Research Article

Measure of Noncompactness for Compact Matrix Operators on Some BK Spaces

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We study the spaces $\omega_p^0$, $\omega_p^1$, and $\omega_\infty^p$ of sequences that are strongly summable to zero, summable, and bounded with index $p \geq 1$ by the Cesàro method of order 1 and establish the representations of the general bounded linear operators from the spaces $\omega_p^0$ into the spaces $\omega_\infty^p$, $\omega_1^p$, and $\omega_1^\infty$. We also give estimates for the operator norm and the Hausdorff measure of noncompactness of such operators. Finally we apply our results to characterize the classes of compact bounded linear operators from $\omega_p^0$ and $\omega_p^1$ into $\omega_\infty^p$ and $\omega_1^p$.

1. Introduction

The spaces $\omega_p^0$, $\omega_p^1$, and $\omega_\infty^p$ of all complex sequences that are strongly summable to zero, strongly summable, and strongly bounded, with index $p \geq 1$ by the Cesàro method of order 1, were first introduced and studied by Maddox [1, 2]. Further recent studies on the spaces $\omega_p(p)$ where the constant index $p$ in $\omega_p^0$ is replaced by a term of a positive sequence $p = (p_k)$ can be found in [3, 4]. Extensive studies of generalizations of the spaces $\omega_p^0$, $\omega_p^1$, and $\omega_\infty^p$ to spaces of sequences of strong weighted means can be found in [5, 6].

In [7], a complete list was given of the characterizations of all matrix transformations from Maddox’s spaces into the classical spaces $\ell_0^\infty$, $\ell_0^1$, and $\ell_1$ of all bounded, convergent, and null sequences and of all absolutely convergent series. The characterizations of matrix transformations from the classical sequence spaces into Maddox’s spaces with index $p = 1$ were established in [8]. Furthermore, some classes of compact bounded linear operators between those spaces were characterized.

Recently, several authors applied the Hausdorff measure of noncompactness to characterize matrix transformations between sequence spaces that are matrix domains of triangles in the classical sequence spaces, for instance, in [9–14].

In this paper, we extend our studies from the normally considered matrix transformations to the general bounded linear operators from $\omega_p^0$ into $\omega_\infty^p$, $\omega_1^p$, and $\omega_1^\infty$. We establish the representations of those operators, deduce estimates for their operator norms and Hausdorff measures of noncompactness, and characterize the corresponding classes of compact bounded linear operators.

2. Notations and Basic Results

In this section we list the notations, concepts, and basic results needed in the paper.

As usual, we denote by $\omega$ and $\phi$ the sets of all complex sequences $x = (x_k)_{k=1}^\infty$ and of all sequences that terminate in zeros; also let $e$ and $e^{(n)}$ for all $n \in \mathbb{N}$ be the sequences with $e_k = 1$ for all $k$ and $e^{(n)}_k = 0$ and $e^{(m)}_k = 0$ for $k \neq n$.

A Banach space $X \subset C$ is a BK space if each coordinate $P_n : X \to C$ with $P_n x = x_n$ for $x = (x_k)_{k=1}^\infty \in X$ is continuous. A BK space $X \supset \phi$ is said to have AK if $x^{[m]} = \sum_{k=1}^m x_k e^{(k)} \to x (m \to \infty)$ for every sequence $x = (x_k)_{k=1}^\infty \in X$.

Let $(X, \|\cdot\|)$ be a normed space and $S_X = \{x \in X : \|x\| = 1\}$ and $\overline{B}_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and closed
unit ball in $X$, respectively. If $X$ and $Y$ are Banach spaces, then we write $\mathcal{B}(X, Y)$ for the space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup\{\|L(x)\| : x \in S_X\}$; we write $X^\ast = \mathcal{B}(X, \mathbb{C})$ for the \textit{continuous dual} of $X$, that is, the space of all continuous linear functionals on $X$ with the norm $\|f\| = \sup\{|f(x)| : x \in S_X\}$. Furthermore, if $(X, \| \cdot \|)$ is a normed sequence space, then we write $\|a\|_X = \sup_{n \in \mathbb{N}} |a_n x_k|$ for $a \in \omega$ provided the expression on the right-hand side exists and is finite which is the case whenever $X$ is a BK space and $a \in X^\beta$ [15, Theorem 7.2.9].

For any subset $X$ of $\omega$, the set $X^\beta = \{a \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x \in X\}$ is called the $\beta$-dual of $X$.

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex numbers, let $X$ and $Y$ be subsets of $\omega$, and let $x \in \omega$. We write $A_n = (a_{nk})_{k=1}^{\infty}$ for the sequence in the $n$th row of $A$, $A_n x = \sum_{k=1}^{\infty} a_{nk} x_k$, $Ax = (A_n x)_{n=1}^{\infty}$ (provided all the series converge) and $(X, Y)$ for the class of all matrices $A$ such that $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$. It is known that if $X$ and $Y$ are BK spaces then every matrix $A \in (X, Y)$ defines an operator $L_A : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$ [15, Theorem 4.2.8] and if, in addition, $X$ has AK then every operator $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $L(x) = Ax$ for all $x \in X$ [16, Theorem 1.9].

Throughout, let $1 \leq p < \infty$, and let $q$ be the conjugate number of $p$; that is, $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$. We write

$$w_0^p = \{ x \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \},$$

$$w^p = \{ x \in \omega : x = x - \xi \cdot e \in w_0^p \text{ for some } \xi \in \mathbb{C} \},$$

$$w_{co}^p = \{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p < \infty \}$$

for the sets of all sequences that are strongly summable to $0$, strongly summable, and strongly bounded, with index $p$, respectively.

The following results are known and can be found in [1] and, for instance, in [7, Proposition 2.1].

For each sequence $x \in w^p$, the $w^p$-limit $\xi$ for which

$$x - \xi \cdot e \in w_0^p$$

is unique. We write $\sum_{\nu} = \sum_{n=0}^{\nu} 1$ for $\nu = 0, 1, \ldots$ and $\max_{\nu} = \max_{2 \leq k \leq 2^{\nu+1} - 1}$.

The sets $w_0^p$, $w^p$, and $w_{co}^p$ are BK space with the equivalent block and sectional norms

$$\|x\|_b = \sup_{n,k} \left( \frac{1}{2^n} \sum_{\nu} |x_{\nu}|^p \right)^{1/p},$$

$$\|x\|_s = \sup_{n} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}.$$
(c) \( A \in (w^p_0, w) \) if and only if (11) holds and

\[
\text{for each } k \in \mathbb{N} \text{ there exists } \alpha_k \in \mathbb{C} \text{ such that } \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |a_{nk} - \alpha_k| = 0.
\]  

(13)

(d) \( A \in (w^p, w_0) \) if and only if (11) and (12) hold and

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} \left| \sum_{k=1}^{\infty} a_{nk} \right| = 0.
\]  

(14)

(e) \( A \in (w^p, w) \) if and only if (11) and (13) hold and

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} \left| \sum_{k=1}^{\infty} a_{nk} - \bar{a} \right| = 0 \text{ for some } \bar{a} \in \mathbb{C}.
\]  

(15)

(f) If \( A \in (X, Y) \) in the cases above, then

\[
\|A\|_{(w^p_0, w_0)} \leq \|L\| \leq 4 \cdot \|A\|_{(w^p_0, w_0)}.
\]  

(16)

Proof. (a) follows from [17, Corollary 1], (6), and (7).

(b) and (c) follow from [15, 8.3.6, p. 123], since \( w^p_0 \) and \( w^p \) are closed subspaces of \( w^p_0 \).

(d) and (e) follow from Parts (b) and (c) by [15, 8.3.7].

(f) follows from [17, equation (2.8)], (6), and (7). \( \square \)

3. Representation of Bounded Linear Operators

Here we establish the representations of the bounded linear operators in \( \mathcal{B}(w^p, Y) \) for \( Y = w_0, w, w_0 \), and give estimates for the operator norms in each case. Throughout, we assume that \( w^p \) and \( Y \) have the block and sectional norms, respectively.

We note that, since \( w^p_0 \) has AK, every \( L \in \mathcal{B}(w^p_0, Y) \) is given by an infinite matrix \( A \in (w^p_0, Y) \), and its operator norm satisfies the inequalities in (16).

**Theorem 2.** (a) One has \( L \in \mathcal{B}(w^p, w_0) \) if and only if there exist a matrix \( A \in (w^p, w_0) \) and a sequence \( b \in w_0 \) such that

\[
L(x) = b \cdot \xi + Ax \quad \forall x \in w^p,
\]

where \( \xi \) is the \( w^p \)-limit of \( x \).

Moreover, one has

\[
\sup_m \left( \frac{1}{m} \max_{n \in [1..m]} \left( \left| \sum_{k=1}^{n} b_k \right| + \left\| \sum_{k=1}^{n} A_n \right\|_{\mathcal{M}_p} \right) \right) \leq \|L\| \leq 4 \cdot \sup_m \left( \frac{1}{m} \max_{n \in [1..m]} \left( \left| \sum_{k=1}^{n} b_k \right| + \left\| \sum_{k=1}^{n} A_n \right\|_{\mathcal{M}_p} \right) \right).
\]  

(18)

(b) One has \( L \in \mathcal{B}(w^p, w) \) if and only if there exist a matrix \( A \in (w^p_0, w) \) and a sequence \( b \in w_0 \), with

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |b_n + A_n e - \tilde{b}| = 0 \quad \text{for some } \tilde{b} \in \mathbb{C}
\]

such that (17) holds; moreover, one has (18).

(c) One has \( L \in \mathcal{B}(w^p, w_0) \) if and only if there exist a matrix \( A \in (w^p_0, w_0) \) and a sequence \( b \in w_0 \) with

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |b_n + A_n e| = 0
\]

such that (17) holds; moreover, one has (18).

Proof. (a) First we assume \( L \in \mathcal{B}(w^p_0, Y) \) and write \( L_n = P_n \circ L \) for \( n = 1, 2, \ldots \) where \( P_n \) denotes the \( n \)-th coordinate. Since \( w^p \) is a BK space, it follows that \( L_n \in (w^p)^* \) for each \( n \), and hence we have by (9)

\[
L_n(x) = b_n \cdot \xi + A_n x \quad \forall x \in w^p
\]

where \( \xi \) is the \( w^p \)-limit of \( x \),

\[
b_n = L_n(e) - \sum_{k=1}^{\infty} L_n(e^{(k)})
\]

(26)

\[
a_{nk} = L_n(e^{(k)}) \quad \forall n \text{ and } k,
\]

(27)

\[
A_n \in \mathcal{M}_p \quad \forall n.
\]

This yields (17); moreover, we have by (10)

\[
\|L_n\| = |b_n| + \|A_n\|_{\mathcal{M}_p} \text{ for } n = 1, 2, \ldots
\]

(23)

It also follows from (21) and \( L(e) \in w_0 \) that

\[
\sup_m \frac{1}{m} \sum_{n=1}^{m} |L_n(e)| = \sup_m \frac{1}{m} \sum_{n=1}^{m} |b_n + A_n e| < \infty
\]

(24)

and so \( b + A_e \in w_0 \). Furthermore, since \( L(x^{(0)}) = A x^{(0)} \) for all \( x^{(0)} \in w^p_0 \), we have \( A \in (w^p, w_0, w) \), and so \( A \in w_0 \) and we obtain \( b = (b + A_e) - A_e \in w_0 \).

Now we show (18). We define \( L_{N_m} : w^p \to \mathbb{C} \) for all \( m \in \mathbb{N} \) and for each subset \( N_m \) of \( \{1, 2, \ldots, m\} \) by

\[
L_{N_m} = \frac{1}{m} \sum_{n \in N_m} L_n
\]

(25)

Then clearly \( L_{N_m} \in (w^p)^* \), and we obtain by a well-known inequality (cf. [18])

\[
|L_{N_m}(x)| \leq \frac{1}{m} \sum_{n=1}^{m} |L_n(x)| \leq 4 \cdot \max_{N_m \subseteq [1..m]} \|L_{N_m}(x)\|
\]

and hence by the first inequality in (26) and by (10)

\[
\|L_{N_m}\| = \frac{1}{m} \left( \left| \sum_{n \in N_m} b_n \right| + \left\| \sum_{n \in N_m} A_n \right\|_{\mathcal{M}_p} \right) \leq \|L\|
\]

(27)
for all \( m \in \mathbb{N} \) and all \( N_m \subset \{1, 2, \ldots, m\} \), and so the first inequality in (18) follows. Also, we obtain for all \( m \in \mathbb{N} \) from the second inequality in (26) and by (10)

\[
\left\| \frac{1}{m} \sum_{n=1}^{m} L_n(x) \right\| \leq 4 \cdot \max_{N_m \subset \{1, \ldots, m\}} \left\| L_{N_m} \right\|
\]

This implies the second inequality in (18).

Conversely, we assume that \( A \in (w_0^p, w_{co}) \), that \( b \in w_{co} \), and that (17) is satisfied. Let \( \epsilon > 0 \), and \( x \in w^p \) be given and let \( \xi \) be the \( w^p \)-limit of \( x \). Then there exists \( M \in \mathbb{N} \) such that \( (1/m) \sum_{n=1}^{m} |x_n - \xi|^p < \epsilon \) for all \( m > M \). Thus we have for all \( m > M \)

\[
|\xi| \leq \left( \frac{1}{m} \sum_{n=1}^{m} |\xi|^p \right)^{1/p} \leq \left( \frac{1}{m} \sum_{n=1}^{m} |x_n|^p \right)^{1/p} + \left( \frac{1}{m} \sum_{n=1}^{m} |x_n - \xi|^p \right)^{1/p} \leq \|x\|_b + \epsilon.
\]

Since \( \epsilon \) was arbitrary, we have

\[
|\xi| \leq \|x\|_b. \tag{30}
\]

We define the map \( g : w^p \to w_{co} \) with \( g(x) = bx \) for all \( x \in w^p \), where \( \xi \in C \) is the \( w^p \)-limit of \( x \). Then \( g \) trivially is linear, and it follows from (30) that

\[
\left\| g(x) \right\|_b = \sup \left\{ \frac{1}{m} \sum_{n=1}^{m} |b_n \xi| \right\} \leq \sup \left\{ \frac{1}{m} \sum_{n=1}^{m} |b_n| \right\} \cdot \|x\|_b, \tag{31}
\]

and, since \( b \in w_{co} \), we obtain \( g \in \mathcal{B}(w^p, w_{co}) \). Furthermore, we have \( A \in (w_0^p, w_{co}) = (w^p, w_{co}) \), and hence \( L_A \in \mathcal{B}(w^p, w_{co}) \), and so, by (17), \( L = g + L_A \in \mathcal{B}(w^p, w_{co}) \).

(b) First we assume \( L \in \mathcal{B}(w^p, w) \). Then \( L \in \mathcal{B}(w, w_{co}) \), and by Part (a) there are \( b \in w_{co} \) and \( A \in (w_0^p, w_{co}) = (w^p, w_{co}) \) such that (17) is satisfied; also clearly (18) is satisfied. It follows from (17), \( L(e) \in w \), and \( L(e^{(k)}) \in w \) for each \( k \) that there exist \( \beta \in C \) and \( \alpha_k \in C \) for \( k = 1, 2, \ldots \), such that

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |L_n(e) - \beta| = \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |b_n + A_n x - \beta| = 0,
\]

which is (19), and

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |L_n(e^{(k)}) - \alpha_k| = \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} |\alpha_{nk} - \alpha_k| = 0 \quad \text{for each } k.
\]

Now it follows from \( A \in (w_0^p, w_{co}) \) and (33) by Proposition 1(c) that \( A \in (w^p, w) \).

Conversely, we assume that \( A \in (w_0^p, w) \), that \( b \in w_{co} \), and that (17) and (19) are satisfied. Then we have \( A \in (w_0^p, w_{co}) \), and so \( L \in \mathcal{B}(w^p, w_{co}) \) by Part (a). Let \( x \in w^p \) be given and let \( \xi \) be the \( w^p \)-limit of \( x \). Then we have \( x^{(0)} = x - \xi \in w_0^p \) and, by (17),

\[
L_n(x) = b_n \xi + \sum_{k=1}^{\infty} \alpha_{nk} x_k = (b_n + A_n x) \xi + A_n x^{(0)}. \tag{34}
\]

Since \( A \in (w_0^p, w) \), the \( w \)-limit of \( \eta_0 \) of \( A x^{(0)} \) exists and we have by (19) and (34)

\[
0 \leq \lim_{m \to \infty} \left( \frac{1}{m} \sum_{n=1}^{m} |L_n(x) - (\beta \xi + \eta_0)| \right) \leq \lim_{m \to \infty} \left( \frac{1}{m} \sum_{n=1}^{m} \left| (b_n + A_n x) \xi - \beta \xi \right| \right) \leq \lim_{m \to \infty} \left( \frac{1}{m} \sum_{n=1}^{m} |A_n x^{(0)} - \eta_0| \right) = 0.
\]

Therefore we have \( L \in \mathcal{B}(w^p, w) \).

(c) The proof of Part (c) is similar to that of Part (b) with \( \beta = 0 \) and \( \alpha_k = 0 \) (\( k = 1, 2, \ldots \)).

\[\square\]

Remark 3. It was shown in the proof of [19, Theorem 3.6] that if \( A \in (w_0^p, w) \) then \( \chi^{(0)} \in \mathcal{M}_p \) with \( \chi_k \) (\( k = 1, 2, \ldots \)) from (33) and in [19, equation (3.14)] that the \( w \)-limit of \( A x^{(0)} \) for any sequence \( x^{(0)} \) in \( w_0^p \) is given by

\[
\eta_0 = \sum_{k=1}^{\infty} \alpha_k x_k^{(0)}. \tag{36}
\]

Let \( L \in \mathcal{B}(w^p, w) \) and \( x \in w^p \) and let \( \xi \) be the \( w^p \)-limit of \( x \); then we obtain by (35) and (36) for the \( w \)-limit of \( L(x) \)

\[
\eta = \beta \cdot \xi + \eta_0 = \beta \cdot \xi + \sum_{k=1}^{\infty} \alpha_k (x_k - \xi) \tag{37}
\]

with \( \beta \) from (19).

4. Compact Operators

In this section, we establish estimates for the Hausdorff measures of noncompactness of linear operators and characterize some classes of compact operators from \( X \) into \( Y \), where \( X = w_0^p, w^p \) and \( Y = w_0^p, w \).

First we recall some useful definitions and results. The reader is referred to the monographs [20–23] for the theory and applications of measures of noncompactness. Let \( X \) and
be Banach spaces and let $L : X \to Y$ be a linear operator. Then $L$ is said to be compact if its domain is all of $X$ and, for every bounded sequence $(x_n)_{n=1}^\infty$ in $X$, the sequence $(L(x_n))_{n=1}^\infty$ has a convergent subsequence in $Y$. We denote the class of such operators by $\mathcal{E}(X, Y)$.

Let $(X, d)$ be a metric space, $B(x, r) = \{ y \in X : d(y, x) < r \}$ denote the open ball of radius $r$ and centre in $x$, and $\mathcal{M}_X$ denote the class of bounded subsets of $M$. Then the map $\chi : \mathcal{M}_X \to [0, \infty)$ with

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subseteq \bigcup_{k=1}^n B(x_k, r_k) \right\},$$

is called the Hausdorff measure of noncompactness.

Let $X$ and $Y$ be Banach spaces and let $X_1$ and $X_2$ be the Hausdorff measures of noncompactness on $X$ and $Y$. Then the operator $L : X \to Y$ is said to be $(X_1, X_2)$-bounded if $L(Q) \subseteq \mathcal{M}_Y$ for every $Q \subseteq \mathcal{M}_X$, and there exists a positive constant $C$ such that $\chi_2(L(Q)) \leq C \cdot \chi_1(Q)$ for every $Q \subseteq \mathcal{M}_X$.

If an operator $L$ is $(X_1, X_2)$-bounded, then

$$\|L\|_{(X_1, X_2)} = \inf \left\{ C > 0 : X_2(L(Q)) \leq C \cdot X_1(Q) \right\}$$

is called the $(X_1, X_2)$-measure of noncompactness of $L$. In particular, if $X_1 = X_2 = X$, then we write $\|L\|_X = \|L\|_{(X, X)}$.

The following useful results are known.

**Proposition 4.** Let $X$ and $Y$ be Banach spaces and $L \in \mathcal{B}(X, Y)$. Then one has

$$\|L\|_X = \chi(L(B_X)) = \chi(L(S_X))$$

(see [23, Theorem 2.25]),

$$L \in \mathcal{C}(X, Y) \iff \|L\|_X = 0$$

(see [23, Corollary 2.26, equation (2.58)]).

We also need the following results which are an immediate consequence of [8, Proposition 3.2 and Lemma 3.5].

**Proposition 5.** (a) Let $P_n : w \to w$ be the projectors onto the linear span of $\{e, e^{(1)}, \ldots, e^{(m)}\}, I : w \to w$ the identity, and $\mathcal{B}_n = I - P_n$ for $n = 0, 1, \ldots$. Then one has for all $Q \subseteq \mathcal{M}_w$

$$\frac{1}{2} \lim_{n \to \infty} \left( \sup_{x \in Q} \|P_n(x)\| \right) \leq \chi(Q) \leq \lim_{n \to \infty} \left( \sup_{x \in Q} \|P_n(x)\| \right).$$

(b) Let $P_n : w_0 \to w_0$ be the projectors onto the linear span of $\{e^{(1)}, e^{(2)}, \ldots, e^{(n)}\}, I : w_0 \to w_0$ the identity, and $\mathcal{B}_n = I - P_n$ for $n = 0, 1, \ldots$. Then one has for all $Q \subseteq \mathcal{M}_{w_0}$

$$\chi(Q) = \lim_{n \to \infty} \left( \sup_{x \in Q} \|P_n(x)\| \right).$$

**Proof.** (a) The inequalities in (42) follow from [8, Proposition 3.2 and Lemma 3.5].

(b) The identity in (42) follows from [8, Proposition 3.2 and Lemma 3.3(a)].

Now we give estimates for the Hausdorff measures of noncompactness of the general operators $L \in \mathcal{B}(w^p, w)$ and $L \in \mathcal{B}(w^p, w_0)$. Let $m, r \in \mathbb{N}$ and $m > r$. Then we write $N(m, r)$ for any subset of the set $\{r + 1, r + 2, \ldots, m\}$.

**Theorem 6.** (a) Let $L \in \mathcal{B}(w^p, w)$. One uses the notations of Theorem 2 and writes $y_n = b_n - \beta + \sum_{k=1}^n \alpha_k n$ for $n = 1, 2, \ldots$ and $C = (c_n)_{n=1}^\infty$ for the matrix with $c_n = a_{nk} - \alpha_k$ for all $n$ and $k$. Then one has

$$\frac{1}{2} \lim_{r \to \infty} \left( \sup_{m} \left( \frac{1}{m} \max_{n \in N(m, r)} \left( \sum_{n \in N(m, r)} \frac{y_n}{C_n} \right) \right) \right) \leq \|L\|_x \leq 4 \cdot \lim_{r \to \infty} \left( \sup_{m} \left( \frac{1}{m} \max_{n \in N(m, r)} \left( \sum_{n \in N(m, r)} \frac{y_n}{C_n} \right) \right) \right).$$

(b) Let $L \in \mathcal{B}(w^p, w_0)$. Then one has

$$\lim_{r \to \infty} \left( \sup_{m} \left( \frac{1}{m} \max_{n \in N(m, r)} \left( \sum_{n \in N(m, r)} \frac{b_n}{C_n} \right) \right) \right) \leq \|L\|_x \leq 4 \cdot \lim_{r \to \infty} \left( \sup_{m} \left( \frac{1}{m} \max_{n \in N(m, r)} \left( \sum_{n \in N(m, r)} \frac{b_n}{C_n} \right) \right) \right).$$

**Proof.** (a) We assume $L \in \mathcal{B}(w^p, w)$. Let $x \in w^p$ be given, $\xi$ be the $w^p$-limit of $x$, and $y = L(x)$. Then we have from Theorem 2(b) and (17) that $y = b \cdot \xi + Ax$, where $A \in (w^p, w)$ and $b \in w_\infty$, and it follows that

$$y_n = b_n \cdot \xi + A_n x = \left( b_n + \sum_{k=1}^m a_{nk} \right) \cdot \xi + A_n (x - \xi \cdot e) \quad \forall n \in \mathbb{N}.$$
\( \mathcal{R}_r(y) = \sum_{n=r+1}^{\infty} (y_n - \eta) e_n \) by (4). Let \( r \in \mathbb{N} \) be given. We write \( f_{n,r}(x) = (\mathcal{R}_r(L(x)))_n \) for all \( n \) and obtain \( f_{n,r}(x) = 0 \) for \( n \leq r \) and, for \( n > r \), from (46) and (37)

\[
f_{n,r}(x) = y_n - \eta = b_n \cdot \xi + A_n x - \left( \tilde{\beta} \cdot \xi + \sum_{k=1}^{\infty} \alpha_k (x_k - \xi) \right)
\]

Since \( f_{n,r} \in (w^p)^* \), we obtain by the same kind of argument as in the proof of (18) that

\[
\sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} y_n \right) + \left\lVert \sum_{n \in N(m,r)} C_n \right\rVert_{\mathcal{M}_p} \right) 
\leq \sup_{x \in S_p, p} \left\lVert \mathcal{R}_r(L(x)) \right\rVert
\]

\[
\leq 4 \cdot m \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} y_n \right) + \left\lVert \sum_{n \in N(m,r)} C_n \right\rVert_{\mathcal{M}_p} \right). \tag{48}
\]

Now the inequalities in (44) follow from (40) and (42).

(b) Now \( \tilde{\beta} = \alpha_k = 0 \) for \( k = 1, 2, \ldots \), and as in the proof of Part (a) we obtain (48) with \( y_n \) and \( C_n \) replaced by \( b_n \) and \( A_n \), respectively, and the inequalities in (45) follow from (40) and (43).

Corollary 7. (a) Let \( L \in \mathcal{B}(w^p_0, w) \). Then \( L \) is given by a matrix \( A \in (w^p_0, w) \) and one has, writing \( C = (c_{nk})_{n,k=1}^{\infty} \) for the matrix with \( c_{nk} = \alpha_k - \alpha_k \) for all \( n \) and \( k \), where \( \alpha_k \) is given by (33),

\[
\frac{1}{2} \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} C_n \right) \right) \right)
\leq \|L\|_X \leq 4 \cdot \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} C_n \right) \right) \right). \tag{49}
\]

(b) Let \( L \in \mathcal{B}(w^p_0, w_0) \). Then one has

\[
\lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} A_n \right) \right) \right)
\leq \|L\|_X \leq 4 \cdot \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} A_n \right) \right) \right). \tag{50}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{To} & w^p_0 & \text{From} & w^p \\
\hline
w_0 & (1) & w^p & (2) \\
w & (3) & w^p & (4) \\
\hline
\end{array}
\]

Proof. (a) The estimates in (49) are easily obtained from [8, Theorem 3.6] with \( \|\cdot\|_{(X,w)} = \|\cdot\|_{(w^p_0,w)} \) defined in (11).

(b) The estimates in (50) follow from those in (45) with \( b_n = 0 \) for all \( n \).

We apply our results and close with the characterizations of the classes \( \mathcal{C}(X,Y) \) for \( X = w^p, w_0^p \) and \( Y = w, w_0 \).

Corollary 8. Let \( L \in \mathcal{B}(X,Y) \). Then the necessary and sufficient conditions for \( L \in \mathcal{C}(X,Y) \) can be read from Table 1 where

\[
(1) \quad \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} A_n \right) \right) \right) = 0,
\]

\[
(2) \quad \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} b_n \right) \right) + \left\lVert \sum_{n \in N(m,r)} A_n \right\rVert_{\mathcal{M}_p} \right) = 0,
\]

\[
(3) \quad \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} C_n \right) \right) \right) = 0,
\]

\[
(4) \quad \lim_{r \to \infty} \left( \sup_m \left( \frac{1}{m \cdot N(m,r)} \left( \sum_{n \in N(m,r)} y_n \right) \right) + \left\lVert \sum_{n \in N(m,r)} C_n \right\rVert_{\mathcal{M}_p} \right) = 0. \tag{51}
\]

Proof. The conditions in (1)–(4) are immediate consequences of (41) and the conditions in (50), (45), (49), and (44), respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References


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