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Research Article

Commutators of Multilinear Calderón-Zygmund Operator and BMO Functions in Herz-Morrey Spaces with Variable Exponents

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We obtain the boundedness of a commutator generated by multilinear Calderón-Zygmund operator and BMO functions in Herz-Morrey spaces with variable exponents.

1. Introduction

In recent decades, variable exponent function spaces have attracted much attention. Since Kováčik and Rákosník [1] clarified fundamental properties of the variable Lebesgue and Sobolev spaces, there are many spaces studied, such as Besov and Trieble-Lizorkin spaces with variable exponents, Hardy spaces with variable exponents, Bessel potential spaces with a variable exponent, and Herz-Morrey spaces with variable exponents; see [2–13]. Recently, multilinear operators and their commutators in variable exponent function spaces are also intensively studied by a significant number of authors, such as multilinear commutators of multilinear singular integral with Lipschitz functions and BMO functions, respectively, in [14–19], multilinear commutators of BMO functions and multilinear singular integral operators with nonsmooth kernels in [20, 21], a vector-estimate of higher order commutators on Herz-Morrey spaces with variable exponent in [22], maximal multilinear commutators and maximal iterated commutators generated by multilinear operators and Lipschitz functions in [23], and weighted estimates for vector-valued commutators of multilinear operators in [24].

Motivated by the above results, in this paper, we will consider the boundedness of a commutator generated by a multilinear Calderón-Zygmund operator and BMO functions on the variable Herz-Morrey spaces.

To state the main result of this paper, we need to recall more notations.

Let T be a multilinear singular integral operator which is initially defined on the m-fold product of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Its values are taken in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ such that, for $x \notin \cap_{j=1}^m \operatorname{supp} f_j$, $\vec{f} = (f_1, \ldots, f_m)$,

$$T\vec{f}(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i)\, dy_1 \cdots dy_m, \quad (1)$$

where $f_1, \ldots, f_m \in L_C^\infty(\mathbb{R}^n)$ (the space of compactly supported bounded functions). Here, the kernel K is a function in $(\mathbb{R}^n)^{m+1}$ away from the diagonal $y_0 = y_1 = \cdots = y_m$ and satisfies the standard estimates

$$\begin{gathered} |K(x, y_1, \ldots, y_m)| \le A(|x - y_1| + \cdots + |x - y_m|)^{-mn}, \\ |K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \\ \le \frac{A|x - x'|^\epsilon}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\epsilon}} \end{gathered} \quad (2)$$

provided that $|x - x'| \le (1/2)\max\{|x - y_1| + \cdots + |x - y_m|\}$, and, for each $1 \le i \le m$,

$$\begin{gathered} |K(x, y_1, \ldots, y_i, \ldots, y_m) - K(x, y_1, \ldots, y_i', \ldots, y_m)| \\ \le \frac{A|y_i - y_i'|^\epsilon}{\left(\sum_{j=1}^m |x - y_j|\right)^{mn+\epsilon}} \end{gathered} \quad (3)$$

provided that $|y_i - y_i'| \le (1/2)\max\{|x-y_1|+\cdots+|x-y_m|\}$, where A and ϵ are positive constants.

Such kernels are called the m-linear Calderón-Zygmund kernels and the collection of such functions is denoted by m-CZK(A,ϵ) in [25].

Let T be as in (1) with an m-CZK(A,ϵ) kernel. If, for some $1 < q_1, q_2, \ldots, q_m < \infty$, T is bounded from $L^{q_1} \times L^{q_2} \times \cdots \times L^{q_m}$ to L^q with $1/q_1 + 1/q_2 + \cdots + 1/q_m = 1/q$, then we say T is an m-linear Calderón-Zygmund operator. Grafakos and Torres in [25] showed that if T is an m-linear Calderón-Zygmund operator, then T is bounded from $L^{q_1} \times L^{q_2} \times \cdots \times L^{q_m}$ to L^q for any $1 < q_1, q_2, \ldots, q_m < \infty$ such that $1/q_1 + 1/q_2 + \cdots + 1/q_m = 1/q$. Then, Grafakos and Torres in [26] obtained weighted norm inequalities for multilinear Calderón-Zygmund operators.

If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (the set of all complex-valued locally integrable functions on $\mathbb{R}^n$), set

$$\|b\|_* := \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|\,dx, \tag{4}$$

where the supremum is taken over all balls in $\mathbb{R}^n$, b_B is the mean of b on B, and what follows $|E|$ is the Lebesgue measure of measurable set E in $\mathbb{R}^n$. A function b is called bounded mean oscillation if $\|b\|_* < \infty$ and BMO$(\mathbb{R}^n)$ is the set of all locally integrable functions b on $\mathbb{R}^n$ such that $\|b\|_* < \infty$.

Let $\vec{b} = (b_1, \ldots, b_m)$ and $b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for $1 \le j \le m$. We will consider the commutator $[\vec{b}, T]$, which is defined for suitable functions $f_1, \ldots, f_m$ by

$$\left[\vec{b}, T\right]\vec{f}(x) := \sum_{\sigma}(-1)^{|\sigma'|} b_\sigma(x)\, T b_{\sigma'}\vec{f}(x); \tag{5}$$

where $\vec{f} = (f_1, \ldots, f_m)$, σ denotes a subset of $\{1, 2, \ldots, m\}$, σ' is the complement of σ in $\{1, 2, \ldots, m\}$, $|\sigma'|$ denotes the number of elements of σ', $b_\sigma(x) = \prod_{j\in\sigma} b_j(x)$, $b_{\sigma'}\vec{f} = (g_1, \ldots, g_m)$, when $j \in \sigma'$, $g_j = b_j f_j$, otherwise, $g_j = f_j$.

Definition 1. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function.

(i) The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable: } \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}. \tag{6}$$

(ii) The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \left\{ f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n \right\}, \tag{7}$$

where, and in what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$.

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}. \tag{8}$$

Letting $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, we denote

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p_+ := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x). \tag{9}$$

The set $\mathscr{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathscr{P}^0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as mentioned above for $p(\cdot) \in \mathscr{P}^0(\mathbb{R}^n)$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$ that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|\,dy, \quad \forall x \in \mathbb{R}^n, \tag{10}$$

where B is a ball and $|B|$ is the volume of B. Let $\mathscr{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. For more on $\mathscr{B}(\mathbb{R}^n)$, see [27, 28].

Definition 2. We say that a function $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant C such that

$$|\alpha(x) - \alpha(y)| \le \frac{C}{\log(e + 1/|x-y|)}, \qquad x, y \in \mathbb{R}^n, \quad |x-y| < \frac{1}{2}. \tag{11}$$

If

$$|\alpha(x) - \alpha(0)| \le \frac{C}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n, \tag{12}$$

then we say that $\alpha(\cdot)$ is log-Hölder continuous at the origin. If there exists $\alpha_\infty \in \mathbb{R}$ such that

$$|\alpha(x) - \alpha_\infty| \le \frac{C}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n, \tag{13}$$

then we say that $\alpha(\cdot)$ is log-Hölder continuous at the infinity. If $\alpha(\cdot)$ is both locally log-Hölder continuous and log-Hölder continuous at the infinity, then we say $\alpha(\cdot)$ is global log-Hölder continuous.

We denote by $\mathscr{P}^{\log}_0(\mathbb{R}^n)$ and $\mathscr{P}^{\log}_\infty(\mathbb{R}^n)$ the class of all exponents $p \in \mathscr{P}(\mathbb{R}^n)$ which is log-Hölder continuous at the origin and at the infinity, respectively.

Next, we recall the definitions of Herz spaces and Herz-Morrey spaces with variable exponents. We use the following notations. For each $k \in \mathbb{Z}$, we define

$$B_k := \left\{x \in \mathbb{R}^n : |x| \le 2^k\right\}, \qquad D_k := B_k \setminus B_{k-1}, \qquad \chi_k := \chi_{D_k}. \tag{14}$$

Definition 3 (see [29, Definition 2]). Let $q \in (0,\infty]$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}} := \left\{ \sum_{k\in\mathbb{Z}} \left\| 2^{k\alpha(\cdot)} f\chi_k \right\|^q_{p(\cdot)} \right\}^{1/q} < \infty, \tag{15}$$

with the usual modifications when $q = \infty$.

Definition 4 (see [30, Definition 2]). Let $0 < q < \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{16}$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \left\| 2^{k\alpha(\cdot)} f\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}. \tag{17}$$

If $\alpha(\cdot)$ is a constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ was defined in [30]. If $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constant and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{q,p}(\mathbb{R}^n)$ is the classical Herz space in [31].

Lemma 5 (see [30, Lemma 1 and (10)]). *If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2 \in (0,1)$ and $C > 0$ such that, for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\begin{aligned} \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &\le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} &\le C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \end{aligned} \tag{18}$$

There is a position to state our result.

Theorem 6. *Let $b_i \in BMO(\mathbb{R}^n)$ and $p(\cdot)$, $p_i(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ satisfy $1/p(x) = 1/p_1(x) + 1/p_2(x) + \cdots + 1/p_m(x)$, $1 < p_i^- \le p_i^+ < n/\lambda_i$, $(p(\cdot)/p_*)' \in \mathscr{B}(\mathbb{R}^n)$ for some $0 < p_* < p_-$, $i = 1,2,\ldots,m$. Let $0 < q_i < \infty$, $0 \le \lambda_i < \infty$ and $\alpha_i(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$ for $i = 1,2,\ldots,m$ with*

$$\lambda_i - n\delta_1 < \alpha_i^- < \alpha_i^+ < n\delta_2, \tag{19}$$

where δ_1, $\delta_2 \in (0,1)$ are the constants appearing in (18). Suppose that $\lambda = \sum_{i=1}^m \lambda_i$, $\alpha(\cdot) = \sum_{i=1}^m \alpha_i(\cdot)$, and $1/q = \sum_{i=1}^m (1/q_i)$. Then

$$\left\| [\vec{b},T](\vec{f}) \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \le C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)} \tag{20}$$

with the constant $C > 0$ independent of $\vec{f} = (f_1, f_2, \ldots, f_m)$.

Remark 7. Let $\lambda_i = 0$, and then the commutator $[\vec{b},T]$ is bounded from the product of variable exponents Herz spaces $\dot{K}^{\alpha_1(\cdot)}_{q_1,p_1(\cdot)}(\mathbb{R}^n) \times \dot{K}^{\alpha_2(\cdot)}_{q_2,p_2(\cdot)}(\mathbb{R}^n) \times \cdots \times \dot{K}^{\alpha_m(\cdot)}_{q_m,p_m(\cdot)}(\mathbb{R}^n)$ to variable exponents Herz space $\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$ when $-n\delta_1 < \alpha_i^- < \alpha_i^+ < n\delta_2$.

Finally, we point out that C denotes a positive constant which may be different at different occurrences.

2. Proof of the Main Result

To give our proof, we need some lemmas.

Lemma 8 (see [32, Proposition 2]). *If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_\infty(\mathbb{R}^n)$, $0 \le \lambda < \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $q \in (0,\infty)$, then*

$$\begin{aligned} &\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \\ &\approx \max\left\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}, \right. \\ &\quad \sup_{L>0, L\in\mathbb{Z}} \left[2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} \right. \\ &\quad \left. \left. + 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} \right] \right\}. \end{aligned} \tag{21}$$

Lemma 9 (see [30, Lemma 2]). *If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that, for all balls B in $\mathbb{R}^n$,*

$$C^{-1} \le \frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C. \tag{22}$$

Lemma 10 (see [29, Lemma 3]). *Let k be a positive integer. Then one has that, for all $b \in BMO(\mathbb{R}^n)$ and all $i, j \in \mathbb{Z}$ with $j > i$,*

$$\begin{aligned} C^{-1}\|b\|^k_* &\le \sup_{B:ball} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\| (b-b_B)^k \chi_B \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|^k_*, \\ \left\| (b-b_{B_i})^k \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C(j-i)^k \|b\|^k_* \left\| \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{23}$$

In fact, if $k = 1$, then, from inequalities (23), for all balls B and $i, j \in \mathbb{Z}$ with $j > i$, we have

$$\begin{aligned} \|(b-b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C\|b\|_* \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\ \left\| (b-b_{B_i}) \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C(j-i)\|b\|_* \left\| \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{24}$$

Lemma 11 (see [1, Theorem 2.1]). *If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{25}$$

where $r_p := 1 + 1/p_+ - 1/p_-$.

Lemma 12 (see [14, Theorem 2.3]). *Let p, p_1, $p_2 \in \mathscr{P}^0(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$. Then there exists a constant C_{p,p_1} independent of the functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_{p,p_1}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \tag{26}$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$.

Lemma 13 (see [14, Corollary 2.2]). *Let T be a 2-linear Calderón-Zygmund operator and let b_1 and b_2 be BMO functions. Let $p(\cdot) \in \mathscr{P}^0(\mathbb{R}^n)$ such that there exists $0 < p_* < p_-$ with $(p(\cdot)/p_*)' \in \mathscr{B}(\mathbb{R}^n)$. If $p_1(\cdot), p_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$, then there exists a constant C independent of functions $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$ for $i = 1, 2$ such that*

$$\|[b_1, b_2, T](f_1, f_2)\|_{L^{p(\cdot)}} \le C\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{p_1(\cdot)}}\|f_2\|_{L^{p_2(\cdot)}}. \tag{27}$$

Proof of Theorem 6. Although our method is suitable for any multilinear operator, for simplicity, we only consider 2-linear operators. Let b_1 and b_2 be BMO functions. Since the set of all bounded compactly supported functions is dense in Herz-Morrey spaces with variable exponents, we let f_1 and f_2 be bounded compactly supported functions; then, for $x \in \mathbb{R}^n$, we write

$$\begin{aligned}
&[b_1, b_2, T](f_1, f_2)(x) \\
&\quad = b_1(x) b_2(x) T(f_1, f_2)(x) - b_1(x) T(f_1, b_2 f_2)(x) \\
&\qquad - b_2(x) T(b_1 f_1, f_2)(x) + T(b_1 f_1, b_2 f_2)(x).
\end{aligned} \tag{28}$$

We write

$$f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x) =: \sum_{l_i=-\infty}^{\infty} f_{l_i}(x), \qquad i = 1, 2. \tag{29}$$

Then, for each $k \in \mathbb{Z}$, if $l_i \le k - 1$, from Lemma 11, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} [b_i(x) - b_i(y_i)] f_{l_i}(y_i)\, dy_i \\
&\quad \le C\left[\int_{\mathbb{R}^n} \left(b_i(x) - b_{B_{l_i}}\right) f_{l_i}(y_i)\, dy_i \right. \\
&\qquad \left. + \int_{\mathbb{R}^n} \left(b_{B_{l_i}} - b_i(y_i)\right) f_{l_i}(y_i)\, dy_i\right] \\
&\quad \le C\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \left(\left|b_i(x) - b_{B_{l_i}}\right| \cdot \|\chi_{l_i}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \right. \\
&\qquad \left. + \left\|\left|b_{B_{l_i}} - b_i\right| \cdot \chi_{l_i}\right\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\right),
\end{aligned} \tag{30}$$

and by using inequalities (18), (22), and (24), it follows that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n} [b_i(\cdot) - b_i(y_i)] f_{l_i}(y_i)\, dy_i \chi_k\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \left(\left\|\left|b_i - b_{B_{l_i}}\right| \chi_{B_k}\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_{l_i}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \right. \\
&\qquad \left. + \left\|\left|b_{B_{l_i}} - b_i\right| \chi_{B_{l_i}}\right\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_k\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right) \\
&\quad \le C\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \left[(k - l_i)\|b_i\|_*\|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{l_i}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \right. \\
&\qquad \left. + \|b_i\|_*\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right] \\
&\quad \le C\|b_i\|_*\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}(k - l_i)\|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\|b_i\|_*\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}(k - l_i)\|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times \|\chi_{B_k}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}} \\
&\quad \le C(k - l_i) 2^{(l_i-k)n\delta_2} 2^{kn}\|b_i\|_*\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{31}$$

Similarly, if $l_i \ge k + 1$, we have

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n} [b_i(\cdot) - b_i(y_i)] f_{l_i}(y_i) dy_i \chi_k\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \left(\left\|\left|b_i - b_{B_k}\right| \chi_{B_k}\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_{l_i}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \right. \\
&\qquad \left. + \left\|\left|b_{B_k} - b_i\right| \chi_{B_{l_i}}\right\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_k\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right) \\
&\quad \le C\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \left[\|b_i\|_*\|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{l_i}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \right. \\
&\qquad \left. + (l_i - k)\|b_i\|_*\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right] \\
&\quad \le C\|b_i\|_*\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}(l_i - k)\|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\|b_i\|_*\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}(l_i - k) \\
&\qquad \times \frac{\|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_{l_i}}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}\|\chi_{B_{l_i}}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_{B_{l_i}}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C(l_i - k) 2^{(k-l_i)n\delta_1} 2^{l_i n}\|b_i\|_*\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{32}$$

As for the case $l_i = k$, we get

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n} \left[b_i(\cdot) - b_i(y_i)\right] f_{l_i}(y_i) dy_i \chi_k\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \left(\left\|\left|b_i - b_{B_k}\right| \chi_{B_k}\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_{l_i}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \right. \\
&\qquad \left. + \left\|\left|b_{B_k} - b_i\right| \chi_{B_{l_i}}\right\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)} \cdot \|\chi_k\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right)
\end{aligned}$$

$$\leq C\|b_i\|_* \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}$$
$$\leq C2^{kn}\|b_i\|_* \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}. \tag{33}$$

By Lemma 8, we know

$$\|[b_1, b_2, T](f_1, f_2)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx \max\{E, F\}, \tag{34}$$

where

$$E := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$F := \sup_{L>0, L\in\mathbb{Z}} \left[2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|[b_1, b_2, T](f_1, f_2)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} + 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q} \times \|[b_1, b_2, T](f_1, f_2)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q} \right]. \tag{35}$$

Since the estimate of F is essentially similar to that of E, so it suffices to prove that E is bounded in Herz-Morrey spaces with variable exponents. It is easy to see that

$$E \leq C\sum_{i=1}^{9} I_i, \tag{36}$$

where

$$I_1 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=-\infty}^{k-2} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_2 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k-1}^{k+1} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_3 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k+2}^{\infty} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_4 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=k-1}^{k+1} \sum_{l_2=-\infty}^{k-2} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_5 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=k-1}^{k+1} \sum_{l_2=k-1}^{k+1} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_6 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{\infty} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_7 := \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=k+2}^{\infty} \sum_{l_2=-\infty}^{k-2} [b_1, b_2, T] \times (f_{l_1}, f_{l_2}) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q},$$

$$I_8 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=k+2}^{\infty} \sum_{l_2=k-1}^{k+1} [b_1, b_2, T] \times (f_{l_1}, f_{l_2})\chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q},$$

$$I_9 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left\| \sum_{l_1=k+2}^{\infty} \sum_{l_2=k+2}^{\infty} [b_1, b_2, T] \times \left(f_{l_1}, f_{l_2}\right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}. \tag{37}$$

Using the symmetry of f_1 and f_2, we only need to estimate I_1, I_2, I_3, I_5, I_6, I_9 because the estimates of I_2, I_3, and I_6 are analogous to those of I_4, I_7, and I_8, respectively. In what follows, we divide it into 6 steps.

Step 1. To estimate the term of I_1, we note that $l_i \le k-2$ for $i = 1, 2$. Thus, for $x \in D_k$, $y_i \in D_{l_i}$,

$$\left|x - y_i\right| \ge \left||x| - \left|y_i\right|\right| > 2^{k-1} - 2^{l_i} \ge 2^{k-2}. \tag{38}$$

Then, for $x \in D_k$, we get

$$\left|K\left(x, y_1, y_2\right)\right| \le C\left(\left|x - y_1\right| + \left|x - y_2\right|\right)^{-2n} \le C2^{-2kn}. \tag{39}$$

Therefore,

$$\begin{aligned} &[b_1, b_2, T]\left(f_{l_1}, f_{l_2}\right)(x) \\ &\quad \le C2^{-2kn} \prod_{i=1}^{2} \int_{\mathbb{R}^n} \left[b_i(x) - b_i(y_i)\right] f_{l_i}(y_i)\, dy_i. \end{aligned} \tag{40}$$

By Lemma 12 and inequality (31), we obtain

$$\begin{aligned} &\left\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=-\infty}^{k-2} [b_1, b_2, T](f_{l_1}, f_{l_2})\chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C \left\| 2^{-2kn} \prod_{i=1}^{2} \left(\sum_{l_i=-\infty}^{k-2} \int_{\mathbb{R}^n} \left[b_i(\cdot) - b_i(y_i)\right] f_{l_i}(y_i)\, dy_i \right) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-2kn} \prod_{i=1}^{2} \sum_{l_i=-\infty}^{k-2} \left\| \int_{\mathbb{R}^n} [b_i(\cdot) - b_i(y_i)] f_{l_i}(y_i) dy_i \chi_k \right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \\ &\le C \prod_{i=1}^{2} \sum_{l_i=-\infty}^{k-2} (k - l_i)\, 2^{(l_i-k)n\delta_2} \|b_i\|_* \left\| f_{l_i} \right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{41}$$

Since $1/q = 1/q_1 + 1/q_2$, it follows that

$$\begin{aligned} I_1 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \times \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\prod_{i=1}^{2} \sum_{l_i=-\infty}^{k-2} (k - l_i)\, 2^{(l_i-k)n\delta_2} \|b_i\|_* \times \left\| f_{l_i} \right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \right) \right\}^{1/q} \\ &\le C \sup_{L\le 0, L\in\mathbb{Z}} \prod_{i=1}^{2} 2^{-L\lambda_i} \|b_i\|_* \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_i=-\infty}^{k-2} (k - l_i)\, 2^{(l_i-k)n\delta_2 + k\alpha_i(0)} \times \left\| f_{l_i} \right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \right)^{q_i} \right\}^{1/q_i} \\ &=: C \sup_{L\le 0, L\in\mathbb{Z}} I_{11}(L)\, I_{12}(L), \end{aligned} \tag{42}$$

where

$$\begin{aligned} I_{1i}(L) &:= 2^{-L\lambda_i} \|b_i\|_* \\ &\quad \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_i=-\infty}^{k-2} (k - l_i)\, 2^{(l_i-k)\mu_i} 2^{l_i\alpha_i(0)} \times \left\| f_{l_i} \right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \right)^{q_i} \right\}^{1/q_i}. \end{aligned} \tag{43}$$

Here, we denote $\mu_i = n\delta_2 - \alpha_i(0)$ for short. It follows from condition (19) that $\mu_i > 0$. To continue calculations for (43), we consider the two cases $0 < q_i \le 1$ and $1 < q_i < \infty$.

If $0 < q_i \le 1$, since, for $a_1, a_2, \ldots \ge 0$,

$$\left(\sum_{h=1}^{\infty} a_h \right)^{q_i} \le \sum_{h=1}^{\infty} a_h^{q_i}, \tag{44}$$

then

$$\begin{aligned} &I_{1i}(L) \\ &\quad \le C2^{-L\lambda_i} \|b_i\|_* \left\{ \sum_{k=-\infty}^{L} \sum_{l_i=-\infty}^{k-2} (k - l_i)^{q_i} 2^{(l_i-k)\mu_i q_i} 2^{l_i\alpha_i(0)^{q_i}} \times \left\| f_{l_i} \right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \right\}^{1/q_i} \end{aligned}$$

$$
\begin{aligned}
&= C2^{-L\lambda_i}\|b_i\|_* \left\{ \sum_{l_i=-\infty}^{L-2} 2^{l_i\alpha_i(0)^{q_i}} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \right. \\
&\qquad \left. \times \sum_{k=l_i+2}^{L} (k-l_i)^{q_i} 2^{(l_i-k)\mu_i q_i} \right\}^{1/q_i} \\
&\le C\|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{l_i=-\infty}^{L-2} 2^{l_i\alpha_i(0)^{q_i}} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \right\}^{1/q_i} \\
&\le C\|b_i\|_* \|f_{l_i}\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}.
\end{aligned} \tag{45}
$$

If $1 < q_i < \infty$, by the Hölder inequality, we obtain

$$
\begin{aligned}
I_{1i}(L) &= 2^{-L\lambda_i}\|b_i\|_* \\
&\quad \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_i=-\infty}^{k-2} (k-l_i) 2^{(l_i-k)\mu_i} 2^{l_i\alpha_i(0)} \right.\right. \\
&\qquad \left.\left. \times \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \right)^{q_i} \right\}^{1/q_i} \\
&\le C2^{-L\lambda_i}\|b_i\|_* \\
&\quad \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_i=-\infty}^{k-2} 2^{(l_i-k)\mu_i(q_i/2)} 2^{l_i\alpha_i(0)q_i} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \right) \right. \\
&\qquad \left. \times \left(\sum_{l_i=-\infty}^{k-2} (k-l_i)^{q_i'} 2^{(l_i-k)\mu_i(q_i'/2)} \right)^{q_i/q_i'} \right\}^{1/q_i} \\
&\le C2^{-L\lambda_i}\|b_i\|_* \\
&\quad \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_i=-\infty}^{k-2} 2^{(l_i-k)\mu_i(q_i/2)} 2^{l_i\alpha_i(0)q_i} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \right) \right\}^{1/q_i} \\
&= C2^{-L\lambda_i}\|b_i\|_* \left(\sum_{l_i=-\infty}^{L-2} 2^{l_i\alpha_i(0)q_i} \right. \\
&\qquad \left. \times \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \sum_{k=l_i+2}^{L} 2^{(l_i-k)\mu_i(q_i/2)} \right)^{1/q_i} \\
&\le C\|b_i\|_* 2^{-L\lambda_i} \left(\sum_{l_i=-\infty}^{L-2} 2^{l_i\alpha_i(0)q_i} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}^{q_i} \right)^{1/q_i} \\
&\le C\|b_i\|_* \|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}.
\end{aligned} \tag{46}
$$

Thus, for any $0 < q_i < \infty$,

$$
\begin{aligned}
I_1 &\le C \sup_{L\le 0, L\in\mathbb{Z}} I_{11}(L) I_{12}(L) \\
&\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
\end{aligned} \tag{47}
$$

Step 2. To estimate I_2, for $x \in D_k$, $y_i \in D_{l_i}$, $i = 1, 2$ and $l_1 \le k-2$, $k-1 \le l_2 \le k+1$, we have

$$
|x - y_2| \ge |x - y_1| \ge \big||x| - |y_1|\big| > 2^{k-2}. \tag{48}
$$

Since $l_1 \le k-2$, it follows from inequality (31) that

$$
\begin{aligned}
&\left\| \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k-1}^{k+1} [b_1, b_2, T](f_{l_1}, f_{l_2}) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C2^{-2kn} \sum_{l_1=-\infty}^{k-2} \left\| \int_{\mathbb{R}^n} \left[b_1(\cdot) - b_1(y_1) \right] f_{l_1}(y_1) dy_1 \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times \sum_{l_2=k-1}^{k+1} \left\| \int_{\mathbb{R}^n} [b_2(\cdot) - b_2(y_2)] f_{l_2}(y_2) dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C \left(\sum_{l_1=-\infty}^{k-2} (k-l_1) 2^{(l_1-k)n\delta_2} \|b_1\|_* \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right) \\
&\qquad \times 2^{-kn} \sum_{l_2=k-1}^{k+1} \left\| \int_{\mathbb{R}^n} [b_2(\cdot) - b_2(y_2)] f_{l_2}(y_2) dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{49}
$$

For $k-1 \le l_2 \le k+1$, combining the above term and inequalities (32) and (33), we have

$$
\begin{aligned}
I_2 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\|b_1\|_*\|b_2\|_* \\
&\quad \times \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{l_1=-\infty}^{k-2} \sum_{l_2=k-1}^{k+1} (k-l_1) 2^{(l_1-k)n\delta_2} \right.\right. \\
&\qquad \left.\left. \times \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} 2^{(l_2-k)n} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\|b_1\|_* \\
&\quad \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_1=-\infty}^{k-2} (k-l_1) 2^{(l_1-k)\mu_1} 2^{l_1\alpha_1(0)} \right.\right. \\
&\qquad \left.\left. \times \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \right\}^{1/q_1}
\end{aligned}
$$

$$\times 2^{-L\lambda_2}\|b_2\|_* \left\{\sum_{k=-\infty}^{L}\left(\sum_{l_2=k-1}^{k+1} 2^{k\alpha_2(0)}2^{(l_2-k)n} \times \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2}\right\}^{1/q_2}$$
$$=: C\sup_{L\le 0, L\in\mathbb{Z}} I_{21}(L)\, I_{22}(L). \tag{50}$$

Here, we used $2^{-n\delta_2} < 1$ and $2^{(l_2-k)n(1-\delta_1)} < 2^{(l_2-k)n}$ in the first inequality.

Obviously,

$$I_{21}(L) = I_{11}(L) \le C\|b_1\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}. \tag{51}$$

Therefore, we only need to estimate $I_{22}(L)$:

$$\begin{aligned} I_{22}(L) &= 2^{-L\lambda_2}\|b_2\|_* \left\{\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2} \times \left(\sum_{l_2=k-1}^{k+1} 2^{(l_2-k)n}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2}\right\}^{1/q_2} \\ &\le C\|b_2\|_* 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2}\|f_2\chi_k\|^{q_2}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{1/q_2} \\ &\le C\|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{52}$$

Step 3. To estimate I_3, for $x \in D_k$, $y_i \in D_{l_i}$, $i = 1, 2$, and $l_1 \le k-2$, $l_2 \ge k+2$, then we have

$$|x-y_1| \ge |x| - |y_1| > 2^{k-2}, \qquad |x-y_2| \ge |y_2| - |x| > 2^{l_2-2}. \tag{53}$$

Thus, for $x \in C_k$, we get

$$\begin{aligned} &[b_1, b_2, T](f_{l_1}, f_{l_2})(x) \\ &\le C2^{-kn}\int_{\mathbb{R}^n}[b_1(x) - b_1(y_1)]\, f_{l_1}(y_1)\, dy_1 2^{-l_2 n} \\ &\times \int_{\mathbb{R}^n}[b_2(x) - b_2(y_2)]\, f_{l_2}(y_2)\, dy_2. \end{aligned} \tag{54}$$

From inequalities (31) and (32), we obtain

$$\begin{aligned} &\left\|\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}[b_1, b_2, T](f_{l_1}, f_{l_2})\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{l_1=-\infty}^{k-2}(k-l_1)\, 2^{(l_1-k)n\delta_2}\|b_1\|_*\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\times \sum_{l_2=k+2}^{\infty}(l_2-k)\, 2^{(k-l_2)n\delta_1}\|b_2\|_*\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{55}$$

Consequently, it follows that

$$\begin{aligned} I_3 &\le C\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\|b_1\|_*\|b_2\|_* \\ &\times \left\{\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{l_1=-\infty}^{k-2}(k-l_1)\, 2^{(l_1-k)n\delta_2}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q} \times \left(\sum_{l_2=k+2}^{\infty}(l_2-k)\, 2^{(k-l_2)n\delta_1}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q}\right\}^{1/q} \\ &\le C\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\|b_1\|_* \\ &\times \left\{\sum_{k=-\infty}^{L}\left(\sum_{l_1=-\infty}^{k-2}(k-l_1)\, 2^{(l_1-k)\mu_1}2^{l_1\alpha_1(0)} \times \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1}\right\}^{1/q_1} \\ &\times 2^{-L\lambda_2}\|b_2\|_* \\ &\times \left\{\sum_{k=-\infty}^{L}\left(\sum_{l_2=k+2}^{\infty}(l_2-k) \times 2^{(k-l_2)(n\delta_1+\alpha_2(0))}2^{l_2\alpha_2(0)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2}\right\}^{1/q_2} \\ &=: C\sup_{L\le 0, L\in\mathbb{Z}} I_{31}(L)\, I_{32}(L). \end{aligned} \tag{56}$$

Note that

$$I_{31}(L) = I_{21}(L) \le C\|b_1\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}, \tag{57}$$

so we only compute $I_{32}(L)$. From inequality (19) and $n\delta_1 + \alpha_2(0) - \lambda_2 > 0$, we obtain

$$\begin{aligned}
&I_{32}(L)\\
&= 2^{-L\lambda_2}\|b_2\|_* \left\{\sum_{k=-\infty}^{L}\left(\sum_{l_2=k+2}^{\infty}(l_2-k)\,2^{(k-l_2)(n\delta_1+\alpha_2(0))}2^{l_2\alpha_2(0)}\right.\right.\\
&\qquad\left.\left.\times\|f_2\chi_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2}\right\}^{1/q_2}\\
&\le C2^{-L\lambda_2}\|b_2\|_*\\
&\quad\times\left\{\sum_{k=-\infty}^{L}\left[\sum_{l_2=k+2}^{\infty}(l_2-k)\,2^{(k-l_2)(n\delta_1+\alpha_2(0))}\,2^{l_2\lambda_2}\right.\right.\\
&\qquad\times 2^{-l_2\lambda_2}\left(\sum_{m=-\infty}^{l_2}2^{m\alpha_2(0)q_2}\right.\\
&\qquad\left.\left.\left.\times\|f_2\chi_m\|^{q_2}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{1/q_2}\right]^{q_2}\right\}^{1/q_2}\\
&\le C2^{-L\lambda_2}\|b_2\|_*\left\{\sum_{k=-\infty}^{L}2^{k\lambda_2q_2}\right.\\
&\qquad\times\left(\sum_{l_2=k+2}^{\infty}(l_2-k)\,2^{(k-l_2)(n\delta_1+\alpha_2(0)-\lambda_2)}\right.\\
&\qquad\left.\left.\times\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2}\right\}^{1/q_2}\\
&\le C\|b_2\|_*2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}2^{k\lambda_2q_2}\right)^{1/q_2}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{58}$$

Therefore,

$$\begin{aligned}
I_3 &\le C\sup_{L\le 0,L\in\mathbb{Z}}I_{31}(L)\,I_{32}(L)\\
&\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{59}$$

Step 4. It turns to estimate the term I_5. Applying the Hölder inequality and Lemma 13, we have

$$\begin{aligned}
I_5 &\le C\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\\
&\quad\times\left\{\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1}\|[b_1,b_2,T]\right.\right.\\
&\qquad\left.\left.\times(f_{l_1},f_{l_2})\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q}\right\}^{1/q}\\
&\le C\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\right.\\
&\qquad\times\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1}\|b_1\|_*\|b_2\|_*\right.\\
&\qquad\left.\left.\times\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q}\right\}^{1/q}\\
&\le C\prod_{i=1}^{2}\|b_i\|_*\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_i}\\
&\quad\times\left\{\sum_{k=-\infty}^{L}2^{k\alpha_i(0)q_i}\left(\sum_{l_i=k-1}^{k+1}\|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right)^{q_i}\right\}^{1/q_i}\\
&\le C\prod_{i=1}^{2}\|b_i\|_*\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_i}\\
&\quad\times\left(\sum_{k=-\infty}^{L}2^{k\alpha_i(0)q_i}\|f_i\chi_k\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\right)^{1/q_i}\\
&\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{60}$$

Step 5. Now it goes to the estimate of I_6.

It is clear that, for $x\in D_k$ and $k-1\le l_1\le k+1$ and $l_2\ge k+2$,

$$\begin{aligned}
&|[b_1,b_2,T](f_1,f_2)(x)|\\
&\quad\le C2^{-kn}\int_{\mathbb{R}^n}[b_1(x)-b_1(y_1)]\,f_1(y_1)\,dy_1\,2^{-l_2n}\\
&\qquad\times\int_{\mathbb{R}^n}[b_2(x)-b_2(y_2)]\,f_2(y_2)\,dy_2.
\end{aligned}\tag{61}$$

Therefore,

$$\begin{aligned}
&\left\|\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}[b_1,b_2,T](f_{l_1},f_{l_2})\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C\|b_1\|_*\|b_2\|_*\\
&\qquad\times\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}2^{(l_1-k)n}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times(l_2-k)\,2^{(k-l_2)n\delta_1}\|b_2\|_*\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

$$
\begin{aligned}
I_6 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \\
&\quad\times \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{\infty} \left\| [b_1, b_2, T] \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \times (f_{l_1}, f_{l_2}) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \|b_1\|_* \|b_2\|_* \\
&\quad\times \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{\infty} 2^{(l_1-k)n} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right.\right. \\
&\qquad\qquad \left.\left.\left. \times (l_2-k)\, 2^{(k-l_2)n\delta_1} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \|b_1\|_* \\
&\quad\times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_1=k-1}^{k+1} 2^{k\alpha_1(0)} 2^{(l_1-k)n} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \right\}^{1/q_1} \\
&\quad\times 2^{-L\lambda_2} \|b_2\|_* \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_2=k+2}^{\infty} (l_2-k)\, 2^{(k-l_2)(n\delta_1+\alpha_2(0))} \right.\right. \\
&\qquad\qquad \left.\left.\left. \times\, 2^{l_2\alpha_2(0)} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_2} \right\}^{1/q_2} \\
&=: C \sup_{L\le 0, L\in\mathbb{Z}} I_{61}(L)\, I_{62}(L).
\end{aligned}
\tag{62}
$$

Here, the estimate of $I_{61}(L)$ is similar to that of $I_{22}(L)$ and $I_{62}(L) = I_{32}(L)$.

Step 6. Finally, we will finish the estimation of the last term I_9. Note that $l_2 \ge k+2$ and $|x-y_i| > 2^{l_i-2}$ for $x \in D_k$, $y_i \in D_{l_i}$, $i = 1, 2$: we get

$$
\begin{aligned}
&[b_1, b_2, T]\left(f_{l_1}, f_{l_2}\right)(x) \\
&\quad\le C 2^{-l_1 n} \int_{\mathbb{R}^n} \left[b_1(x) - b_1(y_1)\right] f_{l_1}(y_1)\, dy_1 \cdot 2^{-l_2 n} \\
&\qquad\times \int_{\mathbb{R}^n} \left[b_2(x) - b_2(y_2)\right] f_{l_2}(y_2)\, dy_2.
\end{aligned}
\tag{63}
$$

Applying the Hölder inequality to the last integral, we obtain

$$
\begin{aligned}
&\left\| \sum_{l_1=k+2}^{\infty} \sum_{l_2=k+2}^{\infty} [b_1, b_2, T](f_{l_1}, f_{l_2})\chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C\|b_1\|_* \|b_2\|_* \\
&\qquad\times \sum_{l_1=k+2}^{\infty} \sum_{l_2=k+2}^{\infty} (l_1-k)\, 2^{(k-l_1)n\delta_1} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\qquad\times (l_2-k)\, 2^{(k-l_2)n\delta_1} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C \prod_{i=1}^{2} \|b_i\|_* \sum_{l_i=k+2}^{\infty} (l_i-k)\, 2^{(k-l_i)n\delta_1} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{64}
$$

Thus,

$$
\begin{aligned}
I_9 &\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \\
&\quad\times \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{l_1=k+2}^{\infty} \sum_{l_2=k+2}^{\infty} \left\| [b_1, b_2, T] \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \times \left(f_{l_1}, f_{l_2}\right) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \\
&\quad\times \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\prod_{i=1}^{2} \|b_i\|_* \sum_{l_i=k+2}^{\infty} (l_i-k) \right.\right. \\
&\qquad\qquad \left.\left.\left. \times 2^{(k-l_i)n\delta_1} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\le C \sup_{L\le 0, L\in\mathbb{Z}} \prod_{i=1}^{2} 2^{-L\lambda_i} \|b_i\|_* \\
&\quad\times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l_i=k+2}^{\infty} (l_i-k)\, 2^{(k-l_i)(n\delta_1+\alpha_i(0))} \right.\right. \\
&\qquad\qquad \left.\left.\left. \times\, 2^{l_i\alpha_i(0)} \|f_{l_i}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \right)^{q_i} \right\}^{1/q_i} \\
&=: C \sup_{L\le 0, L\in\mathbb{Z}} I_{91}(L)\, I_{92}(L) \\
&\le C\|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)} \\
&\quad\times \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{65}
$$

Here, the estimate of $I_{9i}(L)$, $i = 1, 2$, is similar to that of $I_{32}(L)$.

Combing all estimates for I_i, $i = 1, 2, \ldots, 9$ together, we get

$$
E \le C\|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{q_1,p_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),\lambda_2}_{q_2,p_2(\cdot)}(\mathbb{R}^n)}. \tag{66}
$$

This finishes the proof of Theorem 6. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] O. Kováčik and J. Rákosník, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$," *Czechoslovak Mathematical Journal*, vol. 41, no. 4, pp. 592–618, 1991.

[2] A. Almeida and D. Drihem, "Maximal, potential and singular type operators on Herz spaces with variable exponents," *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 781–795, 2012.

[3] A. Almeida, J. Hasanov, and S. Samko, "Maximal and potential operators in variable exponent Morrey spaces," *Georgian Mathematical Journal*, vol. 15, no. 2, pp. 195–208, 2008.

[4] A. Almeida and P. Hästö, "Besov spaces with variable smoothness and integrability," *Journal of Functional Analysis*, vol. 258, no. 5, pp. 1628–1655, 2010.

[5] L. Diening, P. Hästö, and S. Roudenko, "Function spaces of variable smoothness and integrability," *Journal of Functional Analysis*, vol. 256, no. 6, pp. 1731–1768, 2009.

[6] B. H. Dong and J. S. Xu, "New Herz type Besov and Triebel-Lizorkin spaces with variable exponents," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 384593, 27 pages, 2012.

[7] P. Gurka, P. Harjulehto, and A. Nekvinda, "Bessel potential spaces with variable exponent," *Mathematical Inequalities & Applications*, vol. 10, no. 3, pp. 661–676, 2007.

[8] M. Izuki, "Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent," *Mathematical Sciences Research Journal*, vol. 13, no. 10, pp. 243–253, 2009.

[9] H. Kempka, "2-microlocal Besov and Triebel-Lizorkin spaces of variable integrability," *Revista Matemática Complutense*, vol. 22, no. 1, pp. 227–251, 2009.

[10] H. Kempka, "Atomic, molecular and wavelet decomposition of generalized 2-microlocal Besov spaces," *Journal of Function Spaces and Applications*, vol. 8, no. 2, pp. 129–165, 2010.

[11] E. Nakai and Y. Sawano, "Hardy spaces with variable exponents and generalized Campanato spaces," *Journal of Functional Analysis*, vol. 262, no. 9, pp. 3665–3748, 2012.

[12] Y. Sawano, "Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators," *Integral Equations and Operator Theory*, vol. 77, no. 1, pp. 123–148, 2013.

[13] H. Wang and Z. Liu, "The Herz-type Hardy spaces with variable exponent and their applications," *Taiwanese Journal of Mathematics*, vol. 16, no. 4, pp. 1363–1389, 2012.

[14] A.-W. Huang and J.-S. Xu, "Multilinear singular integrals and commutators in variable exponent Lebesgue spaces," *Applied Mathematics—A Journal of Chinese Universities*, vol. 25, no. 1, pp. 69–77, 2010.

[15] X. Tao and H. Zhang, "On the boundedness of multilinear operators on weighted Herz-Morrey spaces," *Taiwanese Journal of Mathematics*, vol. 15, no. 4, pp. 1527–1543, 2011.

[16] W. Wang and J. Xu, "Commutators of multilinear singular integrals with Lipschitz functions on products of variable exponent Lebesgue spaces," *Advances in Mathematics*, vol. 38, no. 6, pp. 669–677, 2009 (Chinese).

[17] W. Wang and J.-s. Xu, "Commutators of multilinear singular integrals with Lipschitz functions," *Communications in Mathematical Research*, vol. 25, no. 4, pp. 318–328, 2009.

[18] J. Xu, "Generalized commutators of multilinear singular integrals," *Proceedings of A. Razmadze Mathematical Institute*, vol. 142, pp. 109–122, 2006.

[19] J. S. Xu, "Multilinear commutators of multilinear singular integrals," *Acta Mathematica Sinica*, vol. 51, no. 5, pp. 1021–1034, 2008.

[20] X. T. Duong, L. Grafakos, and L. Yan, "Multilinear operators with non-smooth kernels and commutators of singular integrals," *Transactions of the American Mathematical Society*, vol. 362, no. 4, pp. 2089–2113, 2010.

[21] J.-l. Lian, J. Li, and H.-x. Wu, "Multilinear commutators of BMO functions and multilinear singular integrals with non-smooth kernels," *Applied Mathematics—A Journal of Chinese Universities*, vol. 26, no. 1, pp. 109–120, 2011.

[22] Z. H. Xuan and L. S. Shu, "Boundedness of higher order commutators on Herz-Morrey spaces with variable exponent," *Journal of Nanjing University Mathematical Biquarterly*, vol. 30, no. 2, pp. 188–196, 2013.

[23] D. X. Chen and S. Z. Mao, "Weighted estimates for maximal commutators of multilinear singular integrals," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 128520, 20 pages, 2012.

[24] L. Tang, "Weighted estimates for vector-valued commutators of multilinear operators," *Proceedings of the Royal Society of Edinburgh A*, vol. 138, no. 4, pp. 897–922, 2008.

[25] L. Grafakos and R. H. Torres, "Multilinear Calderón-Zygmund theory," *Advances in Mathematics*, vol. 165, no. 1, pp. 124–164, 2002.

[26] L. Grafakos and R. H. Torres, "Maximal operator and weighted norm inequalities for multilinear singular integrals," *Indiana University Mathematics Journal*, vol. 51, no. 5, pp. 1261–1276, 2002.

[27] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, Springer, Heidelberg, Germany, 2013.

[28] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, vol. 2017 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2011.

[29] M. Izuki, "Boundedness of commutators on Herz spaces with variable exponent," *Rendiconti del Circolo Matematico di Palermo*, vol. 59, no. 2, pp. 199–213, 2010.

[30] M. Izuki, "Fractional integrals on Herz-Morrey spaces with variable exponent," *Hiroshima Mathematical Journal*, vol. 40, no. 3, pp. 343–355, 2010.

[31] S. Z. Lu, D. C. Yang, and G. Hu, *Herz Type Spaces and their Applications*, Science Press, Bejing, China, 2008.

[32] Z. H. Xuan and L. S. Shu, "Vector-estimate of higher order commutators on Herz-Morrey spaces with variable exponent," *Acta Mathematica Sinica*. In press.