Research Article

Malliavin Derivatives in Spaces with Variable Exponents

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Spaces with variable exponents $L^{p(x)}(H, \mu)$ and $L^{p(x)}(H, \mu; H)$ are introduced. After discussing some approximation results of $L^{p(x)}(H, \mu)$, Sobolev spaces on H with variable exponents are introduced. At last, we define Malliavin derivatives in $L^{p(x)}(H, \mu)$ and discuss some properties of Malliavin derivatives in $L^{p(x)}(H, \mu)$.

1. Introduction

Variable exponent spaces play an important role in the study of some nonlinear problems in natural science and engineering. In decades, there is a rapid development on the subject of variable exponent function spaces. Basic properties of the spaces have been discussed by Kováčik and Rákosník in [1]. Some theories of variable exponent spaces can also be found in [2, 3]. Harjulehto et al. present an overview of applications to differential equations with nonstandard growth in [4]. Diening et al. [5] summarize most of the existing literature of theory of function spaces with variable exponents and applications to partial differential equations. In [6], Aoyama discusses the properties of Lebesgue spaces with variable exponents on a probability space.

Malliavin derivatives have many applications on mathematical finance (see Malliavin and Thalmaier [7] and di Nunno et al. [8]), and we are interested in the behavior of Malliavin derivatives in spaces with variable exponents. In this paper, in Section 2, motivated by [6, 9, 10], we will first introduce $L^{p(x)}(H, \mu)$ and $L^{p(x)}(H, \mu; H)$ and give some approximation results of $L^{p(x)}(H, \mu)$, which are useful in the definition on Malliavin derivatives in the following parts. In Section 3 we discuss gradients in $L^{p(x)}(H, \mu)$ and give properties of the linear operator D . At the end of this section, we define variable exponent Sobolev spaces on H . After the above preparation, we give Malliavin derivatives

in $L^{p(x)}(H, \mu)$ and discuss some properties of the Malliavin derivative operator M in $L^{p(x)}(H, \mu)$ in the last section.

2. Preliminaries

Let H be a separable Hilbert space. A Borel probability measure $N_{m,Q}$ in H with mean m and compact covariance operator Q is called Gaussian measure if the Fourier transform satisfies

$$\int_H e^{i\langle h, x \rangle} N_{m,Q}(dx) = e^{i\langle m, h \rangle} e^{(-1/2)\langle Qh, h \rangle}, \quad \forall x \in H. \quad (1)$$

$N_{m,Q}$ is called nondegenerate if $\text{Ker} Q = \{0\}$. We are given a nondegenerate Gaussian measure $\mu = N_{0,Q}$ in the Hilbert space H . Since the operator Q is compact, there exists a complete orthonormal system $\{e_k\}$ in H and a sequence $\{\lambda_k\}$ of positive numbers such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}. \quad (2)$$

We denote by $C_b(H)$ the space of all mappings $\phi : H \rightarrow \mathbb{R}$, which are both continuous and bounded. $C_b(H)$ is a Banach space with the norm $\|\phi\|_0 = \sup_{x \in H} |\phi(x)|$. And denote by $C_b^k(H)$, $k \in \mathbb{N}$, the space of all mappings $\phi : H \rightarrow \mathbb{R}$ which are continuous and bounded together with their derivatives of order not bigger than k .

Given a variable exponent $p : H \rightarrow [1, \infty]$, it is assumed to be a Borel measurable function. On the set of all

Borel measurable functions, the moduli ρ and $\bar{\rho}$ are defined, respectively, by

$$\begin{aligned}\rho(\varphi) &= \int_{H \setminus H_\infty} |\varphi(x)|^{p(x)} \mu(dx) + \operatorname{ess\,sup}_{x \in H_\infty} |\varphi(x)|, \\ \bar{\rho}(F) &= \int_{H \setminus H_\infty} |F(x)|_H^{p(x)} \mu(dx) + \operatorname{ess\,sup}_{x \in H_\infty} |F(x)|_H,\end{aligned}\quad (3)$$

where $H_\infty = \{x : p(x) = \infty\}$, $\varphi : H \rightarrow \mathbb{R}$, and $F : H \rightarrow H$.

Definition 1. The space $\mathcal{L}^{p(x)}(H, \mu)$ is the set of Borel measurable functions $\varphi : H \rightarrow \mathbb{R}$ such that $\rho(\varphi) < \infty$, and it is endowed with the Luxemburg norm:

$$\|\varphi\|_{L^{p(x)}(H, \mu)} = \inf \left\{ \lambda > 0 : \rho\left(\frac{\varphi}{\lambda}\right) \leq 1 \right\}. \quad (4)$$

Define the equivalence relation $r_1 : \varphi \sim \psi$ if and only if $\|\varphi - \psi\|_{L^{p(x)}(H, \mu)} = 0$. Denote by $L^{p(x)}(H, \mu)$ the quotient $\mathcal{L}^{p(x)}(H, \mu)$ with respect to the equivalence relation r_1 .

Definition 2. The space $\mathcal{L}^{p(x)}(H, \mu; H)$ is the set of Borel measurable functions $F : H \rightarrow H$ such that $\bar{\rho}(F) < \infty$, and it is endowed with the Luxemburg norm:

$$\|F\|_{L^{p(x)}(H, \mu; H)} = \inf \left\{ \lambda > 0 : \bar{\rho}\left(\frac{F}{\lambda}\right) \leq 1 \right\}. \quad (5)$$

Define the equivalence relation $r_2 : F \sim G$ if and only if $\|F - G\|_{L^{p(x)}(H, \mu; H)} = 0$. Denote by $L^{p(x)}(H, \mu; H)$ the quotient $\mathcal{L}^{p(x)}(H, \mu; H)$ with respect to the equivalence relation r_2 .

Proposition 3 (see Lemma 3.2.20 in [5]). *Let variable exponents $p(x), q(x), r(x) \in [1, \infty]$ be such that $1/r(x) = (1/p(x)) + (1/q(x))$, μ -a.e.; then the inequality*

$$\|\varphi\psi\|_{L^{r(x)}(H, \mu)} \leq 2\|\varphi\|_{L^{p(x)}(H, \mu)}\|\psi\|_{L^{q(x)}(H, \mu)} \quad (6)$$

holds for every $\varphi \in L^{p(x)}(H, \mu)$ and $\psi \in L^{q(x)}(H, \mu)$, where in the case $r = p = q = \infty$ we use the convention $r/p = r/q = 1$.

Proposition 4. *The variable exponent p satisfies $1 \leq p^- \leq p(x) \leq p^+ < \infty$, μ -a.e. Let $F \in L^{p(x)}(H, \mu; H)$ and set $F_k(x) = \langle F(x), e_k \rangle$; $k \in \mathbb{N}$. Then $F_k \in L^{p(x)}(H, \mu)$ for all $k \in \mathbb{N}$ and*

$$F(x) = \sum_{k=1}^{\infty} F_k(x) e_k, \quad \mu\text{-a.e.}, \quad (7)$$

where the series is convergent in $L^{p(x)}(H, \mu; H)$.

Proof. For $F_k, k \in \mathbb{N}$,

$$\int_H |F_k(x)|^{p(x)} \mu(dx) \leq \int_H |F(x)|_H^{p(x)} \mu(dx) < \infty, \quad (8)$$

so $F_k \in L^{p(x)}(H, \mu)$. Since $\{e_k\}$ is a complete orthonormal system in H and $F(x) \in H$ for $x \in H$, we have

$$F(x) = \sum_{k=1}^{\infty} F_k(x) e_k, \quad \mu\text{-a.e.} \quad (9)$$

Set $P_n F(x) = \sum_{k=1}^n F_k(x) e_k$, and we have

$$\begin{aligned}\lim_{n \rightarrow \infty} P_n F(x) &= F(x), \quad \mu\text{-a.e.}, \\ |F(x)|_H^{p(x)} &= \left(\sum_{k=1}^{\infty} |F_k(x)|^2 \right)^{p(x)/2} \geq \left(\sum_{k=n+1}^{\infty} |F_k(x)|^2 \right)^{p(x)/2} \\ &= |P_n F(x) - F(x)|_H^{p(x)},\end{aligned}\quad (10)$$

so by Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n \rightarrow \infty} \int_H |P_n F(x) - F(x)|_H^{p(x)} \mu(dx) = 0. \quad (11)$$

Thus, the series are convergent in $L^{p(x)}(H, \mu; H)$. The proof is completed. $\square$

Denote the linear span of all real and imaginary parts functions $\varphi_h(x) = e^{i\langle h, x \rangle}$, $x \in H$, and $h \in H$, by $\mathcal{E}(H)$.

Proposition 5 (see Lemma 2.2 in [9]). *For all $\varphi \in C_b(H)$ there exists a two-index sequence $\{\varphi_{n,k}\} \subset \mathcal{E}(H)$ such that $\|\varphi_{n,k}\|_0 \leq \|\varphi\|_0$, $n, k \in \mathbb{N}$, and*

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \varphi_{n,k}(x) = \varphi(x), \quad x \in H. \quad (12)$$

$C_c(H)$ is the space of all mappings $\varphi : H \rightarrow \mathbb{R}$ which are continuous and have compact support in H . We have the following proposition about $C_c(H)$.

Proposition 6. *If the variable exponent p satisfies $1 \leq p^- \leq p(x) \leq p^+ < \infty$, μ -a.e., then $C_c(H)$ is dense in $L^{p(x)}(H, \mu)$.*

Proof. Note that $S := S(H, \mu)$ is the set of all the simple functions. It is easy to check that $S \subset L^{p(x)}(H, \mu)$. Let $f \in L^{p(x)}(H, \mu)$ with $f \geq 0$. Since f is a Borel measurable function, there exist $\{f_n\} \subset S$ with $0 \leq f_n \nearrow f$, μ -a.e., and $|f_n(x) - f(x)|^{p(x)} \leq |f(x)|^{p(x)}$, so by Lebesgue's Dominated Convergence Theorem, we have $f_n \rightarrow f$ in $L^{p(x)}(H, \mu)$. Thus, f is in the closure of S . If we drop the assumption $f \geq 0$, then we split f into positive and negative parts which belong to the closure of S . Thus S is dense in $L^{p(x)}(H, \mu)$. For any $\varepsilon > 0$ and $g \in L^{p(x)}(H, \mu)$, there exists $s \in S$ such that $\|g - s\|_{L^{p(x)}(H, \mu)} < \varepsilon/2$.

On the other hand, for $p^+ < \infty$, $C_c(H)$ is dense in $L^{p^+}(H, \mu)$. So there exists $\tilde{g} \in C_c(H)$ such that $\|s - \tilde{g}\|_{L^{p^+}(H, \mu)} < \varepsilon/4$. By Proposition 3, we have

$$\|s - \tilde{g}\|_{L^{p(x)}(H, \mu)} \leq 2\|s - \tilde{g}\|_{L^{p^+}(H, \mu)} < \frac{\varepsilon}{2}. \quad (13)$$

Thus, $\|g - \tilde{g}\|_{L^{p(x)}(H, \mu)} \leq \varepsilon$ and $C_c(H)$ is dense in $L^{p(x)}(H, \mu)$. The proof is completed. $\square$

Proposition 7. *If the variable exponent p satisfies $1 \leq p^- \leq p(x) \leq p^+ < \infty$, μ -a.e., then $\mathcal{E}(H)$ is dense in $L^{p(x)}(H, \mu)$.*

Proof. Since $C_c(H) \subset C_b(H)$, by Proposition 6 $C_b(H)$ is dense in $L^{p(x)}(H, \mu)$. So for any $\varepsilon > 0$ and $f \in L^{p(x)}(H, \mu)$, there exists $\varphi \in C_b(H)$ such that

$$\|f - \varphi\|_{L^{p(x)}(H, \mu)} < \frac{\varepsilon}{2}. \quad (14)$$

By Proposition 5 and diagonal rule, there exists a sequence $\{\varphi_n\} \subset \mathcal{E}(H)$ such that $|\varphi_n(x)| \leq \|\varphi\|_0$, $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x), \quad x \in H. \quad (15)$$

By Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n \rightarrow \infty} \int_H |\varphi_n(x) - \varphi(x)| \mu(dx) = 0. \quad (16)$$

And as

$$\begin{aligned} & \int_H |\varphi_n(x) - \varphi(x)|^{p(x)} \mu(dx) \\ & \leq (2\|\varphi\|_0)^{p^+} \int_H |\varphi_n(x) - \varphi(x)| \mu(dx), \end{aligned} \quad (17)$$

we have $\varphi_n \rightarrow \varphi$ in $L^{p(x)}(H, \mu)$. Suppose that $N \in \mathbb{N}$, $\|\varphi_n - \varphi\|_{L^{p(x)}(H, \mu)} < \varepsilon/2$ for $n > N$, and we have $\|\varphi_n - f\|_{L^{p(x)}(H, \mu)} < \varepsilon$ for $n > N$. Thus, $\mathcal{E}(H)$ is dense in $L^{p(x)}(H, \mu)$. $\square$

Proposition 8. Suppose that W_z , $z \in H$, is a white noise function on H . If the variable exponent p satisfies $1 \leq p^- \leq p(x) \leq p^+ < \infty$, μ -a.e., then

$$\|W_z\|_{L^{p(x)}(H, \mu)} \leq 2 \left(\frac{(2m)!}{2^m m!} |z|_H^m \right)^{1/2m}, \quad (18)$$

where $p^+ < 2m$ and $m \in \mathbb{N}$.

Proof. Since W_z is a Gaussian random variable with mean 0 and covariance $|z|_H$, by Proposition 3, we have

$$\begin{aligned} \|W_z\|_{L^{p(x)}(H, \mu)} & \leq 2 \|1\|_{L^{(2mp(x))/(2m-p(x))}(H, \mu)} \|W_z\|_{L^{2m}(H, \mu)} \\ & = 2 \left(\frac{(2m)!}{2^m m!} |z|_H^m \right)^{1/2m}. \end{aligned} \quad (19)$$

The proof is completed. $\square$

3. Variable Exponent Sobolev Spaces on Separable Hilbert Space H

For any $\varphi \in \mathcal{E}(H)$ and $k \in \mathbb{N}$, we denote by $D_k \varphi$ its derivative in the direction of e_k ; that is,

$$D_k \varphi(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\varphi(x + \varepsilon e_k) - \varphi(x)), \quad x \in H. \quad (20)$$

Denote the gradient of φ by $D\varphi$.

We will consider the following linear mapping:

$$D : \mathcal{E}(H) \subset L^{p(x)}(H, \mu) \longrightarrow L^{p(x)}(H, \mu; H), \quad \varphi \longmapsto D\varphi, \quad (21)$$

where $1 \leq p^- \leq p(x) \leq p^+ < \infty$, μ -a.e.

Lemma 9 (see Lemma 2.6 in [9]). Suppose $\varphi, \psi \in \mathcal{E}(H)$; then

$$\int_H D_k \varphi \psi d\mu = - \int_H \varphi D_k \psi d\mu + \frac{1}{\lambda_k} \int_H x_k \varphi \psi d\mu, \quad (22)$$

where $x \in H$ and $x_k = \langle x, e_k \rangle$.

Lemma 10 (see Corollary 2.7 in [9]). Suppose $\varphi, \psi \in \mathcal{E}(H)$ and $z \in Q^{1/2}(H)$; then

$$\int_H \langle D\varphi, z \rangle \psi d\mu = - \int_H \langle D\psi, z \rangle \varphi d\mu + \int_H W_{Q^{-1/2}z} \varphi \psi d\mu. \quad (23)$$

Given a linear operator A , not necessarily closed, if the closure of its graph happens to be the graph of some operator, that operator is called the closure of A , and we say that A is closable. Denote the closure of A by $\overline{A}$. And A is closable if and only if for any sequence $\{x_n\} \subset D(A)$ such that

$$\lim_{n \rightarrow \infty} x_n = 0, \quad \lim_{n \rightarrow \infty} Ax_n = y, \quad (24)$$

one has $y = 0$.

Theorem 11. The mapping D is a closable linear operator.

Proof. Let $\{\varphi_n\} \subset \mathcal{E}(H)$ such that

$$\begin{aligned} \varphi_n & \longrightarrow 0 \quad \text{in } L^{p(x)}(H, \mu), \\ D\varphi_n & \longrightarrow F \quad \text{in } L^{p(x)}(H, \mu; H). \end{aligned} \quad (25)$$

By the definition of closable operators, we only need to prove that $F = 0$.

For any $\psi \in \mathcal{E}(H)$ and $z \in Q^{1/2}(H)$, by Lemma 10, we have

$$\int_H \langle D\varphi_n, z \rangle \psi d\mu = - \int_H \langle D\psi, z \rangle \varphi_n d\mu + \int_H W_{Q^{-1/2}z} \varphi_n \psi d\mu. \quad (26)$$

By Proposition 3, we have

$$\begin{aligned} \left| \int_H \langle D\varphi_n - F, z \rangle \psi d\mu \right| & \leq \int_H |D\varphi_n - F|_H |z|_H |\psi|_H d\mu \\ & \leq 2|z|_H \|D\varphi_n - F\|_{L^{p(x)}(H, \mu; H)} \\ & \quad \times \|\psi\|_{L^{q(x)}(H, \mu)}, \end{aligned}$$

$$\left| \int_H \langle D\psi, z \rangle \varphi_n d\mu \right| \leq 2|z|_H \|\varphi_n\|_{L^{p(x)}(H, \mu)} \|D\psi\|_{L^{q(x)}(H, \mu; H)}, \quad (27)$$

where $(1/p(x)) + (1/q(x)) = 1$; $x \in H$. Since ψ is bounded, by Propositions 3 and 8, we have

$$\begin{aligned} \left| \int_H W_{Q^{-1/2}z} \varphi_n \psi d\mu \right| & \leq 2C \|\varphi_n\|_{L^{p(x)}(H, \mu)} \|W_{Q^{-1/2}z}\|_{L^{q(x)}(H, \mu)} \\ & \leq C_{p^+} |Q^{-1/2}z|_H^{1/2} \|\varphi_n\|_{L^{p(x)}(H, \mu)}. \end{aligned} \quad (28)$$

Thus, $\int_H \langle F, z \rangle \psi d\mu = 0$, as $n \rightarrow \infty$. Since $F_k \in L^{p(x)}(H, \mu)$, for a fixed $k \in \mathbb{N}$, suppose $\{\psi_{kn}\} \subset \mathcal{E}(H)$ such that $\psi_{kn} \rightarrow |F_k|^{p(x)-1} \text{sgn}(F_k)$ in $L^{q(x)}(H, \mu)$. We have

$$\lim_{n \rightarrow \infty} \int_H F_k \psi_{kn} d\mu = 0; \quad (29)$$

that is, $\int_H |F_k|^{p(x)} d\mu = 0$. Thus, we have $F_k(x) = 0$, μ -a.e., and $F(x) = 0$, μ -a.e. The proof is completed. $\square$

We denote by $\bar{D}$ the closable operator of D and by $W^{1,p(x)}(H, \mu)$ the domain of $\bar{D}$.

Proposition 12. *For any $k \in \mathbb{N}$, the linear operator D_k is closable and its closure $\bar{D}_k$ satisfies $\bar{D}_k \varphi = \langle \bar{D} \varphi, e_k \rangle$ for $\varphi \in W^{1,p(x)}(H, \mu)$.*

Proof. For any $k \in \mathbb{N}$, let $\{\varphi_n\} \subset \mathcal{E}(H)$ such that

$$\begin{aligned} \varphi_n &\rightarrow 0 \text{ in } L^{p(x)}(H, \mu), \\ D_k \varphi_n &\rightarrow F_k \text{ in } L^{p(x)}(H, \mu). \end{aligned} \quad (30)$$

By the definition of closable operators, we only need to prove that $F_k = 0$. For any $\psi \in \mathcal{E}(H)$ and $z \in H$, by Lemma 9, we have

$$\int_H D_k \varphi_n \psi d\mu = - \int_H \varphi_n D_k \psi d\mu + \frac{1}{\lambda_k} \int_H x_k \varphi_n \psi d\mu. \quad (31)$$

Similar to the proof of Theorem 11, we have $\int_H F_k \psi d\mu = 0$ as $n \rightarrow \infty$ and $F_k = 0$, μ -a.e. Thus D_k is closable. And

$$\bar{D}_k \varphi = \lim_{n \rightarrow \infty} D_k \varphi_n = \lim_{n \rightarrow \infty} \langle D \varphi_n, e_k \rangle = \langle \bar{D} \varphi, e_k \rangle. \quad (32)$$

The proof is completed. $\square$

4. Malliavin Derivatives in $D^{1,p(x)}(H, \mu)$

For any $\varphi \in \mathcal{E}(H)$, define the linear operator:

$$M_0 : \mathcal{E}(H) \subset L^{p(x)}(H, \mu) \rightarrow L^{p(x)}(H, \mu; H), \quad (33)$$

$$M_0 \varphi(x) = Q^{1/2} D \varphi(x), \quad \varphi \in \mathcal{E}(H), \quad x \in H,$$

where $1 \leq p^- \leq p(x) \leq p^+ < \infty$, μ -a.e. And

$$M_k \varphi(x) = \langle M_0 \varphi(x), e_k \rangle = \lambda_k^{1/2} D_k \varphi(x), \quad (34)$$

where $\varphi \in \mathcal{E}(H)$ and $x \in H$.

Proposition 13 (see Corollary 2.10 in [9]). *Suppose $\varphi, \psi \in \mathcal{E}(H)$ and $z \in H$; then the following identity holds:*

$$\int_H \langle M_0 \varphi, z \rangle \psi d\mu = - \int_H \langle M_0 \psi, z \rangle \varphi d\mu + \int_H W_z \varphi \psi d\mu. \quad (35)$$

In a similar way to Theorem 11, one has Theorem 14.

Theorem 14. *The mapping M_0 is a closable linear operator.*

Proof. Let $\{\varphi_n\} \subset \mathcal{E}(H)$ such that

$$\varphi_n \rightarrow 0 \text{ in } L^{p(x)}(H, \mu), \quad (36)$$

$$M_0 \varphi_n \rightarrow F \text{ in } L^{p(x)}(H, \mu; H).$$

We only need to prove that $F = 0$.

For any $\psi \in \mathcal{E}(H)$ and $z \in H$, by Proposition 13, we have

$$\int_H \langle M_0 \varphi_n, z \rangle \psi d\mu = - \int_H \langle M_0 \psi, z \rangle \varphi_n d\mu + \int_H W_z \varphi_n \psi d\mu. \quad (37)$$

And we have $\int_H \langle F, z \rangle \psi d\mu = 0$, as $n \rightarrow \infty$. Similar to Theorem 11, we have $F(x) = 0$, μ -a.e. The proof is completed. $\square$

We denote by M the closable operator of M_0 and by $D^{1,p(x)}(H, \mu)$ the domain of M . For any $\varphi \in D^{1,p(x)}(H, \mu)$, we call $M\varphi$ the Malliavin derivative of φ .

Proposition 15. *If p satisfies $1 \leq p(x) \leq p^+ < \infty$, μ -a.e., then $W^{1,p(x)}(H, \mu) \subset D^{1,p(x)}(H, \mu)$ and $M\varphi = Q^{1/2} \bar{D} \varphi$ for any $\varphi \in W^{1,p(x)}(H, \mu)$.*

Proof. For any $\varphi \in W^{1,p(x)}(H, \mu)$, there exists $\{\varphi_n\} \subset \mathcal{E}(H)$ such that $\varphi_n \rightarrow \varphi$ in $L^{p(x)}(H, \mu)$. As

$$\begin{aligned} &\int_H |Q^{1/2} D \varphi_n(x) - Q^{1/2} \bar{D} \varphi(x)|_H^{p(x)} \mu(dx) \\ &\leq \int_H \|Q^{1/2}\|^{p(x)} |D \varphi_n(x) - \bar{D} \varphi(x)|_H^{p(x)} \mu(dx) \\ &\leq \max \left\{ \|Q^{1/2}\|, \|Q^{1/2}\|^{p^+} \right\} \int_H |D \varphi_n(x) - \bar{D} \varphi(x)|_H^{p(x)} \mu(dx) \end{aligned} \quad (38)$$

we have $M\varphi = Q^{1/2} \bar{D} \varphi$ and $\varphi \in D^{1,p(x)}(H, \mu)$. $\square$

To prove Proposition 17, we need the following lemma.

Lemma 16 (see Lemma 2.3 in [9]). *For all $\varphi \in C_b^1(H)$ there exists a two-index sequence $\{\varphi_{n,k}\} \subset \mathcal{E}(H)$ such that $\|\varphi_{n,k}\|_0 + \|D\varphi_{n,k}\|_0 \leq \|\varphi\|_0 + \|D\varphi\|_0$, $n, k \in \mathbb{N}$, and*

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \varphi_{n,k}(x) &= \varphi(x), \quad x \in H, \\ \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} D\varphi_{n,k}(x) &= D\varphi(x), \quad x \in H. \end{aligned} \quad (39)$$

Proposition 17. *If p satisfies $1 \leq p(x) \leq p^+ < \infty$, μ -a.e., then one has $g(\varphi) \in D^{1,p(x)}(H, \mu)$ and $Mg(\varphi) = g'(\varphi)M\varphi$ for any $\varphi \in D^{1,p(x)}(H, \mu)$ and $g \in C_b^1(H)$.*

Proof. First, we prove that $g(\psi) \in D^{1,p(x)}(H, \mu)$ for any $\psi \in \mathcal{E}(H)$. By Lemma 16, for $g(\psi)$, there exists a two-index sequence $\{\phi_{n,k}\} \subset \mathcal{E}(H)$ such that $\|\phi_{n,k}\|_0 + \|D\phi_{n,k}\|_0 \leq \|g(\psi)\|_0 + \|Dg(\psi)\|_0$, $n, k \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \phi_{n,k}(x) = g(\psi)(x), \quad x \in H, \quad (40)$$

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} D\phi_{n,k}(x) = Dg(\psi)(x), \quad x \in H.$$

By diagonal rule and Lebesgue's Dominated Convergence Theorem, $\phi_n \rightarrow g(\psi)$ in $L^{p(x)}(H, \mu)$. Since M is closed, we have $Mg(\psi) = \lim_{n \rightarrow \infty} M_0 \phi_n$ and $g(\psi) \in D^{1,p(x)}(H, \mu)$.

Now we will prove the proposition. For $\varphi \in D^{1,p(x)}(H, \mu)$, there exists $\{\varphi_n\} \subset \mathcal{E}(H)$ such that

$$\lim_{n \rightarrow \infty} \varphi_n = \varphi \text{ in } L^{p(x)}(H, \mu). \quad (41)$$

By the first part of this proof, $g(\varphi_n) \in D^{1,p(x)}(H, \mu)$. And

$$Mg(\varphi_n) = Q^{1/2} Dg(\varphi_n) = g'(\varphi_n) Q^{1/2} D\varphi_n = g'(\varphi_n) M\varphi_n. \quad (42)$$

Since $M\varphi_n \in L^{p(x)}(H, \mu; H)$ and $g'(\varphi_n) \in C_b(H)$, we have

$$\lim_{n \rightarrow \infty} Mg(\varphi_n) = \lim_{n \rightarrow \infty} g'(\varphi_n) M\varphi_n = g'(\varphi) M\varphi. \quad (43)$$

As M is closed, $Mg(\varphi) = g'(\varphi) M\varphi$. The proof is completed. $\square$

Proposition 18. Assume that p satisfies $1 \leq p(x) \leq p^+ < \infty$, μ -a.e. For $\varphi, \psi \in D^{1,p(x)}(H, \mu)$; suppose ψ and $M\psi$ are bounded; then $\varphi\psi \in D^{1,p(x)}(H, \mu)$ and $M(\varphi\psi) = \psi M\varphi + \varphi M\psi$.

Proof. First, for $\varphi \in \mathcal{E}(H)$, as $\psi \in D^{1,p(x)}(H, \mu)$, there exists $\{\psi_n\} \subset \mathcal{E}(H)$ such that $\psi_n \rightarrow \psi$ in $L^{p(x)}(H, \mu)$. And since

$$M_0(\varphi\psi_n) = \psi_n M_0\varphi + \varphi M_0\psi_n, \quad (44)$$

we have

$$\lim_{n \rightarrow \infty} M_0(\varphi\psi_n) = \lim_{n \rightarrow \infty} (\psi_n M_0\varphi + \varphi M_0\psi_n) = \psi M_0\varphi + \varphi M\psi. \quad (45)$$

As $\varphi\psi_n \in \mathcal{E}(H)$, $\varphi\psi_n \rightarrow \varphi\psi$ in $L^{p(x)}(H, \mu)$, and $M(\varphi\psi) = \lim_{n \rightarrow \infty} M_0(\varphi\psi_n)$ in $L^{p(x)}(H, \mu; H)$, we have $\varphi\psi \in D^{1,p(x)}(H, \mu)$.

Secondly, for $\varphi \in D^{1,p(x)}(H, \mu)$, there exists $\{\varphi_n\} \subset \mathcal{E}(H)$ such that $\varphi_n \rightarrow \varphi$ in $L^{p(x)}(H, \mu)$. By the first part of the proof,

$$M(\varphi_n\psi) = \psi M_0\varphi_n + \varphi_n M\psi. \quad (46)$$

Since ψ and $M\psi$ are bounded, we have

$$M(\varphi\psi) = \lim_{n \rightarrow \infty} M_0(\varphi_n\psi) = \psi M\varphi + \varphi M\psi, \quad (47)$$

and $\varphi\psi \in D^{1,p(x)}(H, \mu)$. The proof is completed. $\square$

5. Conclusion

In this work some new results on Malliavin derivatives are given. Malliavin derivatives are extended from $D^{1,2}(H, \mu)$ to variable exponent spaces and some properties of Malliavin derivatives in variable exponent spaces are obtained. We introduce $L^{p(x)}(H, \mu)$ and $L^{p(x)}(H, \mu; H)$, and this work generalizes classical variable exponents spaces based on Kováčik and Rákosník [1]. Then, we prove that $\mathcal{E}(H)$ is dense in $L^{p(x)}(H, \mu)$ and by this approximation result we give extension of Malliavin derivatives which are given by P. Malliavin [11]. Some properties of Malliavin derivatives in $D^{1,2}(H, \mu)$ are also discussed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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