Research Article

Estimates of Intrinsic Square Functions on Generalized Weighted Morrey Spaces

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We prove the boundedness of the intrinsic functions on generalized weighted Morrey spaces $M^{p,\varphi}(w)$, including the strong type estimates and weak type estimates. Moreover, we define the k th-order commutators generated by $BMO(\mathbb{R}^n)$ functions and intrinsic functions, and obtain their strong type estimates on $M^{p,\varphi}(w)$. In some cases, we improve previous results.

1. Introduction

It is well known that the theory of Littlewood-Paley square functions plays an important role in harmonic analysis, such as in the study of Fourier multiplier and singular integral operators. About their detailed properties and applications, we refer the readers to [1–3].

For convenience, let us recall some definitions. Suppose $u(x, t) = (P_t * f)(x)$ is the Poisson integral of f , where $P_t(x)$ denotes the Poisson kernel in $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$. The Littlewood-Paley g -function and the Lusin area integral (square function) S_β are defined, respectively, by

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x, t)|^2 t dt \right)^{1/2}, \quad (1)$$

$$S_\beta(f)(x) = \left(\int \int_{\Gamma_\beta(x)} |\nabla u(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2},$$

where $\Gamma_\beta(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \beta t\}$ for any $\beta > 0$. If $\beta = 1$, set $S_\beta(f) = S(f)$. The corresponding g_λ^* -function $g_\lambda^*(f)$ is given by

$$g_\lambda^*(f)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |\nabla u(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}. \quad (2)$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be real and radial and have support contained in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \phi(x) dx = 0$. The continuous square functions $g_\psi(f)$ and $S_{\psi,\beta}(f)$ are also defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |(\psi_t * f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$S_{\psi,\beta}(f)(x) = \left(\int \int_{\Gamma_\beta(x)} |(\psi_t * f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad (3)$$

where ψ_t denotes the usual L^1 dilation of ψ : $\psi_t(y) = t^{-n} \psi(y/t)$.

Recently, Wilson [4] introduced a natural substitute for the above square functions, which he called the intrinsic square function. This function dominates pointwise all the above square functions and is independent of any particular kernel. At the same time, it is not essentially larger than any particular $S_{\psi,\beta}(f)$. For $0 < \alpha \leq 1$, let $\mathcal{E}_\alpha$ be the family of functions ϕ having their support in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \phi(x) dx = 0$, and for all x and y , $|\phi(x) - \phi(y)| \leq |x - y|^\alpha$. If $f \in L_{loc}^1(\mathbb{R}^n)$ and $(y, t) \in \mathbb{R}_+^{n+1}$, we define

$$A_\alpha(f)(y, t) \equiv \sup_{\phi \in \mathcal{E}_\alpha} |(\phi_t * f)(y)|, \quad (4)$$

where $\phi_t(y) = t^{-n}\phi(y/t)$. Then the intrinsic square of f (of order α) is defined by

$$G_{\alpha,\beta}(f)(x) = \left(\int \int_{\Gamma_\beta(x)} (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \quad (5)$$

Here and below, we drop the subscript β if $\beta = 1$. Although the function $G_{\alpha,\beta}(f)$ is depend of kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different $\beta \geq 1$:

$$G_{\alpha,\beta}(f)(x) \leq \beta^{3n/2+\alpha} G_\alpha(f)(x), \quad x \in \mathbb{R}^n. \quad (6)$$

See [4] for more details. The intrinsic Littlewood-Paley g -function and the intrinsic g_λ^* -function are defined by

$$\begin{aligned} g_\alpha(f)(x) &= \left(\int_0^\infty (A_\alpha(f)(y,t))^2 \frac{dt}{t} \right)^{1/2}, \\ g_{\lambda,\alpha}^*(f)(x) &= \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \right. \\ &\quad \left. \times (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \end{aligned} \quad (7)$$

respectively. In [4, 5], Wilson established the boundedness of intrinsic square functions on weighted Lebesgue spaces. Their boundedness on various function spaces and their sharp bounds have received great attentions; see [6–17].

The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ were introduced by Morrey [18] to study the local behavior of solutions to second-order elliptic partial differential equations. Mizuhara [19] introduced generalized Morrey spaces. Later, Guliyev defined the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ with normalized norm. Recently, Komori and Shirai [20] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators on these spaces. Guliyev et al. [21, 22] defined the generalized weighted Morrey spaces $M^{p,\varphi}(w)$ as follows.

Definition 1. Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times \mathbb{R}^+$ and let $w(x)$ be a weight function on $\mathbb{R}^n$. The generalized weighted Morrey space $M^{p,\varphi}(w)$ is the space of all functions $f \in L_{\text{loc}}^{p,w}(\mathbb{R}^n)$ with finite norm

$$\begin{aligned} \|f\|_{M^{p,\varphi}(w)} &= \sup_{\substack{x \in \mathbb{R}^n \\ r > 0}} \frac{1}{\varphi(x, r) w(B(x, r))^{1/p}} \\ &\quad \times \left(\int_{B(x, r)} |f(y)|^p w(y) dy \right)^{1/p}, \end{aligned} \quad (8)$$

where $w(B(x, r)) = \int_{B(x, r)} w(y) dy$. The weak generalized weighted Morrey space $WM^{p,\varphi}(w)$ consists of all functions $f \in WL_{\text{loc}}^{p,w}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{WM^{p,\varphi}(w)} = \sup_{\substack{x \in \mathbb{R}^n \\ r > 0}} \frac{1}{\varphi(x, r) w(B(x, r))^{1/p}} \|f\|_{WL_w^p(B(x, r))}, \quad (9)$$

where

$$\|f\|_{WL_w^p(B(x, r))} = \sup_{t > 0} t \left(\int_{\{y \in B(x, r) : |f(y)| > t\}} w(y) dy \right)^{1/p}. \quad (10)$$

If $w(x) \equiv 1$ and $\varphi(x, r) = r^{(\lambda-n)/p}$ with $0 < \lambda < n$, then $M^{p,\varphi}(w) = L^{p,\lambda}(\mathbb{R}^n)$; if $w(x) \equiv 1$, then $M^{p,\varphi}(1) = M^{p,\varphi}(\mathbb{R}^n)$; if $\varphi(x, r) = w(B(x, r))^{(\kappa-1)/p}$ for $0 < \kappa < 1$, then $M^{p,\varphi}(w) = L^{p,\kappa}(w)$. There are many papers that discussed the conditions on $\varphi(x, r)$ to obtain the boundedness of operators on the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$. For example, see [6, 23–29]. Recently, Guliyev introduced the generalized conditions:

$$\int_r^\infty \frac{\text{ess inf}_{t < s < \infty} \varphi_1(x, s) w(B(x, s))^{1/p}}{w(B(x, t))^{1/p}} \frac{dt}{t} \leq C \varphi_2(x, r), \quad (11)$$

$$\begin{aligned} \int_r^\infty \ln^k \left(e + \frac{t}{r} \right) \frac{\text{ess inf}_{t < s < \infty} \varphi_1(x, s) w(B(x, s))^{1/p}}{w(B(x, t))^{1/p}} \frac{dt}{t} \\ \leq C \varphi_2(x, r), \end{aligned} \quad (12)$$

where C does not depend on x and r . Under these conditions, the boundedness of some classical operators and commutators in generalized weighted Morrey spaces were obtained, respectively. See [8, 9, 21, 22, 30].

Recently, Wang [12, 16] studied the boundedness of the intrinsic functions and their commutators on weighted Morrey spaces. In [6, 17], the authors obtained the boundedness of the intrinsic functions and their commutators on generalized Morrey spaces. Guliyev et al. [7–9] obtained the estimates for vector-valued intrinsic square functions and their k th-order commutators on vector-valued generalized weighted Morrey spaces. In this paper, we prove the boundedness of the intrinsic functions and their k th-order commutators on generalized weighted Morrey spaces $M^{p,\varphi}(w)$ under the conditions (11), (12), and $w \in A_p$, respectively. Our partial results coincide with some results [7–9], but we improve known results in some case. See Remark 7.

The rest of the paper is organized as follows: some definitions and our main results are stated in Section 2. In Section 3, we give some lemmas. Finally, in Section 4, we prove our main theorem. Throughout this paper, $A \leq B$ means that there is a positive constant C independent of all essential variables such that $A \leq CB$. If $A \leq B$ and $B \leq A$, then we write $A \approx B$. We denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$ and $p' = \infty$ if $p = 1$.

2. Definitions and Main Results

In this paper, $B = B(x_0, r)$ denotes the ball with center x_0 and radius r . Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B , and $B^c = \mathbb{R}^n \setminus B$. A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E , we denote the characteristic function of E by χ_E , the Lebesgue measure of E by $|E|$ and the

weighted measure of E by $w(E)$, where $w(E) = \int_E w(x) dx$. Moreover, for a locally integrable function $b(x)$, we define

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y) w(y) dy. \quad (13)$$

For simplicity, we denote $b_{B(x,r),w}$ by $b_{B(x,r)}$ if $w = 1$.

Definition 2 (see [1]). A locally integrable function $b(x)$ is in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_{\substack{x \in \mathbb{R}^n \\ r > 0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty. \quad (14)$$

Definition 3 (see [3]). We say that a weight function $w \in A_p$, $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} &= \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \\ &\quad \times \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty, \end{aligned} \quad (15)$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. A weight function $w \in A_\infty$ if $w \in A_p$ for some $1 < p < \infty$. A weight function $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \cdot \operatorname{ess\,inf}_{x \in B} w(x). \quad (16)$$

Definition 4. Let b be a locally integrable function on $\mathbb{R}^n$ and

$$\begin{aligned} A_{\alpha,b}^k(f)(y,t) \\ \equiv \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)]^k \phi_t(y-z) f(z) dz \right|. \end{aligned} \quad (17)$$

The k th-order commutators $[b, G_\alpha]^k$ and $[b, g_\alpha]^k$ are defined by

$$\begin{aligned} [b, G_\alpha]^k(f)(x) &= \left(\int \int_{\Gamma(x)} (A_{\alpha,b}^k(f)(y,t))^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \\ [b, g_\alpha]^k(f)(x) &= \left(\int_0^\infty (A_{\alpha,b}^k(f)(y,t))^2 \frac{dt}{t} \right)^{1/2}, \end{aligned} \quad (18)$$

respectively. Analogously, the k th-order commutators $[b, g_{\lambda,\alpha}^*]^k$ are defined by

$$\begin{aligned} [b, g_{\lambda,\alpha}^*]^k(f)(x) &= \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \right. \\ &\quad \times (A_{\alpha,b}^k(f)(y,t))^2 \frac{dy dt}{t^{n+1}} \Big)^{1/2}. \end{aligned} \quad (19)$$

Theorem 5. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy condition (11). Then,

$$\begin{aligned} \|G_\alpha(f)\|_{M^{p,\varphi_2}(w)} &\leq \|f\|_{M^{p,\varphi_1}(w)}, \quad 1 < p < \infty, \\ \|G_\alpha(f)\|_{WM^{1,\varphi_2}(w)} &\leq \|f\|_{M^{1,\varphi_1}(w)}, \quad p = 1. \end{aligned} \quad (20)$$

Theorem 6. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfy condition (11). Then for $\lambda > 3 + 2\alpha/n$, we have

$$\begin{aligned} \|g_{\lambda,\alpha}^*(f)\|_{M^{p,\varphi_2}(w)} &\leq \|f\|_{M^{p,\varphi_1}(w)}, \quad 1 < p < \infty, \\ \|g_{\lambda,\alpha}^*(f)\|_{WM^{1,\varphi_2}(w)} &\leq \|f\|_{M^{1,\varphi_1}(w)}, \quad p = 1. \end{aligned} \quad (21)$$

Theorem 6'. Let $0 < \alpha \leq 1$, $1 < p < \infty$, $w \in A_p$ and let (φ_1, φ_2) satisfy condition (11). Then, for $\lambda > \max\{p, 3\}$, we have

$$\|g_{\lambda,\alpha}^*(f)\|_{M^{p,\varphi_2}(w)} \leq \|f\|_{M^{p,\varphi_1}(w)}. \quad (22)$$

Remark 7. If $p > 3 + 2\alpha/n$, comparing Theorem 6 with Theorem 6', we obtain that the conclusion of Theorem 6 is better. So in this sense, our Theorem 6 improves the result in [12]. If $1 < p \leq 3$, the conclusion of Theorem 6' is better than that of Theorem 6. Therefore, we improve the corresponding result in [6, 17]. Here, we point out that Theorem 6' and Theorem 9' can be extended to vector-valued cases; see [7–9].

Theorem 8. Let $0 < \alpha \leq 1$, $1 < p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfy condition (12). Let $b \in BMO(\mathbb{R}^n)$; then

$$\|[b, G_\alpha]^k(f)\|_{M^{p,\varphi_2}(w)} \leq \|b\|_*^k \|f\|_{M^{p,\varphi_1}(w)}. \quad (23)$$

Theorem 9. Let $0 < \alpha \leq 1$, $1 < p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfy condition (12). If $b \in BMO(\mathbb{R}^n)$ and $\lambda > 3 + 2\alpha/n$, then

$$\|[b, g_{\lambda,\alpha}^*]^k(f)\|_{M^{p,\varphi_2}(w)} \leq \|b\|_*^k \|f\|_{M^{p,\varphi_1}(w)}. \quad (24)$$

Theorem 9'. Let $0 < \alpha \leq 1$, $1 < p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfy condition (12). If $b \in BMO(\mathbb{R}^n)$ and $\lambda > \max\{p, 3\}$, then

$$\|[b, g_{\lambda,\alpha}^*]^k(f)\|_{M^{p,\varphi_2}(w)} \leq \|b\|_*^k \|f\|_{M^{p,\varphi_1}(w)}. \quad (25)$$

For $0 < \alpha \leq 1$, Wilson [4] showed that $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Therefore, by Theorems 5 and 8, we have the following.

Corollary 10. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfy condition (11); then

$$\begin{aligned} \|g_\alpha(f)\|_{M^{p,\varphi_2}(w)} &\leq \|f\|_{M^{p,\varphi_1}(w)}, \quad 1 < p < \infty, \\ \|g_\alpha(f)\|_{WM^{1,\varphi_2}(w)} &\leq \|f\|_{M^{1,\varphi_1}(w)}, \quad p = 1. \end{aligned} \quad (26)$$

Corollary 11. Let $0 < \alpha \leq 1$, $1 < p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfy condition (12). If $b \in BMO(\mathbb{R}^n)$, then

$$\| [b, g_\alpha]^k(f) \|_{M^{p, \varphi_2}(w)} \leq \| b \|_*^k \| f \|_{M^{p, \varphi_1}(w)}. \quad (27)$$

Remark 12. If $\varphi(x, r) = w(B(x, r))^{(\kappa-1)/p}$ for $0 < \kappa < 1$, then $M^{p, \varphi}(w) = L^{p, \kappa}(w)$. Let $w \in A_p$ ($1 \leq p < \infty$); the pair (φ_1, φ_2) satisfies the condition (12) with $k \geq 0$. See [30] for its proof. Therefore, Theorems 5, 6', 8, 9' and Corollaries 10 and 11 contain the results in [12, 16].

Remark 13. If $1 < p < \infty$, $w = 1$, and $k = 1$, then Theorems 5–9 and Corollaries 10 and 11 contain the results in [6, 17].

3. Lemmas

For any $x \in \mathbb{R}^n$ and $\beta \geq 1$, the following inequality proved in [4] holds:

$$G_{\alpha, \beta}(f)(x) \leq \beta^{3n/2+\alpha} G_\alpha(f)(x). \quad (28)$$

So we can readily obtain the following.

Lemma 14. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. If w is a weight, we have

$$\| G_{\alpha, 2^j}(f) \|_{L_w^p(\mathbb{R}^n)} \leq 2^{j(3n/2+\alpha)} \| G_\alpha(f) \|_{L_w^p(\mathbb{R}^n)}, \quad \forall j \in \mathbb{Z}^+. \quad (29)$$

By the similar argument as in [12], we can get the following.

Lemma 15. Let $0 < \alpha \leq 1$, $1 < p < \infty$, and $w \in A_p$. Then, the k th-order commutators $[b, G_\alpha]^k$ and $[b, g_{\lambda, \alpha}^*]^k$ are all bounded from $L_w^p(\mathbb{R}^n)$ to itself whenever $b \in BMO(\mathbb{R}^n)$.

In the following, we will give a lemma about the Hardy type operator:

$$H_k(f)(r) := \frac{1}{r} \int_0^r \ln^k \left(e + \frac{r}{t} \right) f(t) d\mu(t), \quad k \geq 0, \quad (30)$$

where μ is a nonnegative Borel measure on $(0, \infty)$; we have the following.

Lemma 16 (see [30]). *The inequality*

$$\operatorname{esssup}_{r>0} H_k(f)(r) W(r) \leq C \operatorname{esssup}_{r>0} f(r) V(r) \quad (31)$$

holds for all nonnegative and nonincreasing f on $(0, \infty)$ if and only if

$$A := \sup_{r>0} \frac{W(r)}{r} \int_0^r \ln^k \left(e + \frac{r}{t} \right) \frac{d\mu(t)}{\operatorname{esssup}_{0<s<t} V(s)} < \infty, \quad (32)$$

and $C \approx A$.

Note that if $k = 0$ and $d\mu(t) = dt$, Lemma 16 was proved in [31].

Lemma 17 (see [30]). Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Suppose $k > 0$ and $r_1, r_2 > 0$. Then,

(i) for $1 \leq p < \infty$, we have

$$\left(\frac{1}{w(B(x, r_1))} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^{kp} w(y) dy \right)^{1/p} \leq C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \| b \|_*^k, \quad (33)$$

where C is independent of b, w, x, r_1 , and r_2 .

(ii) for $1 < p < \infty$, we have

$$\left(\frac{1}{w^{1-p'}(B(x, r_1))} \times \int_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^{kp'} w(y)^{1-p'} dy \right)^{1/p'} \leq C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \| b \|_*^k, \quad (34)$$

where C is independent of b, w, x, r_1 , and r_2 .

Lemma 18 (see [12]). Let $0 < \alpha \leq 1$, $1 < p < \infty$, and $w \in A_p$. Then, for any $j \in \mathbb{Z}^+$, we have

$$\| G_{\alpha, 2^j}(f) \|_{L_w^p(\mathbb{R}^n)} \leq \begin{cases} 2^{jn} \| G_\alpha(f) \|_{L_w^p(\mathbb{R}^n)}, & 1 \leq p \leq 2; \\ 2^{jnp/2} \| G_\alpha(f) \|_{L_w^p(\mathbb{R}^n)}, & 2 < p < \infty. \end{cases} \quad (35)$$

4. Proofs of Main Theorems

Proof of Theorem 5. We will adopt the idea used in [27]. Fix a ball $B = B(x_0, r)$ and decompose $f = f_1 + f_2$, where $f_1 = f \chi_{2B}$ and $f_2 = f \chi_{(2B)^c}$. Then, for $1 < p < \infty$, we have

$$\| G_\alpha(f) \|_{L_w^p(B)} \leq \| G_\alpha(f_1) \|_{L_w^p(B)} + \| G_\alpha(f_2) \|_{L_w^p(B)}. \quad (36)$$

Since $w \in A_p$ for $1 < p < \infty$, we can obtain the following inequality from [5]

$$\| G_\alpha(f_1) \|_{L_w^p(B)} \leq \| f_1 \|_{L_w^p(\mathbb{R}^n)} = \| f \|_{L_w^p(2B)}. \quad (37)$$

For $\| f \|_{L_w^p(2B)}$ and $w \in A_p$ with $1 \leq p < \infty$, a calculation shows that

$$\| f \|_{L_w^p(2B)} \leq [w]_{A_p}^{1/p} w(B)^{1/p} \int_{2r}^\infty \frac{\| f \|_{L_w^p(B(x_0, t))} dt}{w(B(x_0, t))^{1/p} t}. \quad (38)$$

See [30] for its proof. Therefore, for $1 < p < \infty$, we get

$$\| G_\alpha(f_1) \|_{L_w^p(B)} \leq [w]_{A_p}^{1/p} w(B)^{1/p} \int_{2r}^\infty \frac{\| f \|_{L_w^p(B(x_0, t))} dt}{w(B(x_0, t))^{1/p} t}. \quad (39)$$

On the other hand, for $\phi \in \mathcal{E}_\alpha$ with $0 < \alpha \leq 1$, we have

$$|f_2 * \phi_t(y)| \leq t^{-n} \int_{|y-z| \leq t} |f_2(z)| dz. \quad (40)$$

Since $x \in B$, $z \in (2B)^c$, and $(y, t) \in \Gamma(x)$, we have $r \leq |z-x| \leq 2t$. So we obtain

$$\begin{aligned} G_\alpha(f_2)(x) &\leq \left(\int_{\Gamma(x)} \left| t^{-n} \int_{|y-z| \leq t} |f_2(z)| dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\ &\leq \left(\int_{t \geq r/2} \int_{|x-y| < t} \left(\int_{|z-x| \leq 2t} |f_2(z)| dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{1/2} \\ &\leq \left(\int_{t \geq \frac{r}{2}} \left(\int_{|z-x| \leq 2t} |f_2(z)| dz \right)^2 \frac{dt}{t^{2n+1}} \right)^{1/2}. \end{aligned} \quad (41)$$

By Minkowski's inequality and $|z-x| \geq (1/2)|z-x_0|$ for $x \in B$ and $z \in (2B)^c$, we have

$$\begin{aligned} G_\alpha(f_2)(x) &\leq \int_{\mathbb{R}^n} \left(\int_{t > |z-x|/2} \frac{dt}{t^{2n+1}} \right)^{1/2} |f_2(z)| dz \\ &\leq \int_{|z-x_0| \geq 2r} \frac{|f(z)| dz}{|z-x|^{2n}} \leq \int_{(2B)^c} \frac{|f(z)| dz}{|z-x_0|^{2n}} \quad (42) \\ &\approx \int_{(2B)^c} |f(z)| \int_{|z-x_0|}^{\infty} \frac{dt dz}{t^{n+1}}. \end{aligned}$$

It follows from Fubini's theorem, Hölder's inequality, and $w \in A_p$ with $1 \leq p < \infty$ that

$$\begin{aligned} G_\alpha(f_2)(x) &\leq \int_{2r}^{\infty} \int_{B(x_0, t)} \frac{|f(z)| dz dt}{t^{n+1}} \\ &\leq \int_{2r}^{\infty} \|f\|_{L_w^p(B(x_0, t))} \|w^{-1/p}\|_{L^{p'}(B(x_0, t))} \frac{dt}{t^{n+1}} \quad (43) \\ &\leq [w]_{A_p}^{1/p} \int_{2r}^{\infty} \frac{\|f\|_{L_w^p(B(x_0, t))}}{w(B(x_0, t))^{1/p}} \frac{dt}{t}. \end{aligned}$$

Thus, for $1 \leq p < \infty$, the following estimate is valid:

$$\|G_\alpha(f_2)\|_{L_w^p(B)} \leq [w]_{A_p}^{1/p} w(B)^{1/p} \int_{2r}^{\infty} \frac{\|f\|_{L_w^p(B(x_0, t))}}{w(B(x_0, t))^{1/p}} \frac{dt}{t}. \quad (44)$$

Combining the above estimates (36), (39), and (44), we have

$$\|G_\alpha(f)\|_{L_w^p(B)} \leq [w]_{A_p}^{1/p} w(B)^{1/p} \int_{2r}^{\infty} \frac{\|f\|_{L_w^p(B(x_0, t))}}{w(B(x_0, t))^{1/p}} \frac{dt}{t}. \quad (45)$$

Hence, applying the definition of $M^{p, \varphi}(w)$ and substitution of variables, we have

$$\begin{aligned} \|G_\alpha(f)\|_{M^{p, \varphi_2}(w)} &\leq \sup_{\substack{x_0 \in \mathbb{R}^n \\ r > 0}} \varphi_2(x_0, r^{-1})^{-1} \int_0^r \frac{\|f\|_{L_w^p(B(x_0, s^{-1}))}}{w(B(x_0, s^{-1}))^{1/p}} \frac{ds}{s}. \end{aligned} \quad (46)$$

Suppose $W(r) = \varphi_2(x_0, r^{-1})^{-1} r$ and $V(x) = \varphi_1(x_0, r^{-1})^{-1} w(B(x_0, r^{-1}))^{-1/p}$. Since (φ_1, φ_2) satisfies condition (11), we can verify that $W(r), V(r)$ satisfy condition (32). Obviously, $\|f\|_{L_w^p(B(x_0, s^{-1}))}$ is decreasing on variable s . So, by Lemma 16 with $k = 0$, we can conclude

$$\|G_\alpha(f)\|_{M^{p, \varphi_2}(w)} \leq \|f\|_{M^{p, \varphi_1}(w)}. \quad (47)$$

Let $p = 1$. From $w \in A_1$, [5], (38) and (44), it follows that

$$\begin{aligned} \|G_\alpha(f)\|_{WL_w^1(B)} &\leq \|G_\alpha(f_1)\|_{WL_w^1(B)} + \|G_\alpha(f_2)\|_{WL_w^1(B)} \\ &\leq \|f_1\|_{L_w^1(\mathbb{R}^n)} + \|G_\alpha(f_2)\|_{L_w^1(B)} \quad (48) \\ &\leq w(B) \int_{2r}^{\infty} \frac{\|f\|_{L_w^1(B(x_0, t))}}{w(B(x_0, t))} \frac{dt}{t}. \end{aligned}$$

By the definition of $WM_{1, \varphi}(w)$ and Lemma 16 with $k = 0$, we get

$$\begin{aligned} \|G_\alpha(f)\|_{WM_{1, \varphi_2}(w)} &\leq \sup_{\substack{x_0 \in \mathbb{R}^n \\ r > 0}} r \varphi_2(x_0, r^{-1})^{-1} \frac{1}{r} \int_0^r \frac{\|f\|_{L_w^1(B(x_0, s^{-1}))}}{w(B(x_0, s^{-1}))} \frac{ds}{s} \quad (49) \\ &\leq \|f\|_{M^{1, \varphi_1}(w)}. \end{aligned}$$

□

Proof of Theorem 6. By the definition of $g_{\lambda, \alpha}^*(f)$, we have

$$\begin{aligned} [g_{\lambda, \alpha}^*(f)(x)]^2 &= \int_0^\infty \int_{|x-y| < t} \left(\frac{t}{t + |x-y|} \right)^{n\lambda} \\ &\quad \times (A_\alpha f(y, t))^2 \frac{dydt}{t^{n+1}} \\ &\quad + \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t \leq |x-y| < 2^j t} \left(\frac{t}{t + |x-y|} \right)^{n\lambda} \\ &\quad \times (A_\alpha f(y, t))^2 \frac{dydt}{t^{n+1}} \\ &\leq (G_\alpha f(x))^2 + \sum_{j=1}^\infty 2^{-jn\lambda} (G_{\alpha, 2^j}(f)(x))^2. \end{aligned} \quad (50)$$

Thus,

$$\begin{aligned} \|g_{\lambda, \alpha}^*(f)\|_{M^{p, \varphi_2}(w)} &\leq \|G_\alpha(f)\|_{M^{p, \varphi_2}(w)} \\ &\quad + \sum_{j=1}^\infty 2^{-jn\lambda/2} \|G_{\alpha, 2^j}(f)\|_{M^{p, \varphi_2}(w)}. \end{aligned} \quad (51)$$

By Theorem 5, we have

$$\|G_\alpha(f)\|_{M^{p, \varphi_2}(w)} \leq \|f\|_{M^{p, \varphi_1}(w)}. \quad (52)$$

It remains to estimate $\|G_{\alpha,2^j}(f)\|_{M^{p,w_2}(\mathbb{R}^n)}$. To estimate it, we will divide $\|G_{\alpha,2^j}(f)\|_{L_w^p(B)}$ into two terms. As before,

$$\|G_{\alpha,2^j}(f)\|_{L_w^p(B)} \leq \|G_{\alpha,2^j}(f_1)\|_{L_w^p(B)} + \|G_{\alpha,2^j}(f_2)\|_{L_w^p(B)}, \quad (53)$$

where $f = f_1 + f_2$ and $f_1 = f\chi_{2B}$. For the first term, by Lemma 14, Theorem 5, and (38), we can easily deduce

$$\begin{aligned} \|G_{\alpha,2^j}(f_1)\|_{L_w^p(B)} &\leq 2^{j(3n/2+\alpha)} \|G_\alpha(f_1)\|_{L_w^p(\mathbb{R}^n)} \\ &\leq 2^{j(3n/2+\alpha)} \|f\|_{L_w^p(2B)} \\ &\leq 2^{j(3n/2+\alpha)} w(B)^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \end{aligned} \quad (54)$$

Now, we estimate the second term. For $\phi \in \mathcal{C}_\alpha$ with $0 < \alpha \leq 1$, we have

$$\begin{aligned} G_{\alpha,2^j}(f_2)(x) &\leq \left(\int_0^\infty \int_{|x-y|\leq 2^j t} \left(\int_{|z-y|\leq t} |f_2(z)| dz \right)^2 \frac{dy dt}{t^{3n+1}} \right)^{1/2} \quad (55) \end{aligned}$$

Notice that $|x-y| \leq 2^j t$ and $|z-y| \leq t$, so $|z-x| \leq 2^{j+1}t$. Thus,

$$\begin{aligned} G_{\alpha,2^j}(f_2)(x) &\leq \left(\int_0^\infty \int_{|x-y|\leq 2^j t} \left(\int_{|z-x|\leq 2^{j+1}t} |f_2(z)| dz \right)^2 \frac{dy dt}{t^{3n+1}} \right)^{1/2} \\ &\leq \left(\int_0^\infty \left(\int_{|z-x|\leq 2^{j+1}t} |f_2(z)| dz \right)^2 \frac{2^{jn} dt}{t^{2n+1}} \right)^{1/2} \\ &\leq 2^{jn/2} \int_{\mathbb{R}^n} \left(\int_{t\geq |z-x|/2^{j+1}} |f_2(z)|^2 \frac{dt}{t^{2n+1}} \right)^{1/2} dz \\ &\leq 2^{3jn/2} \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x|} dz. \end{aligned} \quad (56)$$

Because $|z-x| \geq (1/2)|z-x_0|$ for $x \in B = B(x_0, r)$ and $|z-x_0| > 2r$, by Fubini's theorem and Hölder's inequality,

$$\begin{aligned} G_{\alpha,2^j}(f_2)(x) &\leq 2^{3jn/2} \int_{|z-x_0|>2r} \frac{|f(z)| dz}{|z-x_0|^n} \\ &\approx 2^{3jn/2} \int_{|z-x_0|>2r} |f(z)| \int_{|z-x_0|}^\infty \frac{dt dz}{t^{n+1}} \\ &\leq 2^{3jn/2} \int_{2r}^\infty \int_{B(x_0,t)} \frac{|f(z)| dz dt}{t^{n+1}} \\ &\leq 2^{3jn/2} [w]_{A_p}^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \end{aligned} \quad (57)$$

So

$$\|G_{\alpha,2^j}(f_2)\|_{L_w^p(B)} \leq 2^{3jn/2} w(B)^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \quad (58)$$

Combining (53), (54), and (58), we have

$$\|G_{\alpha,2^j}(f)\|_{L_w^p(B)} \leq 2^{j(3n/2+\alpha)} w(B)^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \quad (59)$$

Thus, by changing of variables and Lemma 16 with $k = 0$, we get

$$\begin{aligned} \|G_{\alpha,2^j}(f)\|_{M^{p,\varphi_2}(w)} &\leq 2^{j(3n/2+\alpha)} \sup_{\substack{x_0 \in \mathbb{R}^n \\ r>0}} \varphi_2(B(x_0, r))^{-1} \\ &\quad \times \int_0^r \frac{\|f\|_{L_w^p(B(x_0,t^{-1}))}}{w(B(x_0,t^{-1}))^{1/p}} \frac{dt}{t} \\ &\leq 2^{j(3n/2+\alpha)} \|f\|_{M^{p,\varphi_1}(w)}. \end{aligned} \quad (60)$$

Therefore, using (51), (52), and (60), it follows that

$$\begin{aligned} \|\mathcal{G}_{\lambda,\alpha}^*(f)\|_{M^{p,\varphi_2}(w)} &\leq \|f\|_{M^{p,\varphi_1}(w)} \left(1 + \sum_{j=1}^\infty 2^{jn(\alpha/n-\lambda/2+3/2)} \right) \\ &\leq \|f\|_{M^{p,\varphi_1}(w)}, \end{aligned} \quad (61)$$

where the series are convergent since $\lambda > 3 + 2\alpha/n$. $\square$

Proof of Theorem 6'. By Lemma 18, using the arguments as the proofs of Theorem 6, we can finish the proof. The details are omitted here. $\square$

Proof of Theorem 8. Fix a ball $B = B(x_0, r)$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$. Then,

$$\begin{aligned} \|[b, G_\alpha]^k(f)\|_{L_w^p(B)} &\leq \|[b, G_\alpha]^k(f_1)\|_{L_w^p(B)} \\ &\quad + \|[b, G_\alpha]^k(f_2)\|_{L_w^p(B)}. \end{aligned} \quad (62)$$

By Lemma 15 and (38), we have that

$$\begin{aligned} \|[b, G_\alpha]^k(f_1)\|_{L_w^p(B)} &\leq \|b\|_*^k \|f_1\|_{L_w^p(\mathbb{R}^n)} \leq \|b\|_*^k \|f\|_{L_w^p(2B)} \\ &\leq \|b\|_*^k [w]_{A_p}^{1/p} w(B)^{1/p} \\ &\quad \times \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \end{aligned} \quad (63)$$

To estimate $\|[b, G_\alpha]^k(f_2)\|_{L_w^p(B)}$, we divide $[b, G_\alpha]^k(f_2)$ into two parts:

$$\begin{aligned} & [b, G_\alpha]^k(f_2)(x) \\ & \leq \left(\int_{\Gamma(x)} \int_{\phi \in \mathcal{C}_\alpha} \sup_{\mathbb{R}^n} [b(x) - b_{B,w}]^k \right. \\ & \quad \times \left. \phi_t(y-z) f_2(z) dz \right|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2} \\ & + \left(\int_{\Gamma(x)} \int_{\phi \in \mathcal{C}_\alpha} \sup_{\mathbb{R}^n} [b_{B,w} - b(z)]^k \right. \\ & \quad \times \left. \phi_t(y-z) f_2(z) dz \right|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2} \\ & := J + JJ. \end{aligned} \quad (64)$$

First, $J = |b(x) - b_{B,w}|^k G_\alpha(f_2)(x)$. From the proof of Theorem 5, we know

$$G_\alpha(f_2)(x) \leq [w]_{A_p}^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \quad (65)$$

Hence, using Lemma 17, we obtain

$$\begin{aligned} \|J\|_{L_w^p(B)} & \leq [w]_{A_p}^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t} \\ & \times \left(\int_B |b(x) - b_{B,w}|^{kp} w(x) dx \right)^{1/p} \\ & \leq \|b\|_*^k w(B)^{1/p} \int_{2r}^\infty \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \end{aligned} \quad (66)$$

For JJ , note that $\phi \in \mathcal{C}_\alpha$ with $0 < \alpha \leq 1$ and $|x-z| < 2t$; thus, by Minkowski's inequality,

$$\begin{aligned} JJ & \leq \left(\int_{\Gamma(x)} \int_{|x-z|<2t} |b(z) - b_{B,w}|^k |f_2(z)| dz \right|^2 \frac{dydt}{t^{3n+1}} \Big)^{1/2} \\ & \leq \left(\int_0^\infty \int_{|x-z|<2t} |b(z) - b_{B,w}|^k |f_2(z)| dz \right|^2 \frac{dt}{t^{2n+1}} \Big)^{1/2} \\ & \leq \int_{|z-x_0|>2r} |b(z) - b_{B,w}|^k \frac{|f(z)| dz}{|x-z|^n} \\ & \leq \int_{|z-x_0|>2r} |b(z) - b_{B,w}|^k \frac{|f(z)| dz}{|z-x_0|^n} \\ & \approx \int_{|z-x_0|>2r} |b(z) - b_{B,w}|^k \int_{|z-x_0|}^\infty \frac{|f(z)| dt dz}{t^{n+1}}. \end{aligned} \quad (67)$$

Applying Fubini's theorem, Hölder's inequality, and Lemma 17, we have

$$\begin{aligned} JJ & \approx \int_{2r}^\infty \int_{2r \leq |z-x_0| < t} |b(z) - b_{B,w}|^k \frac{|f(z)| dz dt}{t^{n+1}} \\ & \leq \int_{2r}^\infty \left(\int_{B(x_0,t)} |b(z) - b_{B,w}|^{kp'} w(z)^{1-p'} dz \right)^{1/p'} \\ & \quad \times \frac{\|f\|_{L_w^p(B(x_0,t))}}{t^{n+1}} dt \\ & \leq \|b\|_*^k \int_{2r}^\infty \left(1 + \ln \frac{t}{r} \right)^k \|w^{-1/p}\|_{L^{p'}(B(x_0,t))} \\ & \quad \times \frac{\|f\|_{L_w^p(B(x_0,t))}}{t^{n+1}} dt \\ & \leq [w]_{A_p}^{1/p} \|b\|_*^k \int_{2r}^\infty \ln^k \left(e + \frac{t}{r} \right) \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}, \end{aligned} \quad (68)$$

where we use $w \in A_p$ in the last inequality. Combining the previous estimates for J and JJ , we obtain

$$\begin{aligned} \|[b, G_\alpha]^k(f_2)\|_{L_w^p(B)} & \leq \|b\|_*^k w(B)^{1/p} \\ & \times \int_{2r}^\infty \ln^k \left(e + \frac{t}{r} \right) \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \end{aligned} \quad (69)$$

Therefore, it follows from the above estimates (62), (63), and (69) that

$$\begin{aligned} \|[b, G_\alpha]^k(f)\|_{L_w^p(B)} & \leq \|b\|_*^k w(B)^{1/p} \\ & \times \int_{2r}^\infty \ln^k \left(e + \frac{t}{r} \right) \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t}. \end{aligned} \quad (70)$$

By substitution of variables, we obtain

$$\begin{aligned} & \|[b, G_\alpha]^k(f)\|_{M^{p,\varphi_2}(w)} \\ & \leq \|b\|_*^k \sup_{\substack{x_0 \in \mathbb{R}^n \\ r>0}} \varphi_2(x_0, r)^{-1} \\ & \quad \times \int_{2r}^\infty \ln^k \left(e + \frac{t}{r} \right) \frac{\|f\|_{L_w^p(B(x_0,t))}}{w(B(x_0,t))^{1/p}} \frac{dt}{t} \\ & \leq \|b\|_*^k \sup_{\substack{x_0 \in \mathbb{R}^n \\ r>0}} \varphi_2(x_0, r^{-1})^{-1} \\ & \quad \times \int_0^r \ln^k \left(e + \frac{r}{s} \right) \frac{\|f\|_{L_w^p(B(x_0,1/s))}}{w(B(x_0,1/s))^{1/p}} \frac{ds}{s}. \end{aligned} \quad (71)$$

Let $W(r) = \varphi_2(x_0, r^{-1})^{-1}r$ and let $V(r) = \varphi_1(x_0, r^{-1})^{-1}w(B(x_0, r^{-1}))^{-1/p}$. Since (φ_1, φ_2) satisfy condition (12), we can

easily prove that $W(r)$ and $V(r)$ satisfy condition (32). Then, by Lemma 16, we obtain the following estimate:

$$\left\| [b, G_\alpha]^k(f) \right\|_{M^{p, \varphi_2}(w)} \leq \|b\|_*^k \|f\|_{M^{p, \varphi_1}(w)}. \quad (72)$$

□

Proof of Theorem 9. We can prove it by using the the argument as in the proofs of Theorems 6 and 8. Here, the details are omitted. □

Proof of Theorem 9'. The method of proof is the same as those of Theorems 6' and 9. Here, we ignore its proof. □

Conflict of Interests

The authors declare that they have no conflict of interests.

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