounded, we have $\lambda_s \neq 0$ for all $s \in J$. Putting $s = s_1$ and $s = s_2$ in (61) and using the triangle inequality with the resulting inequalities we have

$$|(\lambda_{s_1} - \lambda_{s_2})h(t)| \leq 2\epsilon \left(\frac{1}{|h(s_1)|} + \frac{1}{|h(s_2)|} \right) \quad (62)$$

for all $s_1, s_2 \in J$ and $t \in S$. Since h is unbounded, from (62) we have $\lambda_{s_1} = \lambda_{s_2}$. Thus, $\lambda := \lambda_s$ is independent of $s \in J$. Thus, we have

$$g(s) = \lambda h(s) \quad (63)$$

for all $s \in J$. Furthermore, since both g and h are unbounded, it follows from (59) that $g(s_0) = 0$ if and only if $h(s_0) = 0$. Thus, (63) holds for all $s \in S$. Now, using the triangle inequality we have

$$\begin{aligned} &|g(t+s)g(r) - g(t)g(s+r)| \\ &\leq \left| \frac{[g(t+s)h(u) - f(t+s+u)]g(r)}{h(u)} \right| \\ &\quad + \left| \frac{[f(t+s+u) - g(t)h(s+u)]g(r)}{h(u)} \right| \\ &\quad + \left| \frac{[g(r)h(s+u) - f(r+s+u)]g(t)}{h(u)} \right| \\ &\quad + \left| \frac{[f(r+s+u) - g(r)h(s+u)]g(t)}{h(u)} \right| \\ &\leq 2\epsilon \left(\frac{|g(r)| + |g(t)|}{h(u)} \right) \end{aligned} \quad (64)$$

for all $t, s, r \in S$ and $u \in J$. Since h is unbounded, it follows from (64) that

$$g(t+s)g(r) = g(t)g(s+r) \quad (65)$$

for all $t, s, r \in S$. By Lemma 13, there exists $\lambda_1 \in \mathbb{C}$ such that

$$g(t) = \lambda_1 m(t) \quad (66)$$

for all $t \in S$, or

$$g(r) = 0 \quad (67)$$

for all $r \in S + S$. If (66) holds, putting (63) and (66) in (48) we have

$$|f(t+s) - \lambda_1^2 \lambda m(t+s)| \leq \epsilon \quad (68)$$

for all $t, s \in S$, which gives (57) with $\lambda_2 = \lambda \lambda_1$. If (67) holds, it remains to determine the values of $g(t)$ for $t \in S \setminus (S + S)$. Let $s = t_1 + s_1 = t_2 + s_2$. Then using (48) and (63) we have

$$\begin{aligned} |f(s) - \lambda g(t_1)g(s_1)| &\leq \epsilon, \\ |f(s) - \lambda g(t_2)g(s_2)| &\leq \epsilon. \end{aligned} \quad (69)$$

Using the triangle inequality we have

$$|g(t_1)g(s_1) - g(t_2)g(s_2)| \leq \frac{2\epsilon}{|\lambda|}. \quad (70)$$

This completes the proof. $\square$

As a direct consequence of the above result we have the following.

Corollary 15. Assume that S is 2-divisible or has a neutral element. Suppose that $f, g, h : S \rightarrow \mathbb{C}$ satisfy the inequality (48). Then either there exist positive constants C_1, C_2, C_3 such that

$$|g(t)| \leq C_1, \quad |h(t)| \leq C_2, \quad |f(t)| \leq C_3, \quad \forall t \in S, \quad (71)$$

or else there exist an unbounded exponential function m and nonzero constants $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\begin{aligned} g(t) &= \lambda_1 m(t), \quad \forall t \in S, \\ h(t) &= \lambda_2 m(t), \quad \forall t \in S, \end{aligned} \quad (72)$$

$$|f(t) - \lambda_1 \lambda_2 m(t)| \leq \epsilon, \quad \forall t \in S.$$

Remark 16. In particular, if S is a group with identity 0, it is shown in [14] that every bounded function (f, g, h) satisfying (48) satisfies

$$\begin{aligned} \|f(x) - |g(0)h(0)|\| &\leq 3\epsilon, \\ \|g(x) - |g(0)|\| &\leq \frac{2\epsilon}{M_h}, \\ \|h(x) - |h(0)|\| &\leq \frac{2\epsilon}{M_g} \end{aligned} \quad (73)$$

for all $x \in S$, where $M_g = \sup_{x \in S} |g(x)|$ and $M_h = \sup_{x \in S} |h(x)|$.

As a consequence of Theorem 14 we obtain the general solutions of the functional equation as follows:

$$f(t+s) = g(t)h(s), \quad \forall t, s \in S. \quad (74)$$

Corollary 17. Let $f, g, h : S \rightarrow \mathbb{C}$ be unbounded functions. Then f, g, h satisfies the functional equation (74) if and only if either there exist a nonzero exponential function m and nonzero constants $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\begin{aligned} g(t) &= \lambda_1 m(t), \quad h(t) = \lambda_2 m(t), \quad \forall t \in S \\ f(t) &= \lambda_1 \lambda_2 m(t), \quad \forall t \in S + S \\ f(t) &: \text{arbitrary}, \quad \forall t \notin S + S \end{aligned} \quad (75)$$

or else g satisfies

$$g(t) = 0, \quad \forall t \in S + S \quad (76)$$

$$g(t)g(s) = g(r)g(u), \quad \forall t+s = r+u,$$

and h and f satisfy, respectively,

$$h(t) = \lambda g(t), \quad \forall t \in S \quad (77)$$

for some $\lambda \in \mathbb{C}$, and

$$\begin{aligned} f(t) &= 0, \quad \forall t \in S + S + S, \\ f(t) &: \text{arbitrary}, \quad \forall t \in S \setminus (S + S). \end{aligned} \quad (78)$$

In particular, S is 2-divisible or has a neutral element; then the nonzero solutions of (74) are given by (75).

Remark 18. In particular, if S is a group, it follows from Remark 16 that every bounded solution (f, g, h) of (74) has the form

$$\begin{aligned} |f(x)| &= |g(0)h(0)|, \quad |g(x)| = |g(0)|, \\ |h(x)| &= |h(0)| \end{aligned} \quad (79)$$

for all $x \in S$. Furthermore, S is 2-divisible and $f, g, h : S \rightarrow \mathbb{R}$; it can be shown that

$$\begin{aligned} f(x) &= g(0)h(0), \quad g(x) = g(0), \\ h(x) &= h(0) \end{aligned} \quad (80)$$

for all $x \in S$.

We call the solutions (f, g, h) of form (75) *regular solutions* and call the solutions (f, g, h) of forms (76), (77), and (78) *irregular solutions*.

Example 19. Let n_0 be a positive integer and let S be the set of all integers $k \geq n_0$. Let $f, g, h : S \rightarrow \mathbb{C}$ satisfy (74). We first exhibit the solutions of form (75). Let $m : S \rightarrow \mathbb{C}$ be a nonzero exponential function. If $m(k_0) = 0$ for some $k_0 \in S$, then for each $k \in S$ we have

$$m(k)^{k_0} = m(k_0 \cdot k) = m(k_0)^k = 0. \quad (81)$$

Thus, we have $m(k) \neq 0$ for all $k \in S$. Since $m(k+2)m(k) = m(2k+2) = m(k+1)^2$ for all $k \in S$ we have

$$\frac{m(k+2)}{m(k+1)} = \frac{m(k+1)}{m(k)}, \quad k \in S. \quad (82)$$

Thus, we have

$$\frac{m(k+1)}{m(k)} = \frac{m(n_0+1)}{m(n_0)} := \gamma, \quad k \in S. \quad (83)$$

Putting $k = n_0, n_0+1, \dots, n_0+p-1$ in (83) and multiplying the results in both sides we have

$$m(k) = m(n_0) \gamma^{k-n_0}, \quad p \in S. \quad (84)$$

Now, since $m(n_0+1)^{n_0} = m(n_0(n_0+1)) = m(n_0)^{n_0+1}$ we have

$$\gamma^{n_0} = \left(\frac{m(n_0+1)}{m(n_0)} \right)^{n_0} = m(n_0). \quad (85)$$

From (84) and (85) we have

$$m(k) = \gamma^k, \quad k \in S. \quad (86)$$

Therefore, the regular solutions (f, g, h) of (74) are given by

$$\begin{aligned} g(t) &= \lambda_1 \gamma^t, \quad h(t) = \lambda_2 \gamma^t, \quad t \geq k_0, \\ f(t) &= \lambda_1 \lambda_2 \gamma^t, \quad t \geq 2k_0, \\ f(s) &: \text{arbitrary}, \quad k_0 \leq s \leq 2k_0 - 1 \end{aligned} \quad (87)$$

for some $\lambda_1, \lambda_2, \gamma \in \mathbb{C}$.

Now, we exhibit the irregular solutions of (74). By (76) we have $g(k) = 0$ for all $k \geq 2k_0$, and if $g(k) = 0$ for some $k \geq n_0 + 2$, then we have

$$g(k-1)^2 = g(k-2)g(k). \quad (88)$$

Thus, if $g(k) = 0$ for some $k \geq n_0 + 2$, then $g(k-1) = 0$, which implies $g(n) = 0$ for all $n \geq n_0 + 1$. Now, if $n \geq 2n_0 + 1$, then we have

$$f(n) = \lambda g(n_0 + 1)g(n - n_0 - 1) = 0. \quad (89)$$

Therefore the irregular solutions of (74) are given by

$$\begin{aligned} g(n_0) &= a, & h(n_0) &= b, & g(n) &= h(n) = 0, \\ & & & & n &\geq n_0 + 1, \end{aligned} \quad (90)$$

$$f(n_0 + k) = c_k, \quad k = 0, 1, 2, \dots, n_0 - 1,$$

$$f(2n_0) = ab, \quad f(n) = 0, \quad n \geq 2n_0 + 1,$$

where $a, b, c_k \in \mathbb{C}$, $k = 1, 2, \dots$, are arbitrary complex numbers.

Finally, we consider the stability of functional equation (49).

Theorem 20. Let $g : S \rightarrow \mathbb{C}$ satisfy

$$|g(t+s)g(r) - g(t)g(s+r)| \leq \epsilon \quad (91)$$

for all $t, s, r \in S$. Then either g is bounded or there exist a nonzero constant $\lambda \in \mathbb{C}$ and a nonzero exponential function such that

$$g(t) = \lambda m(t), \quad \forall t \in S \quad (92)$$

or else

$$\begin{aligned} g(t) &= 0, \quad \forall t \in S + S, \\ g(t) &: \text{arbitrary}, \quad \forall t \in S \setminus (S + S). \end{aligned} \quad (93)$$

In particular, S is 2-divisible or has a neutral element; then every solution of (91) is bounded or given by (92).

Proof. In view of (91) we can write

$$\begin{aligned} &|g(t+s)g(r) - g(t)g(s+r)| \\ &\leq \left| g(t+s)g(r) - \frac{g(t)g(s+u)g(r)}{g(u)} \right| \\ &\quad + \left| \frac{g(t)g(s+u)g(r)}{g(u)} - \frac{g(t)g(s+r)g(u)}{g(u)} \right| \\ &\leq \epsilon \left(\frac{|g(r)| + |g(t)|}{|g(u)|} \right) \end{aligned} \quad (94)$$

for all $t, s, r, u \in S$ such that $g(u) \neq 0$. Since g is unbounded, from (94) we have

$$g(t+s)g(r) = g(t)g(s+r) \quad (95)$$

for all $t, s, r \in S$. Using Lemma 13, we get the result. This completes the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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