Research Article

Multiplicity of Solutions for an Elliptic Problem with Critical Sobolev-Hardy Exponents and Concave-Convex Nonlinearities

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Received 29 September 2013; Accepted 25 January 2014; Published 6 March 2014

Academic Editor: Shusen Ding

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We study the existence of multiple solutions for the following elliptic problem: $-\Delta_p u - \mu(|u|^{p-2}u/|x|^p) = (|u|^{p^*(t)-2}/|x|^t)u + \lambda(|u|^{q-2}/|x|^s)u$, $u \in W_0^{1,p}(\Omega)$. We prove that if $1 \leq q < p < N$, then there is a μ_0 , such that for any $\mu \in (0, \mu_0)$, the above mentioned problem possesses infinitely many weak solutions. Our result generalizes a similar result (Azorero and Alonso, 1991).

1. Introduction and Main Results

In this paper, we study the existence of multiple solutions to the following elliptic problem:

$$-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(t)-2}u}{|x|^t} + \lambda \frac{|u|^{q-2}u}{|x|^s}, \quad x \in \Omega, \quad (1)$$

$$u(x) = 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a smooth bounded domain containing the origin 0, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian of u , $1 < p < N$, $0 \leq \mu < \bar{\mu} \equiv ((N-p)/p)^p$, $0 \leq t$, $s < p$, $1 \leq q < p$, and $p^*(t) = p(N-t)/(N-p)$ is the critical Sobolev-Hardy exponent; note that $p^*(0) = p^* \equiv pN/(N-p)$ is the critical Sobolev exponent.

Problem (1) is related to the well-known Sobolev-Hardy inequalities [1]:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(t)}}{|x|^t} dx \right)^{p/p^*(t)} \leq C \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad (2)$$

$$\forall u \in W_0^{1,p}(\Omega).$$

As $t = p$, $p^*(t) = p$, then the well-known Hardy inequality holds [1, 2]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \frac{1}{\mu} \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall u \in W_0^{1,p}(\Omega). \quad (3)$$

In this paper, we use $L^q(\Omega, |x|^s)$ to denote the usual weighted $L^q(\Omega)$ space with the weight $|x|^s$. In $W_0^{1,p}(\Omega)$, for $\mu \in [0, \bar{\mu})$, we use the norm

$$\|u\| = \|u\|_{W_0^{1,p}(\Omega)} = \left(\int_{\Omega} \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx \right)^{1/p}. \quad (4)$$

By (3), this norm is equivalent to the usual norm $(\int_{\Omega} |\nabla u|^p dx)^{1/p}$. By the Hardy inequality and the Sobolev-Hardy inequality, for $0 \leq \mu < \bar{\mu}$, $0 \leq t < p$, we can define the following best constants:

$$A_{\mu,t}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\Omega} (|u|^{p^*(t)}/|x|^t) dx \right)^{p/p^*(t)}}. \quad (5)$$

Note that $A_{\mu,0}$ is the best constant in the Sobolev inequality, that is,

$$A_{\mu,0}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^p - \mu (|u|^p/|x|^p)) dx}{\left(\int_{\Omega} |u|^{p^*} dx \right)^{p/p^*}}. \quad (6)$$

The energy functional of (1) is defined as follows:

$$I(u) = \frac{1}{p} \int_{\Omega} \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx - \frac{1}{p^*(t)} \int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx - \frac{\lambda}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx. \quad (7)$$

Then, $I(u)$ is well defined on $W_0^{1,p}(\Omega)$ and belongs to $C^1(W_0^{1,p}(\Omega), \mathfrak{R})$. The solutions of problem (1) are then the critical points of the functional I .

In recent years, the quasilinear problems related to Hardy inequality and Sobolev-Hardy inequality have been studied by some authors [3–7]. Ghoussoub and Yuan [5] studied problem (1) with $\mu = 0$, $s = 0$, and $p \leq q \leq p^*$ and proved the existence results of positive solutions and sign-changing solutions. Kang in [3, 4] studied (1,1) when $p \leq q \leq p^*(s)$ and verified the existence of positive solutions of (1) when the parameters p, q, s, λ, μ satisfy suitable conditions. To the best of our knowledge, there are few results of problem (1) involving the p -sublinear of $1 \leq q < p < N$. We are only aware of the works [6–9] which studied the existence and multiplicity of solution of problem (1) involving weight functions. Azorero and Alonso [9] studied problem (1) with $s = t = \mu = 0$, $1 < q < p$ and proved that there exists λ_1 , such that (1) has infinitely many solutions for $\lambda \in (0, \lambda_1)$. Hsu [7] studied problem (1) and proved that there exists $\Lambda_0 > 0$ such that (1) has at least two positive solutions for $\lambda \in (0, \Lambda_0)$. In this paper, we study (1) and extend the results of [7, 9].

Throughout this paper, let R_0 be the positive constant such that $\Omega \subset B(0, R_0)$, where $B(0, R_0) = \{x \in \mathfrak{R}^N : |x| < R_0\}$. By Holder inequalities, for all $u \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} & \int_{\Omega} \frac{|u|^q}{|x|^s} dx \\ & \leq \left(\int_{B(0, R_0)} |x|^{(tq - sp^*(t))/(p^*(t) - q)} dx \right)^{(p^*(t) - q)/p^*(t)} \\ & \quad \times \left(\int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx \right)^{q/p^*(t)} \\ & \leq \left(N\omega_N \int_0^{R_0} r^{((tq - sp^*(t))/(p^*(t) - q) + N - 1)} dr \right)^{(p^*(t) - q)/p^*(t)} \\ & \quad \times \left(\int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx \right)^{q/p^*(t)} \\ & = \left(\frac{N\omega_N R_0^{((tq - sp^*(t))/(p^*(t) - q) + N)}}{q(t - N) + p^*(t)(N - s)} \right)^{(p^*(t) - q)/p^*(t)} \end{aligned}$$

$$\begin{aligned} & \times \left(\int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx \right)^{q/p^*(t)} \\ & = c_0 \left(\int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx \right)^{q/p^*(t)}, \end{aligned} \quad (8)$$

where $\omega_N = 2\pi^{N/2}/N\Gamma(N/2)$ is the volume of the unit ball in $\mathfrak{R}^N$. The following inequality comes from the paper [7]:

$$\int_{\Omega} \frac{|u|^q}{|x|^s} dx \leq \left(\frac{N\omega_N R_0^{N-s}}{N-s} \right)^{(p^*(s) - q)/p^*(s)} A_{\mu, s}^{-q/p} \|u\|^q. \quad (9)$$

Now we are ready to state our main results.

Theorem 1. *If $\Omega \subset \mathfrak{R}^N$ is a bounded domain in $\mathfrak{R}^N$, and $1 \leq q < p < N$, then there is a $\mu_0 > 0$ such that problem (1) possesses infinitely many weak solutions in $W_0^{1,p}(\Omega)$ for any $\mu \in (0, \mu_0)$.*

2. The Palais-Smale Condition

Let X be a Banach space and X^{-1} be the dual space of X . The functional $I \in C^1(X, \mathfrak{R})$ is said to satisfy the Palais-Smale condition at level c $((PS)_c)$, if any sequence $(u_n) \subset X$ satisfying

$$I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0 \quad \text{strongly in } X^{-1} \text{ as } n \rightarrow \infty \quad (10)$$

contains a subsequence converging in X to a critical point of the functional I . In this paper, we will take $X = W_0^{1,p}(\Omega)$.

Lemma 2. *Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a Palais-Smale sequence for I defined by (7), that is,*

$$I(u_n) \rightarrow c, \quad (11)$$

$$I'(u_n) \rightarrow 0 \quad \text{in } W^{-1,p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (12)$$

If $1 \leq q < p$ and $c < (p - t)/p(N - t)(A_{\mu, t})^{(N-t)/(p-t)} - K\lambda^{p^*(t)/(p^*(t) - q)}$, and K depends on p, q, N , then, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$, strongly convergent in $W_0^{1,p}(\Omega)$.

Proof. By (11) and (12), it is easy to prove that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. Passing to a subsequence if necessary, we may assume that, as $n \rightarrow \infty$,

$$\begin{aligned} u_n & \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n & \rightharpoonup u \quad \text{weakly in } L^{p^*(t)}(\Omega, |x|^{-t}), \\ u_n & \rightharpoonup u \quad \text{weakly in } L^p(\Omega, |x|^{-p}), \\ u_n & \rightarrow u \quad \text{strongly in } L^q(\Omega, |x|^{-s}), \\ u_n & \rightarrow u \quad \text{strongly in } L^p(\Omega), \\ u_n & \rightarrow u \quad \text{a.e. in } \Omega. \end{aligned} \quad (13)$$

Then, $u \in W_0^{1,p}(\Omega)$ is a solution of problem (1). By the concentration compactness principle (see [10, 11]), there exists a subsequence, still denoted by u_n , at the most countable set J , a set of different points $\{x_j\}_{j \in J} \subset \Omega \setminus \{0\}$, sets of nonnegative real numbers $\{\tilde{\mu}_j\}_{j \in J \cup \{0\}}$, $\{\tilde{\nu}_j\}_{j \in J \cup \{0\}}$, and nonnegative real numbers $\tilde{\tau}_0$ and $\tilde{\gamma}_0$, such that

$$\begin{aligned} |\nabla u_n|^p &\rightharpoonup d\tilde{\mu} \geq |\nabla u|^p + \sum_{j \in J} \tilde{\mu}_j \delta_{x_j} + \tilde{\mu}_0 \delta_0, \\ |u_n|^{p^*} &\rightharpoonup d\tilde{\nu} = |u|^{p^*} + \sum_{j \in J} \tilde{\nu}_j \delta_{x_j} + \tilde{\nu}_0 \delta_0, \\ \frac{|u_n|^{p^*(t)}}{|x|^t} &\rightharpoonup d\tilde{\tau} = \frac{|u|^{p^*(t)}}{|x|^t} + \tilde{\tau}_0 \delta_0, \\ \frac{|u_n|^p}{|x|^p} &\rightharpoonup d\tilde{\gamma} = \frac{|u|^p}{|x|^p} + \tilde{\gamma}_0 \delta_0, \end{aligned} \quad (14)$$

where δ_x is the Dirac mass at x .

Case 1 ($t = s = 0$ and $p^*(t) = p^*$). We claim that J is finite, and, for any $j \in J$, either

$$\tilde{\nu}_j = 0 \quad \text{or} \quad \tilde{\nu}_j \geq (A_{\mu,0})^{N/p}. \quad (15)$$

In fact, let $\varepsilon > 0$ be small enough such that $0 \in \bar{B}(x_j, \varepsilon)$ and $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \emptyset$ for $i \neq j$, $i, j \in J$. We consider $\varphi_j \in C_0^\infty(\mathbb{R}^N)$, such that

$$\begin{aligned} \varphi_j &\equiv 1 \quad \text{on } B\left(x_j, \frac{\varepsilon}{2}\right), \\ \varphi_j &\equiv 0 \quad \text{on } B(x_j, \varepsilon)^c, \\ |\nabla \varphi_j| &\leq \frac{4}{\varepsilon}, \\ 0 &\leq \varphi_j \leq 1. \end{aligned} \quad (16)$$

It is clear that the sequence $\{\varphi_j u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Note that

$$\begin{aligned} \langle I'(u_n), u_n \varphi_j \rangle &= \int_{\Omega} |\nabla u_n|^p \varphi_j dx \\ &\quad + \int_{\Omega} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi_j dx \\ &\quad - \mu \int_{\Omega} \frac{|u_n|^p}{|x|^p} \varphi_j dx \\ &\quad - \int_{\Omega} |u_n|^{p^*} \varphi_j dx - \lambda \int_{\Omega} |u_n|^q \varphi_j dx. \end{aligned} \quad (17)$$

By (13), (16), and the Holder inequality, we obtain

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\Omega} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi_j dx \right| \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{B(x_j, \varepsilon)} |u|^{p^*} dx \right)^{1/p^*} = 0, \\ \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\Omega} \frac{|u_n|^p}{|x|^p} \varphi_j dx \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{B_\varepsilon(x_j)} \frac{|u_n|^p}{(|x_j| - \varepsilon)^p} \varphi_j dx \right| = 0. \end{aligned} \quad (18)$$

From (12)~(18), we get that

$$0 = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle I'(u_n), u_n \varphi_j \rangle \geq \tilde{\mu}_j - \tilde{\nu}_j. \quad (19)$$

By the Sobolev inequality, $A_{0,0} \nu_j^{p/p^*} \leq \tilde{\mu}_j$, hence, we deduce that

$$\tilde{\nu}_j = 0 \quad \text{or} \quad \tilde{\nu}_j \geq (A_{0,0})^{N/p}, \quad (20)$$

which implies that J is finite.

Now we consider the possibility of concentration at the origin. Let $\varepsilon > 0$ be small enough such that $x_j \in B(0, \varepsilon)$, $\forall j \in J$. Take $\varphi^0 \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} \varphi^0 &\equiv 1 \quad \text{on } B\left(0, \frac{\varepsilon}{2}\right), \\ \varphi^0 &\equiv 0 \quad \text{on } B(0, \varepsilon)^c, \\ |\nabla \varphi^0| &\leq \frac{4}{\varepsilon}. \end{aligned} \quad (21)$$

By (13) and (14), we also get that

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle I'(u_n), u_n \varphi^0 \rangle \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \varphi^0 d\tilde{\mu} - \int_{\Omega} \varphi^0 d\tilde{\nu} - \int_{\Omega} \varphi^0 d\tilde{\tau} \right. \\ &\quad \left. - \lambda \int_{\Omega} \varphi^0 \frac{|u|^q}{|x|^s} dx \right) \\ &\geq \tilde{\mu}_0 - \mu \tilde{\gamma}_0 - \tilde{\nu}_0. \end{aligned} \quad (22)$$

By the definition of $A_{\mu,0}$, we deduce that

$$A_{\mu,0} \tilde{\gamma}_0^{p/p^*} \leq \tilde{\mu}_0 - \mu \tilde{\gamma}_0. \quad (23)$$

From (22) we have

$$A_{\mu,0} \tilde{\gamma}_0^{p/p^*} \leq \tilde{\mu}_0 - \mu \tilde{\gamma}_0 \leq \tilde{\nu}_0, \quad (24)$$

which implies that $\tilde{\nu}_0 = 0$ or

$$\tilde{\nu}_0 \geq (A_{\mu,0})^{N/p}. \quad (25)$$

We will prove that (25) and $\tilde{v}_j \geq (A_{0,0})^{N/p}$ are not possible. By (13) and (14),

$$\begin{aligned}
 c &= \lim_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} \left\{ I(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle \right\} \\
 &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{N} \int_{\Omega} |u_n|^{p^*} dx + \lambda \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\Omega} |u_n|^q dx \right\} \\
 &\geq \frac{1}{N} \left(\int_{\Omega} |u|^{p^*} dx + \sum_{j \in J} \tilde{v}_j + \tilde{v}_0 \right) + \lambda \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\Omega} |u|^q dx \\
 &\geq \frac{1}{N} \int_{\Omega} |u|^{p^*} dx + \frac{1}{N} \min \left\{ (A_{\mu,0})^{N/p}, (A_{0,0})^{N/p} \right\} \\
 &\quad + \lambda \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\Omega} |u|^q dx \\
 &\geq \frac{1}{N} \int_{\Omega} |u|^{p^*} dx + \frac{1}{N} (A_{\mu,0})^{N/p} \\
 &\quad + \lambda \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\Omega} |u|^q dx.
 \end{aligned} \tag{26}$$

By applying the Holder inequality at (26), we have

$$\begin{aligned}
 c &\geq \frac{1}{N} (A_{\mu,0})^{N/p} + \frac{1}{N} \int_{\Omega} |u|^{p^*} dx \\
 &\quad - \lambda \left(\frac{1}{q} - \frac{1}{p} \right) |\Omega|^{(p^*-q)/p^*} \left(\int_{\Omega} |u|^{p^*} dx \right)^{q/p^*}.
 \end{aligned} \tag{27}$$

Let $f_1(x) = c_1 x^{p^*} - \lambda c_2 x^q$, $c_1 = 1/N$, $c_2 = (1/q) - (1/p)$. This function obtains its absolute minimum (for $x > 0$) at point $x_0 = (\lambda c_2 q / p^* c_1)^{1/(p^*-q)}$. That is,

$$f_1(x) \geq f_1(x_0) = -K_1 \lambda^{p^*/(p^*-q)}, \tag{28}$$

where

$$\begin{aligned}
 K_1 &= c_2^{p/(p^*-q)} c_1^{-q/(p^*-q)} \\
 &\quad \times \left(\left(\frac{q}{p^*} \right)^{q/(p^*-q)} - \left(\frac{q}{p^*} \right)^{p^*/(p^*-q)} \right) > 0,
 \end{aligned} \tag{29}$$

because of $1 < q < p < Np/(N-p)$. But this result contradicts the hypothesis. Then, $\tilde{v}_j = 0 \ \forall j \in J \cup \{0\}$ and we conclude.

Case 2 ($0 < t < p$, then $p < p^*(t) < p^*$). We only need to consider the possibility of concentration at the origin. Let $\varepsilon > 0$ be small enough such that $B(0, \varepsilon) \subset \Omega$. Take φ^0 a smooth cut-off function centered at the origin such that $0 \leq \varphi^0 \leq 1$,

$\varphi^0 = 1$ for $|x| \leq \varepsilon/2$, $\varphi^0 = 0$ for $|x| \geq \varepsilon$, and $|\nabla \varphi^0| \leq 4/\varepsilon$. By (13) and (14), we get that

$$\begin{aligned}
 0 &= \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle I'(u_n), u_n \varphi^0 \rangle \\
 &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \varphi^0 d\tilde{\mu} - \mu \int_{\Omega} \varphi^0 d\tilde{\gamma} - \int_{\Omega} \varphi^0 d\tilde{\tau} - \lambda \int_{\Omega} \varphi^0 \frac{|u|^q}{|x|^s} dx \right) \\
 &\geq \tilde{\mu}_0 - \mu \tilde{\gamma}_0 - \tilde{\tau}_0.
 \end{aligned} \tag{30}$$

By the definition of $A_{\mu,t}$, we deduce that

$$A_{\mu,t} \tilde{\tau}_0^{p/p^*(t)} \leq \tilde{\mu}_0 - \mu \tilde{\gamma}_0 \tag{31}$$

From (30), we have

$$A_{\mu,t} \tilde{\tau}_0^{p/p^*(t)} \leq \tilde{\tau}_0, \tag{32}$$

which implies that $\tilde{\tau}_0 = 0$ or

$$\tilde{\tau}_0 \geq (A_{\mu,t})^{(N-t)/(p-t)}. \tag{33}$$

We will prove (33) is not possible. From the above arguments and (8), we conclude that

$$\begin{aligned}
 c &= \lim_{n \rightarrow \infty} J(u_n) = \lim_{n \rightarrow \infty} \left\{ J(u_n) - \frac{1}{p} \langle J(u_n), u_n \rangle \right\} \\
 &\geq \frac{p-t}{p(N-t)} \int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx + \frac{p-t}{p(N-t)} (A_{\mu,t})^{(N-t)/(p-t)} \\
 &\quad + \lambda c_0 \left(\frac{1}{q} - \frac{1}{p} \right) \left(\int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx \right)^{q/p^*(t)}.
 \end{aligned} \tag{34}$$

Let $f_2(x) = c_3 x^{p^*(t)} - \lambda c_4 x^q$, $c_3 = (p-t)/p(N-t)$, $c_4 = c_0((1/q) - (1/p))$. This function obtains its absolute minimum at point $x_0 = (\lambda c_4 q / p^*(t) c_3)^{1/(p^*(t)-q)}$. That is,

$$f_2(x) \geq f_2(x_0) = -K_2 \lambda^{p^*(t)/(p^*(t)-q)}. \tag{35}$$

But this result contradicts the hypothesis. Hence, up to a subsequence, we obtain that $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$.

Thus, the proof of the Lemma is completed. $\square$

3. Existence of Infinitely Many Solutions

In this section, we will prove our main result of Theorem 1. We first recall some concepts and results in minimax theory.

Let X be a Banach space, and Σ denote all closed subsets of $X - \{0\}$ which are symmetric with respect to the origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\begin{aligned}
 \gamma(A) &= \min \{k \in \mathbb{N} : \exists \phi \in C(A; \mathbb{R}^k \setminus \{0\}), \phi(x) = -\phi(-x)\}, \\
 &\tag{36}
 \end{aligned}$$

if the minimum exists, and if such a minimum does not exist, then we define $\gamma(A) = \infty$. The main properties of the genus are contained in the following lemma (see [12] for the details).

Lemma 3. Let $A, B \in \Sigma$. Then one has the following.

- (1) If there exists $f \in C(A, B)$, odd, then $\gamma(A) \leq \gamma(B)$.
- (2) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
- (3) If there exists an odd homeomorphism between A and B , then $\gamma(A) = \gamma(B)$.
- (4) If S^{N-1} is the sphere in $\mathbb{R}^N$, then $\gamma(S^{N-1}) = N$.
- (5) Consider $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
- (6) If $\gamma(B) < +\infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.
- (7) If A is compact, then $\gamma(A) < +\infty$, and there exists $\delta > 0$ such that $\gamma(N_\delta(A)) = \gamma(A)$, where $N_\delta(A) = \{x \in X : d(x, A) \leq \delta\}$.
- (8) If X_0 is a subspace of X with codimension k , and $\gamma(A) < k$, then $A \cap X_0 \neq \emptyset$.

Let X be a Banach space and E be a C^1 functional on X . Denote $E^c = \{u \in X \mid E(u) \leq c\}$, $\Sigma_k = \{C \subset W_0^{1,p}(\Omega) - \{0\}, C \text{ is closed}, C = -C, \gamma(C) \geq k\}$.

Given the functional I , under the hypothesis $1 < q < p < n$, using Sobolev's equality and (9), we obtain

$$I(u) \geq \frac{1}{p} \|u\|^p - \frac{1}{p^*(t) A_{\mu,t}^{p^*(t)/p}} \|u\|^{p^*(t)} - \frac{\lambda c_0}{q} \|u\|^q. \quad (37)$$

If we define for $x \geq 0$

$$h(x) = \frac{1}{p} x^p - \frac{1}{p^*(t) A_{\mu,t}^{p^*(t)/p}} x^{p^*(t)} - \frac{\lambda c_0}{q} x^q, \quad (38)$$

then

$$I(u) \geq h(\|u\|). \quad (39)$$

Because $p^*(t) > p$ and $h(x) \rightarrow -\infty$, as $x \rightarrow +\infty$, it is easy to see that there exists $0 < \mu_0 \leq 1$ such that, if $0 < \mu \leq \mu_0$, h attains its positive maximum.

From the structure of $h(x)$, we see that there are constants $0 < R_0 < R_1$, such that $h(R_0) = h(R_1) = 0$, $h(R) \leq 0$ if $R < R_0$, $h(R) > 0$ if $R_0 < R < R_1$, and $h(R) < 0$ if $R > R_1$. Following [9], let $\tau : \mathbb{R}^+ \rightarrow [0, 1] \in C^\infty$ be nonincreasing, such that

$$\tau(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq R_0, \\ 0, & \text{if } x \geq R_1, \end{cases} \quad (40)$$

and let $\varphi(u) = \tau(\|u\|)$; we consider the truncated functional

$$\begin{aligned} J(u) &= \frac{1}{p} \int_{\Omega} \left(|\nabla u|^p - \mu \frac{u^p}{|x|^p} \right) dx \\ &\quad - \frac{1}{p^*(t)} \int_{\Omega} \frac{|u|^{p^*(t)} \varphi(u)}{|x|^t} dx - \frac{\lambda}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx. \end{aligned} \quad (41)$$

Similar to (39), we have $J(u) \geq \bar{h}(\|u\|)$, where

$$\bar{h}(x) = \frac{1}{p} x^p - \frac{\tau(x)}{p^*(t) A_{\mu,t}^{p^*(t)/p}} x^{p^*(t)} - \frac{\lambda c_0}{q} x^q. \quad (42)$$

Clearly, $\bar{h}(x) \geq h(x)$ for $x \geq 0$ and $\bar{h}(x) = h(x)$ if $0 \leq x \leq R_0$, $\bar{h}(x) \geq h(x) \geq 0$, if $R_0 < R < R_1$, and if $x > R_1$, $\bar{h}(x) = (1/p)x^p - (\lambda c_0/q)x^q$ is strictly increasing, and so $\bar{h}(x) > 0$, if $x > R_1$. Consequently, $\bar{h}(x) \geq 0$ for $x \geq R_0$.

Lemma 4. (1) Consider $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$.

(2) If $J(u) \leq 0$, then $\|u\| \leq R_0$ and $I(v) = J(v)$ for all v in a small enough neighborhood of u .

(3) There exists $\lambda_0 > 0$, such that if $0 < \lambda < \lambda_0$, then J verifies a local Palais-Smale condition for $c \leq 0$.

Proof. (1) and (2) are immediate. To prove (3), observe that all Palais-Smale sequences for J with $c \leq 0$ must be bounded; then, by Lemma 2, if λ verifies $((p-t)/p(N-t))(A_{\mu,t})^{(N-t)/(p-t)} - K\lambda^{p^*(t)/(p^*(t)-q)} \geq 0$, there exists a convergent subsequence. $\square$

Now, we use the idea in [9] to construct negative critical values of J via genus.

Lemma 5. Given $n \in \mathbb{N}$, there is an $\varepsilon(n) > 0$, such that

$$\gamma(\{u \in W_0^{1,p}(\Omega) : J(u) \leq -\varepsilon(n)\}) \geq n. \quad (43)$$

Proof. Fix n ; let E_n be an n -dimensional subspace of $W_0^{1,p}(\Omega)$; we take $u_n \in E_n$ with norm $\|u_n\| = 1$ for $0 < \rho < R_0$; we have

$$\begin{aligned} J(\rho u_n) &= I(\rho u_n) \\ &= \frac{1}{p} \rho^p - \frac{1}{p^*(t)} \rho^{p^*(t)} \\ &\quad \times \int_{\Omega} \frac{|u_n|^{p^*(t)}}{|x|^t} dx - \frac{\lambda}{q} \rho^q \int_{\Omega} \frac{|u_n|^q}{|x|^s} dx. \end{aligned} \quad (44)$$

Since E_n is a space of finite dimension, all the norms in E_n are equivalent. If we define

$$\alpha_n = \inf \left\{ \int_{\Omega} \frac{|u|^{p^*(t)}}{|x|^t} dx : u \in E_n, \|u\| = 1 \right\} > 0, \quad (45)$$

$$\beta_n = \inf \left\{ \int_{\Omega} \frac{|u|^q}{|x|^s} dx : u \in E_n, \|u\| = 1 \right\} > 0,$$

we have

$$J(\rho u_n) \leq \frac{1}{p} \rho^p - \frac{\alpha_n}{p^*(t)} \rho^{p^*(t)} - \frac{\lambda \beta_n}{q} \rho^q, \quad (46)$$

and we can choose ε (which depends on n), and $\eta < R_0$, such that $J(\eta u) \leq -\varepsilon$ if $u \in E_n$ and $\|u\| = 1$.

Let $S_\eta = \{u \in W_0^{1,p}(\Omega) : \|u\| = \eta\}$. Consider $S_\eta \cap E_n \subset \{u \in W_0^{1,p}(\Omega) : J(u) \leq -\varepsilon\}$; therefore, by Lemma 3, we see that

$$\gamma(\{u \in W_0^{1,p}(\Omega) : J(u) \leq -\varepsilon(n)\}) \geq \gamma(S_\eta \cap E_n) = n. \quad (47)$$

We are now in a position to prove our main result. $\square$

Proof of Theorem 1. Let $\Sigma_k = \{C \subset W_0^{1,p}(\Omega) : \{0\}, C \text{ is closed, } C = -C, \gamma(C) \geq k\}$, $c_k = \inf_{C \in \Sigma_k} \sup_{u \in C} J(u)$, $K_c = \{u \in W_0^{1,p}(\Omega), J'(u) = 0, J(u) = c\}$, and suppose that $0 < \lambda < \lambda_0$ where λ_0 is the constant given by Lemma 4. We claim that if $k, r \in \mathbb{N}$ are such that $c = c_k = c_{k+1} = \dots = c_{k+r}$, then $\gamma(K_c) \geq r + 1$.

In fact, denote $J^{-\varepsilon} = \{u \in W_0^{1,p}(\Omega) : J(u) \leq -\varepsilon\}$; by Lemma 5, we see that for any $k \in \mathbb{N}$, there is a $\varepsilon(k) > 0$, such that $\gamma(J^{-\varepsilon(k)}) \geq k$. Since J is continuous and even, $J^{-\varepsilon(k)} \in \Sigma_n$ and $c_k \leq -\varepsilon(n) < 0$. As J is bounded from below, we see that $c_k > -\infty$ for all $k \in \mathbb{N}$.

Suppose that $c = c_k = c_{k+1} = \dots = c_{k+r} < 0$; then J satisfies $(PS)_c$ condition by Lemma 2, and it is easy to see that K_c is a compact set.

If $\gamma(K_c) \leq r$, then there is a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \leq r$ by Lemma 3. Since $c < 0$, we can also assume that the closed set $U \subset J^0$. Since J satisfies $(PS)_c$ condition for $c < 0$, by the Deformation Lemma, there is an odd homeomorphism,

$$\eta : W_0^{1,p}(\Omega) \longrightarrow W_0^{1,p}(\Omega) \quad (48)$$

such that $\eta(J^{c+\delta} - U) \subset J^{c-\delta}$ for some δ with $0 < \delta < -c$.

Since $c = c_{k+r} = \inf_{C \in \Sigma_k} \sup_{u \in C} J(u)$, there exists an $A \in \Sigma_{k+r}$, such that

$$\sup_{u \in C} J(u) < c + \delta, \quad \text{i.e., } A \subset J^{c+\delta}, \quad (49)$$

$$\eta(A - U) \subset \eta(J^{c+\delta} - U) \subset J^{c-\delta}. \quad (50)$$

But by Lemma 3 and $\gamma(U) \leq r$, we have

$$\begin{aligned} \gamma(\overline{A - U}) &\geq \gamma(A) - \gamma(U) \geq n, \\ \gamma(\eta \overline{A - U}) &\gamma(\overline{A - U}) \geq n. \end{aligned} \quad (51)$$

Hence, $\eta \overline{A - U} \in \Sigma_k$ and $\sup_{u \in \eta \overline{A - U}} J(u) \geq c_k = c$, which contradicts to (50). So we have proved that $\gamma(K_c) \geq r + 1$.

Now if for all $k \in \mathbb{N}$, we have $\Sigma_{k+1} \subset \Sigma_k$, $c_k \leq c_{k+1} < 0$. If all c_k are distinct, then $\gamma(K_{c_k}) \geq 1$, and we see that $\{c_k\}$ is a sequence of distinct critical values of J ; if for some k_0 , there is a $r \geq 1$ such that

$$c = c_{k_0} = c_{k_0+1} = \dots = c_{k_0+r}, \quad (52)$$

then

$$\gamma(K_{c_{k_0}}) \geq r + 1, \quad (53)$$

which shows that $K_{c_{k_0}}$ contains infinitely many distinct elements.

Since $J(u) = I(u)$ if $J(u) < 0$, we see that there are infinitely many critical points of $I(u)$. The theorem is proved. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported by Ningbo Scientific Research Foundation (2009B21003), K. C. Wong Magna Fund in Ningbo University, NSF of Hebei Province (A2013209278), and National Natural Science Foundation of China (nos. 61271398, 11220248 and 11175092).

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