Research Article

Commutators of Multilinear Calderón-Zygmund Operator and BMO Functions in Herz-Morrey Spaces with Variable Exponents

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We obtain the boundedness of a commutator generated by multilinear Calderón-Zygmund operator and BMO functions in Herz-Morrey spaces with variable exponents.

1. Introduction

In recent decades, variable exponent function spaces have attracted much attention. Since Kováčik and Rákosník [1] clarified fundamental properties of the variable Lebesgue and Sobolev spaces, there are many spaces studied, such as Besov and Triebel-Lizorkin spaces with variable exponents, Hardy spaces with variable exponents, Bessel potential spaces with a variable exponent, and Herz-Morrey spaces with variable exponents; see [2–13]. Recently, multilinear operators and their commutators in variable exponent function spaces are also intensively studied by a significant number of authors, such as multilinear commutators of multilinear singular integral with Lipschitz functions and BMO functions, respectively, in [14–19], multilinear commutators of BMO functions and multilinear singular integral operators with nonsmooth kernels in [20, 21], a vector-estimate of higher order commutators on Herz-Morrey spaces with variable exponent in [22], maximal multilinear commutators and maximal iterated commutators generated by multilinear operators and Lipschitz functions in [23], and weighted estimates for vector-valued commutators of multilinear operators in [24].

Motivated by the above results, in this paper, we will consider the boundedness of a commutator generated by a multilinear Calderón-Zygmund operator and BMO functions on the variable Herz-Morrey spaces.

To state the main result of this paper, we need to recall more notations.

Let $T$ be a multilinear singular integral operator which is initially defined on the $m$-fold product of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Its values are taken in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ such that, for $x \notin \bigcap_{j=1}^m \text{supp } f_j$, $\hat{f} = (f_1, \ldots, f_m)$,

$$ T \hat{f}(x) = \int_{\mathbb{R}^{mn}} K(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) \, dy_1 \cdots dy_m, \quad (1) $$

where $f_1, \ldots, f_m \in L^\infty_0(\mathbb{R}^n)$ (the space of compactly supported bounded functions). Here, the kernel $K$ is a function in $(\mathbb{R}^n)^{m+1}$ away from the diagonal $y_0 = y_1 = \cdots = y_m$ and satisfies the standard estimates

$$ |K(x, y_1, \ldots, y_m)| \leq A (|x-y_1| + \cdots + |x-y_m|)^{-mn}, $$

$$ |K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq \frac{A |x-x'|^p}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\varepsilon}} \quad (2) $$

provided that $|x-x'| \leq (1/2) \max(|x-y_1| + \cdots + |x-y_m|)$, and, for each $1 \leq i \leq m$,

$$ |K(x, y_1, \ldots, y_i, \ldots, y_m) - K(x, y_1, \ldots, y_i', \ldots, y_m)| \leq \frac{A |y_i - y_i'|^p}{(\sum_{j=1}^m |x-y_j|)^{mn+\varepsilon}} \quad (3) $$

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provided that $|x-x'| \leq (1/2) \max(|x-y_1| + \cdots + |x-y_m|)$, and, for each $1 \leq i \leq m$,

$$ |K(x, y_1, \ldots, y_i, \ldots, y_m) - K(x, y_1, \ldots, y_i', \ldots, y_m)| \leq \frac{A |y_i - y_i'|^p}{(\sum_{j=1}^m |x-y_j|)^{mn+\varepsilon}} \quad (3) $$
provided that \(|y_i - y'_i| \leq (1/2) \max|\{x - y_1| + \cdots + |x - y_m|\}|, \)
where \(A\) and \(e\) are positive constants.

Such kernels are called the \(m\)-linear Calderón-Zygmund kernels and the collection of such functions is denoted by \(m\)-CZK(A, e) in \([25]\).

Let \(T\) be as in (1) with an \(m\)-CZK(A, e) kernel. If, for some \(1 < q_1, q_2, \ldots, q_m < \infty, T\) is bounded from \(L^{q_1} \times L^{q_2} \times \cdots \times L^{q_m}\) to \(L^q\) with \(1/q_1 + 1/q_2 + \cdots + 1/q_m = 1/q\), then we say \(T\) is an \(m\)-linear Calderón-Zygmund operator. Grafakos and Torres in \([25]\) showed that if \(T\) is an \(m\)-linear Calderón-Zygmund operator, then \(T\) is bounded from \(L^{q_1} \times L^{q_2} \times \cdots \times L^{q_m}\) to \(L^q\) for any \(1 < q_1, q_2, \ldots, q_m < \infty\) such that \(1/q_1 + 1/q_2 + \cdots + 1/q_m = 1/q\). Then, Grafakos and Torres in \([26]\) obtained weighted norm inequalities for multilinear Calderón-Zygmund operators.

If \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) (the set of all complex-valued locally integrable functions on \(\mathbb{R}^n\)), set
\[
\|b\|_* := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx,
\]
where the supremum is taken over all balls in \(\mathbb{R}^n\), \(b_B\) is the mean of \(b\) on \(B\), and what follows \([E]\) is the Lebesgue measure of measurable set \(E\) in \(\mathbb{R}^n\). A function \(b\) is called bounded mean oscillation if \(\|b\|_* < \infty\) and \(\text{BMO}(\mathbb{R}^n)\) is the set of all locally integrable functions \(b\) on \(\mathbb{R}^n\) such that \(\|b\|_* < \infty\).

Let \(\mathring{b} = (b_1, \ldots, b_m)\) and \(b_j \in \text{BMO}(\mathbb{R}^n)\) for \(1 \leq j \leq m\).

We will consider the commutator \([\mathring{b}, T]\), which is defined for suitable functions \(f_1, \ldots, f_m\) by
\[
[\mathring{b}, T]f(x) := \sum_{\sigma} (-1)^{\sigma'} b_\sigma (x) T b_{\sigma'} f(x),
\]
where \(\sigma = (s_1, \ldots, s_m)\), \(\sigma'\) denotes a subset of \(\{1, 2, \ldots, m\}\), \(\sigma'\) denotes the complement of \(\sigma\) in \(\{1, 2, \ldots, m\}\), \(|\sigma'|\) denotes the number of elements of \(\sigma'\), \(b_\sigma (x) = \prod_{j \in \sigma} b_j (x), b_{\sigma'} = (g_1, \ldots, g_m), \) when \(j \notin \sigma', g_j = b_j f_j, \) otherwise, \(g_j = f_j\).

Definition 1. Let \(p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)\) be a measurable function.

(i) The Lebesgue space with variable exponent \(L^{p(\cdot)}(\mathbb{R}^n)\) is defined by
\[
L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \; \text{is measurable:} \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty \right\}
\]
for some \(\lambda > 0\).

(ii) The space \(L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)\) is defined by
\[
L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) := \left\{ f : f|_K \in L^{p(\cdot)}(\mathbb{R}^n) \; \text{for all compact subsets} \; K \subset \mathbb{R}^n \right\},
\]
where, and in what follows, \(\chi_S\) denotes the characteristic function of a measurable set \(S \subset \mathbb{R}^n\).

\(L^{p(\cdot)}(\mathbb{R}^n)\) is a Banach function space when equipped with the norm
\[
\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

Letting \(p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)\), we denote
\[
p_+ := \inf_{x \in \mathbb{R}^n} p(x), \quad p_+ := \inf_{x \in \mathbb{R}^n} p(x).
\]

The set \(\mathcal{P}(\mathbb{R}^n)\) consists of all \(p(\cdot)\) satisfying \(p_+ > 1\) and \(p_+ < \infty\); \(\mathcal{P}^0(\mathbb{R}^n)\) consists of all \(p(\cdot)\) satisfying \(p_+ > 0\) and \(p_+ < \infty\). \(L^{p(\cdot)}\) can be similarly defined as mentioned above for \(p(\cdot) \in \mathcal{P}(\mathbb{R}^n)\). \(p(\cdot)\) means the conjugate exponent of \(p(\cdot)\) that means \(1/p(\cdot) + 1/p'(\cdot) = 1\).

Let \(f \in L^{1}_{\text{loc}}(\mathbb{R}^n)\). Then the standard Hardy-Littlewood maximal function of \(f\) is defined by
\[
Mf (x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy, \quad \forall x \in \mathbb{R}^n,
\]
Definition 3 (see [29, Definition 2]). Let $q \in (0, \infty]$, $p(\cdot) \in P(R^n)$, and $\alpha(\cdot) : R^n \to R$ with $\alpha \in L^\infty(R^n)$. The homogeneous Herz space $K^\alpha_{q,p}(R^n)$ is defined as the set of all $f \in L^p_{loc}(R^n \setminus \{0\})$ such that
\[
\left\| f \right\|_{K^\alpha_{q,p}(R^n)} := \left\{ \sum_{k \in Z} \left\| 2^{k\alpha} f \chi_k \right\|_{p(\cdot)} \right\}^{1/q} < \infty, \tag{15}
\]
with the usual modifications when $q = \infty$.

Definition 4 (see [30, Definition 2]). Let $q < \infty$, $p(\cdot) \in P(R^n)$, $0 \leq \lambda < \infty$, and $\alpha(\cdot) : R^n \to R$ with $\alpha \in L^\infty(R^n)$. The Herz-Morrey space $MK^\alpha_{q,p}(R^n)$ with variable exponents is defined by
\[
MK^\alpha_{q,p}(R^n) := \left\{ f \in L^p_{loc}(R^n \setminus \{0\}) : \left\| f \right\|_{MK^\alpha_{q,p}(R^n)} < \infty \right\}, \tag{16}
\]
where
\[
\left\| f \right\|_{MK^\alpha_{q,p}(R^n)} := \sup_{L \in Z} 2^{-LA} \left( \sum_{k=-\infty}^L \left\| 2^{k\alpha} f \chi_k \right\|_{L^p(\cdot)(R^n)} \right)^{1/q} \tag{17}
\]
If $\alpha(\cdot)$ is a constant, then $MK^\alpha_{q,p}(R^n) = MK^\alpha_{\lambda}(R^n)$ was defined in [30]. If $\lambda = 0$, then $MK^\alpha_{q,p}(R^n) = MK^\alpha_{q,p}(R^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constant and $\lambda = 0$, then $MK^\alpha_{q,p}(R^n) = MK^\alpha_{\lambda}(R^n)$ is the classical Herz space in [31].

Lemma 5 (see [30, Lemma 1 and (10)]). If $p(\cdot) \in B(R^n)$, then there exist constants $\delta_1, \delta_2 \in (0, 1)$ and $C > 0$ such that, for all balls $B \in R^n$ and all measurable subsets $S \subset B$,
\[
\begin{align*}
\left\| x \right\|_{L^p(\cdot)(R^n)} &\leq C \left\| \frac{S}{|B|} \right\|_{\delta_1}, \tag{18} \\
\left\| x \right\|_{L^p(\cdot)(R^n)} &\leq C \left\| \frac{S}{|B|} \right\|_{\delta_2}.
\end{align*}
\]
There is a position to state our result.

Theorem 6. Let $b_1 \in BMO(R^n)$ and $p(\cdot), p_i(\cdot) \in B(R^n)$ satisfy $1/p(x) = 1/p_1(x) + 1/p_2(x) + \cdots + 1/p_m(x)$, $1 < p_i \leq p_i^* < n/A_i$, $(p_i(\cdot)/p_i^*)^i \in B(R^n)$ for some $0 < p_* < p_\ast$, $i = 1, 2, \ldots, m$. Let $\lambda_i < q_i < \infty$, $0 \leq \lambda_i < \infty$ and $\alpha(\cdot) \in L^\infty(R^n) \cap \mathcal{S}_{0}^\log\infty(R^n) \cap \mathcal{S}_{\infty}^\log\infty(R^n)$ for $i = 1, 2, \ldots, m$ with
\[
\lambda_j - n\delta_1 < \alpha^i_j < \alpha^i_+ < n\delta_2, \tag{19}
\]
where $\delta_1, \delta_2 \in (0, 1)$ are the constants appearing in (18). Suppose that $\lambda = \sum_{i=1}^m \lambda_i$, $\alpha = \sum_{i=1}^m \alpha_i$, and $1/q = \sum_{i=1}^m (1/q_i)$. Then
\[
\left\| f \right\|_{MK^\alpha_{q,p}(R^n)} \leq C \left\| b \right\| \left\| f \right\|_{MK^\alpha_{q,p}(R^n)} \tag{20}
\]
with the constant $C > 0$ independent of $\tilde{f} = (f_1, f_2, \ldots, f_m)$.

Remark 7. Let $\lambda_j = 0$, and then the commutator $[\tilde{b}, T]$ is bounded from the product of variable exponents Herz spaces $K^\alpha_{q_1,p_1}(R^n) \times K^\alpha_{q_2,p_2}(R^n) \times \cdots \times K^\alpha_{q_m,p_m}(R^n)$ to variable exponents Herz space $K^\alpha_{q_1,p_1}(R^n)$ when $-n\delta_1 < \alpha^i_+ < n\delta_2$.

Finally, we point out that $C$ denotes a positive constant which may be different at different occurrences.

2. Proof of the Main Result

To give our proof, we need some lemmas.

Lemma 8 (see [32, Proposition 2]). If $\alpha(\cdot) \in L^\infty(R^n) \cap \mathcal{S}_{0}^\log\infty(R^n) \cap \mathcal{S}_{\infty}^\log\infty(R^n)$, $0 \leq \lambda < \infty$, $p(\cdot) \in B(R^n)$, and $q \in (0, \infty)$, then
\[
\left\| f \right\|_{MK^\alpha_{q,p}(R^n)} \leq C \left\{ \sup_{L \in Z} 2^{-LA} \left( \sum_{k=-\infty}^L \left\| 2^{k\alpha} f \chi_k \right\|_{L^p(\cdot)(R^n)} \right)^{1/q} \right\}, \tag{21}
\]
where $C$ is a positive constant.

Lemma 9 (see [30, Lemma 2]). If $p(\cdot) \in B(R^n)$, then there exists a constant $C > 0$ such that, for all balls $B \in R^n$,
\[
C^{-1} \leq \left\| \frac{X_B}{X_{L^p(\cdot)(R^n)}} \right\|_{\lambda} \leq C. \tag{22}
\]

Lemma 10 (see [29, Lemma 3]). Let $k$ be a positive integer. Then one has that, for all $b \in BMO(R^n)$ and all $i, j \in Z$ with $j > i$,
\[
C^{-1} \left\{ \left\| b \right\| \right\}^k \leq \left\{ \frac{1}{R_{ball}} \left\| \frac{X_B}{X_{L^p(\cdot)(R^n)}} \right\| \left( \left\| b - b_B \right\| X_B \right) \right\} \leq C \left\{ \left\| b \right\| \right\}^k \tag{23}
\]
where $\left\| b \right\|$ is the $k$-th norm of $b$.

In fact, if $k = 1$, then, from inequalities (23), for all balls $B$ and $i, j \in Z$ with $j > i$, we have
\[
\left\| \left( b - b_B \right) X_B \right\|_{L^p(\cdot)(R^n)} \leq C \left\| b \right\| \left\| X_B \right\|_{L^p(\cdot)(R^n)} \tag{24}
\]
and
\[
\left\| \left( b - b_B \right) X_B \right\|_{L^p(\cdot)(R^n)} \leq C \left\| b \right\| \left\| X_B \right\|_{L^p(\cdot)(R^n)} \tag{25}
\]
where $\left\| b \right\|$ is the $k$-th norm of $b$.
Lemma 12 (see [14, Theorem 2.3]). Let $p, p_1, p_2 \in \mathcal{P}(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$. Then there exists a constant $C_{p,p_1}$ independent of the functions $f$ and $g$ such that

$$\|fg\|_{L^p(\mathbb{R}^n)} \leq C_{p,p_1} \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}$$

(26)

holds for every $f \in L^{p_1}(\mathbb{R}^n)$ and $g \in L^{p_2}(\mathbb{R}^n)$.

Lemma 13 (see [14, Corollary 2.2]). Let $T$ be a 2-linear Calderón-Zygmund operator and let $b_1$ and $b_2$ be BMO functions. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that there exists $0 < p_0 < p_\ast$ with $(p(\cdot)/p_\ast)^{\prime} \in \mathcal{B}(\mathbb{R}^n)$. If $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$, then there exists a constant $C$ independent of functions $f_i \in L^{p_i}(\mathbb{R}^n)$ for $i = 1, 2$ such that

$$\|b_1, b_2, T\|_{L^{p_1}(\mathbb{R}^n)} \leq C \|b_1\|_{L^{p_0}(\mathbb{R}^n)} \|b_2\|_{L^{p_0}(\mathbb{R}^n)} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_1}(\mathbb{R}^n)}$$

(27)

Proof of Theorem 6. Although our method is suitable for any multilinear operator, for simplicity, we only consider 2-linear operators. Let $b_1$ and $b_2$ be BMO functions. Since the set of all bounded compactly supported functions is dense in Herz-Morrey spaces with variable exponents, we let $f_1$ and $f_2$ be bounded compactly supported functions; then, for $x \in \mathbb{R}^n$, we write

$$[b_1, b_2, T](f_1, f_2)(x) = b_1(x)b_2(x)T(f_1, f_2)(x) - b_1(x)T(b_1f_1, f_2)(x)$$

(28)

$$- b_2(x)T(b_1f_1, b_2f_2)(x).$$

We write

$$f_i(x) = \sum_{i=\infty}^{\infty} f_i(x) \chi_i(x) =: \sum_{i=\infty}^{\infty} f_i(x), \quad i = 1, 2.$$  

(29)

Then, for each $k \in \mathbb{Z}$, if $l_i \leq k - 1$, from Lemma 11, we obtain

$$\int_{\mathbb{R}^n} [b_1(x) - b_1(y)] f_i(y) \, dy$$

$$\leq C \int_{\mathbb{R}^n} [b_1(x) - b_1(y)] f_i(y) \, dy$$

$$+ \int_{\mathbb{R}^n} [b_2 - b_1(y)] f_i(y) \, dy$$

$$\leq C \|f_i\|_{L^{p_1}(\mathbb{R}^n)} \|b_1 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

(30)

and by using inequalities (18), (22), and (24), it follows that

$$\int_{\mathbb{R}^n} [b_1(x) - b_1(y)] f_i(y) \, dy \chi_i \leq C \|f_i\|_{L^{p_1}(\mathbb{R}^n)} \|b_1 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$+ \|b_2 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$\leq C \|f_i\|_{L^{p_1}(\mathbb{R}^n)} \|b_1 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$+ \|b_2 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

(31)

Similarly, if $l_i \geq k + 1$, we have

$$\int_{\mathbb{R}^n} [b_1(x) - b_1(y)] f_i(y) \, dy \chi_i$$

$$\leq C \|f_i\|_{L^{p_1}(\mathbb{R}^n)} \|b_1 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$+ \|b_2 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$\leq C \|b_i\|_{L^{p_1}(\mathbb{R}^n)} \|f_i\|_{L^{p_1}(\mathbb{R}^n)} (l_i - k) \|b_1\|_{L^{p_1}(\mathbb{R}^n)} \|b_2 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

(32)

As for the case $l_i = k$, we get

$$\int_{\mathbb{R}^n} [b_1(x) - b_1(y)] f_i(y) \, dy \chi_i \leq C \|f_i\|_{L^{p_1}(\mathbb{R}^n)} \|b_1 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$+ \|b_2 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$\leq C \|f_i\|_{L^{p_1}(\mathbb{R}^n)} \|b_1 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

$$+ \|b_2 - b_1\|_{L^{p_1}(\mathbb{R}^n)} \|\chi_i\|_{L^{p_1}(\mathbb{R}^n)}$$

(33)
where
\[ E := \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-\lambda \cdot L} \]
\[ \times \left( \sum_{k=-\infty}^{L} \sum_{l_1=-\infty}^{L} \sum_{l_2=-\infty}^{L} [b_1, b_2, T] \right) \]
\[ \times (f_1, f_2) \chi_k \left\| \right\|_{L^1(\mathbb{R}^n)}^{1/q} \]
\[ F := \sup_{L > 0, L \in \mathbb{Z}} 2^{-\lambda \cdot L} \left( \sum_{k=-\infty}^{L} \sum_{l_1=-\infty}^{L} \sum_{l_2=-\infty}^{L} [b_1, b_2, T] (f_1, f_2) \chi_k \left\| \right\|_{L^1(\mathbb{R}^n)}^{1/q} \right) \]
\[ + 2^{-\lambda \cdot L} \left( \sum_{k=0}^{L} \sum_{l_1=-\infty}^{L} \sum_{l_2=-\infty}^{L} [b_1, b_2, T] (f_1, f_2) \chi_k \left\| \right\|_{L^1(\mathbb{R}^n)}^{1/q} \right) \]
\[ (35) \]

Since the estimate of $F$ is essentially similar to that of $E$, so it suffices to prove that $E$ is bounded in Herz-Morrey spaces with variable exponents. It is easy to see that
\[ E \leq C \sum_{i=1}^{9} I_i \]
\[ (36) \]

where
\[ I_1 := \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-\lambda \cdot L} \]
\[ \times \left( \sum_{k=-\infty}^{L} \sum_{l_1=-\infty}^{L} \sum_{l_2=-\infty}^{L} [b_1, b_2, T] \right) \]
\[ \times (f_1, f_2) \chi_k \left\| \right\|_{L^1(\mathbb{R}^n)}^{1/q} \]
\[
I_8 := \sup_{L \leq 0, L \in Z} 2^{-LA} \left( \sum_{k=0}^{l} 2^{ka(0)q} \left( \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} [b_1, b_2, T] \right) \right) \frac{1}{q_i} \times (f_{i_1}, f_{i_2}) \chi_k \left\| \right\|_{L^{p_i}(\mathbb{R}^n)}^{1/q_i},
\]

\[
I_9 := \sup_{L \leq 0, L \in Z} 2^{-LA} \left( \sum_{k=0}^{l} 2^{ka(0)q} \left( \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} [b_1, b_2, T] \right) \right) \frac{1}{q_i} \times (f_{i_1}, f_{i_2}) \chi_k \left\| \right\|_{L^{p_i}(\mathbb{R}^n)}^{1/q_i}.
\]

Using the symmetry of \( f_{i_1} \) and \( f_{i_2} \), we only need to estimate \( I_1, I_2, I_3, I_4, I_5, I_6 \), because the estimates of \( I_2, I_3, \) and \( I_4 \) are analogous to those of \( I_5, I_7, \) and \( I_8 \), respectively. In what follows, we divide it into 6 steps.

**Step 1.** To estimate the term of \( I_1 \), we note that \( l_i \leq k - 2 \) for \( i = 1, 2 \). Thus, for \( x \in D_k, y_i \in D_{l_i} \),

\[
|x - y_i| \geq |x| - |y_i| > 2^{k-1} - 2^l \geq 2^{-2}. \tag{38}
\]

Then, for \( x \in D_k \), we get

\[
|K(x, y_1, y_2)| \leq C|x - y_1| + |x - y_2| \leq 2^{2k}. \tag{39}
\]

Therefore,

\[
[b_1, b_2, T] (f_{i_1}, f_{i_2}) (x) \leq C 2^{-2k} \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i. \tag{40}
\]

By Lemma 12 and inequality (31), we obtain

\[
\left\| \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} [b_1, b_2, T] (f_{i_1}, f_{i_2}) \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)} \leq C 2^{-2k} \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i \chi_k \leq C 2^{-2k} \left\| \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} [b_1, b_2, T] (f_{i_1}, f_{i_2}) \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)}
\]

\[
\leq C 2^{-2k} \left\| \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)} \leq C 2^{-2k} \left\| \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)} \leq C 2^{-2k} \left\| \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)}.
\]

Since \( 1/q = 1/q_1 + 1/q_2 \), it follows that

\[
I_1 \leq C \sup_{L \leq 0, L \in Z} 2^{-LA} \left( \sum_{k=0}^{l} 2^{ka(0)q} \left[ \prod_{i=1}^{2} \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} (k - l_i) 2^{(l_i-k)\mu_i} \right] \right) \frac{1}{q_i} \times \left\| f_{i_1} \right\|_{L^{p_i}(\mathbb{R}^n)}^{1/q_i} \]

\[
\leq C \sup_{L \leq 0, L \in Z} \left( \sum_{k=0}^{l} 2^{ka(0)q} \left[ \prod_{i=1}^{2} \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} (k - l_i) 2^{(l_i-k)\mu_i} \right] \right) \frac{1}{q_i} \times \left\| f_{i_1} \right\|_{L^{p_i}(\mathbb{R}^n)}^{1/q_i}
\]

\[
= C \sup_{L \leq 0, L \in Z} \left( I_{11} (L) I_{12} (L) \right), \tag{42}
\]

where

\[
I_{11} (L) := 2^{-LA} \left( \sum_{k=0}^{l} 2^{ka(0)q} \left[ \prod_{i=1}^{2} \sum_{l_1+k+2 \leq l_2=\infty}^{\infty} (k - l_i) 2^{(l_i-k)\mu_i} \right] \right) \frac{1}{q_i} \times \left\| f_{i_1} \right\|_{L^{p_i}(\mathbb{R}^n)}^{1/q_i}
\]

\[
\leq C 2^{-2k} \left\| \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)}
\]

\[
\leq C 2^{-2k} \left\| \sum_{i=1}^{2} \int_{\mathbb{R}^n} [b_i (x) - b_i (y_i)] f_{i_1} (y_i) dy_i \chi_k \right\|_{L^{p_i}(\mathbb{R}^n)} \cdot \left\| f_{i_1} \right\|_{L^{p_i}(\mathbb{R}^n)}^{1/q_i}\].

(41)
Thus, for any $0 < q_i < \infty$,

$$I_1 \leq C \sup_{L \leq 0, L \in Z} I_{11}(L) I_{12}(L) \leq C \|b_1\|_\ast \|b_2\|_\ast \|f_1\|_{L^{p_1}((R^n)')} \|f_2\|_{\mathcal{M}^w_{q_1,p_1}((R^n)')}.$$  

(47)

Step 2. To estimate $I_2$, for $x \in D_{k_i}, y_1 \in D_{l_i}, i = 1, 2$ and $l_1 \leq k - 2, k - 1 \leq l_2 \leq k + 1$, we have

$$|x - y_2| \geq |x - y_1| \geq |x| - |y_1| > 2^{k-2}.$$  

(48)

Since $l_1 \leq k - 2$, it follows from inequality (31) that

$$\left\| \sum_{k=1}^{k+1} \sum_{l_2=k-1}^{k} [b_1(b) - b_1(y_1)] f_1(y_1) dy_1 \right\|_{L^{p_1}((R^n)')} \leq C \sum_{l_2=k-1}^{k} \sum_{l_1}^{k} \sum_{l_2=k-1}^{k} \sum_{l_1}^{k} 2^{(k-l_1)nd} \|b_1\|_\ast \|f_1\|_{L^{p_1}((R^n)')} \|f_2\|_{L^{p_1}((R^n)')}.$$  

(49)

For $k - 1 \leq l_2 \leq k + 1$, combining the above term and

$$I_2 \leq C \sup_{L \leq 0, L \in Z} 2^{-L1_i} \|b_1\|_\ast \|b_2\|_\ast \|f_1\|_{L^{p_1}((R^n)')} \|f_2\|_{L^{p_1}((R^n)')}.$$  

(46)
\[ \times 2^{-L1}\|b_k\| \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_1=0}^{k+1} 2^{kn_0}2^{L-k+n} \right) \times \|f_2\|_{L^{p_2}(\mathbb{R}^n)} \right\}^{1/q_2} \]
\[ =: C \sup_{L \leq 0, L \in \mathbb{Z}} I_{21}(L) I_{22}(L). \quad (50) \]

Here, we used \(2^{-\delta} < 1\) and \(2^{(L-k)n(1-\delta_1)} < 2^{(L-k)n}\) in the first inequality. Obviously,

\[ I_{21}(L) = I_{11}(L) \leq C \|b_1\| \|f_1\|_{\text{MK}^{\alpha_1, \lambda_1}_{q_1, p_1}(\mathbb{R}^n)}. \quad (51) \]

Therefore, we only need to estimate \(I_{22}(L)\):

\[ I_{22}(L) \]
\[ = 2^{-L1}\|b_2\| \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_2=0}^{k+1} 2^{kn_0} \right) \times \left( \sum_{l_2=0}^{k+1} 2^{L-k+n} \right) \right\}^{1/q_2} \]
\[ \leq C \|b_2\| \times 2^{-L1} \left( \sum_{k=-\infty}^{L} \left( \sum_{l_1=0}^{k} 2^{kn_0} \right) \right) \times \left( \sum_{l_2=0}^{k+1} 2^{L-k+n} \right) \]
\[ \leq C \|b_2\| \times \|f_2\|_{\text{MK}^{\alpha_1, \lambda_1}_{q_1, p_1}(\mathbb{R}^n)}. \quad (52) \]

**Step 3.** To estimate \(I_{31}\), for \(x \in D_{1i}, y_i \in D_{1i}, i = 1, 2,\) and \(l_1 \leq k - 2, l_2 \geq k + 2,\) then we have

\[ |x - y_1| \geq |x| - |y_1| > 2^{k-2}, \quad |x - y_2| \geq |y_2| - |x| > 2^{k-2}. \quad (53) \]

Thus, for \(x \in C_1\), we get

\[ [b_1, b_2, T] (f_{1i}, f_{2i})(x) \]
\[ \leq C 2^{-kn} \int_{\mathbb{R}^n} [b_1 (x) - b_1 (y_1)] f_{1i} (y_1) dy_1 2^{-kn} \]
\[ \times \int_{\mathbb{R}^n} [b_2 (x) - b_2 (y_2)] f_{2i} (y_2) dy_2. \quad (54) \]

From inequalities (31) and (32), we obtain

\[ \|
\]
\[ \leq C \sum_{l_1=-\infty}^{k-2} (k-l_1) 2^{(l_1-k)n_0} \|b_1\| \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \]
\[ \times \sum_{l_2=k+2}^{\infty} (l_2 - k) 2^{(k-l_2)n_0} \|b_2\| \|f_2\|_{L^{p_2}(\mathbb{R}^n)}. \]

Consequently, it follows that

\[ I_3 \leq C \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L1} \|b_1\| \|b_2\|. \]
\[ \]
\[ \times \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_1=-\infty}^{k-2} (k-l_1) 2^{(l_1-k)n_0} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \right) \right\}^{q_1} \]
\[ \times \left\{ \sum_{l_2=k+2}^{\infty} (l_2 - k) 2^{(k-l_2)n_0} \|f_2\|_{L^{p_2}(\mathbb{R}^n)} \right\}^{q_2} \]
\[ \times 2^{-L1} \|b_2\|. \]
\[ \times \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_1=-\infty}^{k-2} (k-l_1) 2^{(l_1-k)n_0} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \right) \right\}^{q_1} \]
\[ \times \left\{ \sum_{l_2=k+2}^{\infty} (l_2 - k) 2^{(k-l_2)n_0} \|f_2\|_{L^{p_2}(\mathbb{R}^n)} \right\}^{q_2} \]
\[ =: C \sup_{L \leq 0, L \in \mathbb{Z}} I_{31}(L) I_{32}(L). \quad (56) \]

Note that

\[ I_{31}(L) = I_{21}(L) \leq C \|b_1\| \|f_1\|_{\text{MK}^{\alpha_1, \lambda_1}_{q_1, p_1}(\mathbb{R}^n)}. \quad (57) \]
so we only compute $I_{32}(L)$. From inequality (19) and $\alpha_1 + \alpha_2(0) - \lambda_2 > 0$, we obtain

$$I_{32}(L) = 2^{-L \lambda_2} \|b_2\|_* \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_2=k+2}^{\infty} (l_2 - k) 2^{(k-l_2)(\alpha_1 + \alpha_2(0))} \right) \frac{q_1}{q_2} \right\}^{1/q_2} \times \left\{ \sum_{l_2=k-1}^{k+1} \frac{\|f_2\|_{L^p(\mathbb{R}^n)}}{\|b_2\|_*} \right\}^{1/q_2}

\leq C 2^{-L \lambda_2} \|b_2\|_* \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_2=k+2}^{\infty} (l_2 - k) 2^{(k-l_2)(\alpha_1 + \alpha_2(0)-\lambda_2)} \right) \frac{q_1}{q_2} \right\}^{1/q_2} \times \left\{ \sum_{l_2=k-1}^{k+1} \frac{\|f_2\|_{L^p(\mathbb{R}^n)}}{\|b_2\|_*} \right\}^{1/q_2}

\leq C 2^{-L \lambda_2} \|b_2\|_* \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_2=k+2}^{\infty} 2^{(l_2-k)\alpha_1} \right) \frac{q_1}{q_2} \right\}^{1/q_2} \times \left\{ \sum_{l_2=k-1}^{k+1} \frac{\|f_2\|_{L^p(\mathbb{R}^n)}}{\|b_2\|_*} \right\}^{1/q_2}

\leq C \|b_2\|_* 2^{-L \lambda_1} \left( \sum_{k=-\infty}^{L} 2^{k \lambda_2} \right) \frac{q_1}{q_2} \times \left\{ \sum_{l_2=k-1}^{k+1} \frac{\|f_2\|_{L^p(\mathbb{R}^n)}}{\|b_2\|_*} \right\}^{1/q_2}

\leq C \|b_2\|_* \|f_2\|_{\mathcal{M}^{(k+1)\lambda_2}_{q_2,p_2}([\mathbb{R}^n])}

(58)

Therefore,

$$I_3 \leq C \sup_{L \in \mathbb{N}} I_{31}(L) I_{32}(L)

\leq C \|b_1\|_* \|b_2\|_* \|f_1\|_{\mathcal{M}^{\alpha_1+\alpha_2}_{q_1,p_1}([\mathbb{R}^n])} \|f_2\|_{\mathcal{M}^{(k+1)\lambda_2}_{q_2,p_2}([\mathbb{R}^n])}

(59)

Step 4. It turns to estimate the term $I_5$. Applying the Hölder inequality and Lemma 13, we have

$$I_5 \leq C \sup_{L \in \mathbb{N}} 2^{-L \lambda_1} \left\{ \sum_{k=-\infty}^{L} 2^{k \alpha_1} \left( \sum_{l_1=k+1}^{L} \sum_{l_2=k+2}^{L} \|b_1, b_2, T\| \right) \right\}^{1/q_2} \times \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|f_1, f_2\|_{L^p(\mathbb{R}^n)} \right\}^{1/q_2}

\leq C \sup_{L \in \mathbb{N}} 2^{-L \lambda_1} \left\{ \sum_{k=-\infty}^{L} 2^{k \alpha_1} \left( \sum_{l_1=k+1}^{L} \sum_{l_2=k+2}^{L} \|b_1, b_2, T\| \right) \right\}^{1/q_2} \times \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|f_1, f_2\|_{L^p(\mathbb{R}^n)} \right\}^{1/q_2}

\leq C \|b_1\|_* \|b_2\|_* \left\{ \sum_{k=-\infty}^{L} 2^{k \alpha_1} \left( \sum_{l_1=k+1}^{L} \sum_{l_2=k+2}^{L} \|b_1, b_2, T\| \right) \right\}^{1/q_2} \times \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|f_1, f_2\|_{L^p(\mathbb{R}^n)} \right\}^{1/q_2}

(60)

Step 5. Now it goes to the estimate of $I_6$.

It is clear that, for $x \in D_k$ and $k-1 \leq l_1 \leq k+1$ and $l_2 \geq k+2$,

$$\|b_1, b_2, T\| (f_1, f_2) (x)

\leq C 2^{-k \alpha_1} \int_{\mathbb{R}^n} [b_1 (x) - b_1 (y_1)] f_1 (y_1) d y_1 2^{-l_2 \alpha_2} \int_{\mathbb{R}^n} [b_2 (x) - b_2 (y_2)] f_2 (y_2) d y_2.

(61)

Therefore,

$$\left\| \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} [b_1, b_2, T] (f_1, f_2) \chi_k \right\|_{L^{p_1}(\mathbb{R}^n)}$$

\leq C \|b_1\|_* \|b_2\|_* \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|b_1, b_2, T\| \right\}^{1/q_2} \times \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|f_1, f_2\|_{L^p(\mathbb{R}^n)} \right\}^{1/q_2}

\leq C \|b_1\|_* \|b_2\|_* \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|b_1, b_2, T\| \right\}^{1/q_2} \times \left\{ \sum_{l_1=k-1}^{k+1} \sum_{l_2=k+2}^{L} \|f_1, f_2\|_{L^p(\mathbb{R}^n)} \right\}^{1/q_2} \times (l_2 - k) 2^{(l_2-k) \delta_1} \|b_2\|_* \|f_2\|_{L^p(\mathbb{R}^n)}\]
\[ I_6 \leq C \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-LA} \times \left\{ \sum_{k=-\infty}^{L} 2^{kn(0)} \left( \sum_{l_k=k-1}^{k+1} \sum_{l_{k+2}=k+2}^{\infty} \left\| b_1, b_2, T \right\| \right) \| f_{i_1} \|_{L^{p_1}(\mathbb{R}^n)} \times (f_{i_2}, f_{i_3}) \|_{L^q(\mathbb{R}^n)} \right\}^{1/q} \]

\[ \leq C \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-LA} \| b_1 \|_* \| b_2 \|_* \]

\[ \times \left\{ \sum_{k=-\infty}^{L} 2^{kn(0)} \left( \sum_{l_k=k-1}^{k+1} \sum_{l_{k+2}=k+2}^{\infty} 2^{(l_k-k)n} \| f_{i_1} \|_{L^{p_1}(\mathbb{R}^n)} \times (l_k-k) 2^{(l_k-k)n} \| f_{i_2} \|_{L^{p_1}(\mathbb{R}^n)} \right) \right\}^{1/q} \]

\[ \leq C \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-LA} \| b_1 \|_* \]

\[ \times \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_k=k-1}^{k+1} \sum_{l_{k+2}=k+2}^{\infty} (l_k-k) 2^{(l_k-k)n} \| f_{i_1} \|_{L^{p_1}(\mathbb{R}^n)} \times 2^{(l_k-k)n} \| f_{i_2} \|_{L^{p_1}(\mathbb{R}^n)} \right) \right\}^{1/q} \]

\[ \leq C \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{i=1}^{2} 2^{-LA} \| b_2 \|_* \]

\[ \times \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_k=k-1}^{k+1} \sum_{l_{k+2}=k+2}^{\infty} (l_k-k) 2^{(l_k-k)n} \| f_{i_1} \|_{L^{p_1}(\mathbb{R}^n)} \times 2^{(l_k-k)n} \| f_{i_2} \|_{L^{p_1}(\mathbb{R}^n)} \right) \right\}^{1/q} \]

Thus,

\[ I_9 \leq C \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-LA} \times \left\{ \sum_{k=-\infty}^{L} 2^{kn(0)} \left( \sum_{l_k=k-1}^{k+1} \sum_{l_{k+2}=k+2}^{\infty} (l_k-k) 2^{(l_k-k)n} \| f_{i_1} \|_{L^{p_1}(\mathbb{R}^n)} \times (f_{i_2}, f_{i_3}) \|_{L^q(\mathbb{R}^n)} \right) \right\}^{1/q} \]

\[ \leq C \sup_{L \leq 0, L \in \mathbb{Z}} \prod_{i=1}^{2} 2^{-LA} \| b_2 \|_* \]

\[ \times \left\{ \sum_{k=-\infty}^{L} \left( \sum_{l_k=k-1}^{k+1} \sum_{l_{k+2}=k+2}^{\infty} (l_k-k) 2^{(l_k-k)n} \| f_{i_1} \|_{L^{p_1}(\mathbb{R}^n)} \times 2^{(l_k-k)n} \| f_{i_2} \|_{L^{p_1}(\mathbb{R}^n)} \right) \right\}^{1/q} \]

\[ \leq C \sup_{L \leq 0, L \in \mathbb{Z}} \| b_1 \|_* \| b_2 \|_* \| f_{i_1} \|_{\text{MK}_{\alpha^2(\cdot),\lambda^2}^{(1,1)}(\mathbb{R}^n)} \times \| f_{i_2} \|_{\text{MK}_{\alpha^2(\cdot),\lambda^2}^{(1,1)}(\mathbb{R}^n)} \]

\[ \leq C \sup_{L \leq 0, L \in \mathbb{Z}} \| b_1 \|_* \| b_2 \|_* \| f_{i_1} \|_{\text{MK}_{\alpha^2(\cdot),\lambda^2}^{(1,1)}(\mathbb{R}^n)} \times \| f_{i_2} \|_{\text{MK}_{\alpha^2(\cdot),\lambda^2}^{(1,1)}(\mathbb{R}^n)} \]

Here, the estimate of \( I_{61} \) is similar to that of \( I_{62} \) and \( I_{62} (L) = I_{62} (L) \).

\[ \text{Step 6.} \] Finally, we will finish the estimation of the last term \( I_9 \).

Note that \( l_2 \geq k + 2 \) and \( |x - y| > 2^{l_2-2} \) for \( x \in D_k \), \( y_1 \in D_{l_1} \), \( i = 1, 2 \); we get

\[ [b_1, b_2, T] \left( f_{i_1}, f_{i_2} \right) (x) \leq C 2^{-ln} \int_{\mathbb{R}^n} \left[ b_1 (x) - b_1 (y_1) \right] f_{i_1} (y_1) dy_1 \cdot 2^{-ln} \]

\[ \times \int_{\mathbb{R}^n} \left[ b_2 (x) - b_2 (y_2) \right] f_{i_2} (y_2) dy_2. \]

Applying the Hölder inequality to the last integral, we obtain

\[ \left\| \sum_{l_1=k-1}^{k+2} \sum_{l_2=k+2}^{\infty} \left[ b_1, b_2, T \right] (f_{i_1}, f_{i_2}) \|_{L^{p_1}(\mathbb{R}^n)} \right\| \leq C \| b_1 \|_* \| b_2 \|_* \]

This finishes the proof of Theorem 6. \( \square \)
Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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