Research Article

Estimates of L_p Modulus of Continuity of Generalized Bounded Variation Classes

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Some sharp estimates of the L_p ($1 \leq p < \infty$) modulus of continuity of classes of Λ_φ -bounded variation are obtained. As direct applications, we obtain estimates of order of Fourier coefficients of functions of Λ_φ -bounded variation, and we also characterize some sufficient and necessary conditions for the embedding relations $H_p^\omega \subset \Lambda_\varphi BV$. Our results include the corresponding known results of the class ΛBV as a special case.

1. Introduction and Main Results

To generalize the notion of functions of bounded variation, Wiener [1] introduced the class BV_β ($\beta > 1$) of functions of β -bounded variation. Young [2] introduced the notion of functions of φ -bounded variation, and Waterman [3] studied a class of Λ -bounded variations. Combining the notion of Λ -bounded variation with that of φ -bounded variation, Leindler [4] introduced the class $\Lambda_\varphi BV$ of functions of Λ_φ -bounded variation, and both classes of Λ -bounded variation and φ -bounded variation are its special cases. Actually the class $\Lambda_\varphi BV$ first appeared in Schramm and Waterman's paper [5], and some restrictions are imposed on φ in their definition. Here we adopt Leindler's definition.

Definition 1. Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a nondecreasing function with $\varphi(0) = 0$, and let $\Lambda = \{\lambda_k\}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty} (1/\lambda_k) = +\infty$. Let Γ be the set of all sequences of nonoverlapping subintervals $[a_k, b_k]$ in $[a, b]$. If for any $\Delta = \{(a_k, b_k) \subset [a, b], k = 1, 2, \dots, n, n \in \mathbb{Z}^+\} \in \Gamma$, a real valued function $f : [a, b] \rightarrow \mathbb{R}$ satisfies the condition

$$\sum_{k=1}^n \frac{\varphi(|f(b_k) - f(a_k)|)}{\lambda_k} < \infty, \quad (1)$$

then f is said to be of Λ_φ -bounded variation, and this fact is denoted by $f \in \Lambda_\varphi BV$. And the quantity

$$V_{\Lambda_\varphi}(f; [a, b]) := \sup_{\Delta \in \Gamma} \left\{ \sum_{k=1}^n \frac{\varphi(|f(b_k) - f(a_k)|)}{\lambda_k} \right\} \quad (2)$$

is said to be Λ_φ -total variation of f .

In the special case when $\varphi(x) = x^\beta$ ($\beta \geq 1$), f is said to be of Λ_β -bounded variation, and we write $f \in \Lambda_\beta BV$ and $V_{\Lambda_\beta}(f; [a, b]) = V_{\Lambda_\varphi}(f; [a, b])$, and if $\beta = 1$, f is said to be of Λ -bounded variation, and we denote $f \in \Lambda BV$ and $V_\Lambda(f; [a, b]) = V_{\Lambda_1}(f; [a, b])$.

In the case $\Lambda = \{1\}$, we get the class of φ -bounded variation, and f is said to be of φ -bounded variation, and we denote $V_\varphi(f; [a, b]) = V_\Lambda(f; [a, b])$. More specifically, when $\varphi(x) = x^\beta$ ($\beta \geq 1$), we say that f is of β -bounded variation, and we denote $f \in BV_\beta$ and $V_\beta(f; [a, b]) = V_\varphi(f; [a, b])$. The class BV_β is also called the Wiener class and BV_1 is the well-known class of bounded variation BV .

It is easily seen from the definition that $\Lambda_\varphi BV$ functions are bounded; that is, $\Lambda_\varphi BV[a, b] \subseteq B[a, b]$, and the discontinuities of a $\Lambda_\varphi BV$ function are simple and therefore at most denumerable, where $B[a, b]$ denotes the class of bounded real valued functions on $[a, b]$.

Let $\Lambda = \{\lambda_k\}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty} (1/\lambda_k) = +\infty$. If a continuous

and nondecreasing function $\lambda(s)$ on $[0, \infty)$ such that $\lambda(s) \equiv \lambda_1$, $0 \leq s \leq 1$ and $\lambda(k) = \lambda_k$, $k = 1, 2, \dots$, then we say that $\lambda(s)$ generates Λ . By the nondecreasing property of Λ , it is easily verified that if $\lambda(s)$ generates Λ , then

$$\frac{1}{2} \int_0^n \frac{ds}{\lambda(s)} \leq \sum_{k=1}^n \frac{1}{\lambda_k} \leq \int_0^n \frac{ds}{\lambda(s)}, \quad n \geq 1. \quad (3)$$

Let $\lambda(x)$ be a nonnegative real-valued function on $[0, \infty)$. If there exists $a > 0$ such that $\lambda(2x) \leq a\lambda(x)$, $x \in (0, A)$ for some positive constant A , then we say that $\lambda(x)$ satisfies the condition Δ_2 . If $\lambda(x)$ generates $\Lambda = \{\lambda_k\}$ and satisfies the condition Δ_2 , we say that Λ satisfies the condition Δ_2 . Obviously the condition Δ_2 here is a very weak restriction on $\lambda(x)$ and Λ .

Let $\omega(t)$ be a modulus of continuity, that is, a continuous and nondecreasing function on $[0, +\infty)$ satisfying $\omega(0) = 0$ and $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$ for nonnegative t_1 and t_2 . As usual, for $1 \leq p \leq \infty$, denote by $H_p^\omega \equiv H_p^{\omega(t)}$ the class of functions for which $\|f\|_{H_p^\omega} := \sup_{t>0} (\omega(f, t)_p / \omega(t)) < \infty$, where

$$\omega(f; t)_p := \begin{cases} \sup_{0 \leq h \leq t} \left\{ \int_a^b |f(x+h) - f(x)|^p dx \right\}^{1/p}, & 1 \leq p < \infty, \\ \sup_{0 \leq h \leq t} \sup_{x \in [a, b]} |f(x+h) - f(x)|, & p = \infty \end{cases} \quad (4)$$

is the L_p modulus of continuity of f . We write H^ω instead of H_∞^ω and H_p^α ($0 < \alpha \leq 1$) instead of $H_p^{\omega_p^\alpha} \equiv \text{Lip}(\alpha, p)$, the Lipschitz class, for brevity.

Functions of classes BV_β , φBV , ΛBV , and $\Lambda_\varphi BV$ are considered in trigonometric Fourier series and some of them share good approximative properties (see [1–3, 6–11], etc.). What we mention here is the following theorem proved by Shiba [12], Schramm and Waterman [5], and Wang [13]:

Theorem A. (a) If $f \in \Lambda_\beta BV$, $1 \leq \beta < 2r$, $1 \leq r < \infty$, and

$$\frac{\sum_{n=1}^{\infty} \left(\sum_{k=1}^n (1/\lambda_k) \right)^{-1/2r} (\omega_{\beta+(2-\beta)s}(f; \pi/n))^{1-\beta/2r}}{n^{1/2}} < \infty, \quad (5)$$

where $1/r + 1/s = 1$, then the Fourier series of f converges absolutely.

(b) If φ is Δ_2 , $f \in \Lambda_\varphi BV$, $1 \leq \beta < 2r$, $1 \leq r < \infty$, and

$$\frac{\sum_{n=1}^{\infty} \left[\varphi^{-1} \left(\left(\sum_{k=1}^n \left(\frac{1}{\lambda_k} \right) \right)^{-1} \omega_{\beta+(2-\beta)s} \left(f; \frac{\pi}{n} \right) \right) \right]^{1/2r}}{n^{1/2}} < \infty, \quad (6)$$

where $1/r + 1/s = 1$, then the Fourier series of f converges absolutely.

Embedding relations between various generalized bounded variation classes and the class H_p^ω (or the Lipschitz class H_p^α) are also investigated in recent years. It is well known

that $H_p^1 \subset H_1^1 = BV \subset \Lambda BV$. For $f \in \Lambda BV$, the estimates of L_p modulus of continuity of f had been given in [13] for $p = 1$ and in [9, 14] for $1 < p < \infty$. Furthermore, Goginava in [15] and Li and Wang in [9] proved that, for $1 \leq p < \infty$, $0 < \alpha, \delta \leq 1$,

$$\{n^\delta\} BV \subset H_p^\alpha \quad \text{iff } \alpha \leq \min \left\{ \frac{1}{p}, 1 - \delta \right\}. \quad (7)$$

For more detailed results on this topic, we refer readers to [4, 9, 10, 14–25].

In this paper, we obtain some sharp estimates of L_p ($1 \leq p < \infty$) modulus of continuity of the classes $\Lambda_\varphi BV$ in the case of that φ is convex. More specifically, our results include estimates of L_1 modulus of continuity of the classes $\Lambda_\varphi BV$, estimates of L_p ($1 \leq p < \infty$) modulus of continuity of the classes $\Lambda_\beta BV$, and specially estimates of L_p ($1 \leq p < \infty$) modulus of the classes $\{(n+1)^\alpha \ln^\gamma(n+1)\}_\beta BV$ ($0 \leq \alpha \leq 1, \beta \geq 1, \gamma \in \mathbb{R}$). Our results extend and include the corresponding known results of the class ΛBV as a special case and are also sharp in most cases. As direct applications, we obtain estimates of order of Fourier coefficients of functions of Λ_φ -bounded variation, and we also characterize some sufficient and necessary conditions for the embedding relations $H_p^\omega \subset \Lambda_\varphi BV$ and $H_p^\omega \subset \varphi BV$.

Now we state our main results, and in what follows, without loss of generality, we always assume that $[a, b] = [0, 2\pi]$, and functions in various generalized bounded variation classes are 2π periodic.

Theorem 2. Let $\Lambda_\varphi BV$ be the class of functions of Λ_φ -bounded variation, and let $\lambda(s)$ generate Λ , where φ is convex and φ^{-1} is the inverse function of φ . Then, for $f \in \Lambda_\varphi BV$,

$$\omega(f; t)_1 \leq 4\pi\varphi^{-1} \left(\frac{V_{\Lambda_\varphi}(f; [0, 2\pi])}{\int_0^{1/t} (ds/\lambda(s))} \right), \quad (8)$$

and this estimate is sharp in the sense of order provided that Λ satisfies the condition Δ_2 .

Corollary 3 is a direct result of Theorem 2 by choosing $\varphi(x) = x$ and $\Lambda = \{1\}$, respectively.

Corollary 3. (a) Let ΛBV be the class of functions of Λ -bounded variation, and let $\lambda(s)$ generate Λ . Then, for $f \in \Lambda BV$,

$$\omega(f; t)_1 \leq \frac{4\pi V_\Lambda(f; [0, 2\pi])}{\int_0^{1/t} (ds/\lambda(s))}, \quad (9)$$

and this estimate is sharp in the sense of order provided that Λ satisfies the condition Δ_2 .

(b) Let φBV be the class of functions of φ -bounded variation, where φ is convex. Then, for $f \in \varphi BV$,

$$\omega(f; t)_1 \leq 4\pi\varphi^{-1}(tV_\varphi(f; [0, 2\pi])), \quad (10)$$

and this estimate is sharp in the sense of order.

The first part of Corollary 3(a) is due to Wang [13]. For special Λ 's, one has

Corollary 4. Let $f \in \Lambda_\varphi BV$ and φ be convex. Then

(a) for $\Lambda = \{n^\alpha\}$ ($0 \leq \alpha < 1$),

$$\omega(f; t)_1 \leq c\varphi^{-1}\left(t^{1-\alpha}V_{\Lambda_\varphi}(f; [0, 2\pi])\right); \quad (11)$$

(b) for $\Lambda = \{n\}$,

$$\omega(f; t)_1 \leq c\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f; [0, 2\pi])}{\ln(1/t)}\right); \quad (12)$$

(c) for $\Lambda = \{(n+1)\ln(n+1)\}$,

$$\omega(f; t)_1 \leq c\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f; [0, 2\pi])}{\ln \ln(1/t)}\right); \quad (13)$$

(d) for $\Lambda = \{\ln^\gamma(n+1)\}$ ($\gamma > 0$),

$$\omega(f; t)_1 \leq c\varphi^{-1}\left(t \ln^{-\gamma}\left(\frac{1}{t}\right)V_{\Lambda_\varphi}(f; [0, 2\pi])\right); \quad (14)$$

(e) for $\Lambda = \{(n+1)\ln^\gamma(n+1)\}$ ($-\infty < \gamma < 1$),

$$\omega(f; t)_1 \leq c\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f; [0, 2\pi])}{\ln^{1-\gamma}(1/t)}\right); \quad (15)$$

(f) for $\Lambda = \{(n+1)^\alpha \ln^\gamma(n+1)\}$ ($0 < \alpha < 1, \gamma \in \mathbb{R} \setminus \{0\}$),

$$\omega(f; t)_1 \leq c\varphi^{-1}\left(t^{1-\alpha} \ln^{-\gamma}\left(\frac{1}{t}\right)V_{\Lambda_\varphi}(f; [0, 2\pi])\right). \quad (16)$$

The above estimates are sharp in the sense of order.

Let $f(x)$ be an integrable function on $[0, 2\pi]$ and let its Fourier coefficients be defined as follows:

$$\begin{aligned} a_n(f) &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt, \\ b_n(f) &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt, \\ c_n(f) &= a_n(f) + ib_n(f). \end{aligned} \quad (17)$$

We note that

$$\begin{aligned} |c_n(f)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(t) - f\left(t + \frac{\pi}{n}\right) \right| dt \\ &\leq c\omega_1\left(f; \frac{1}{n}\right), \quad c > 0. \end{aligned} \quad (18)$$

Theorem 2 implies the following estimates of Fourier coefficients.

Corollary 5. (a) Let $\Lambda_\varphi BV$ be the class of functions of Λ_φ -bounded variation, and let $\lambda(s)$ generate Λ , where φ is convex and φ^{-1} is the inverse function of φ . Then, for $f \in \Lambda_\varphi BV$,

$$\left| \begin{matrix} a_n(f) \\ b_n(f) \end{matrix} \right| = O\left(\varphi^{-1}\left(\frac{1}{\int_0^n (ds/\lambda(s))}\right)\right), \quad n \geq 1. \quad (19)$$

(b) Let φBV be the class of functions of φ -bounded variation, where φ is convex and φ^{-1} is the inverse function of φ . Then for $f \in \varphi BV$

$$\left| \begin{matrix} a_n(f) \\ b_n(f) \end{matrix} \right| = O\left(\varphi^{-1}\left(\frac{1}{n}\right)\right), \quad n \geq 1. \quad (20)$$

Corollary 5(a) includes Wang's result [13] for the class ΛBV as a special case of $\varphi(x) = x$.

Theorem 6. Let $\Lambda_\beta BV$ ($\beta \geq 1$) be class of functions of Λ_β -bounded variation, and let $\lambda(s)$ generate $\Lambda = \{\lambda_n\}$. Set $\phi(z) = \int_0^z (ds/\lambda(s))$ and assume $f \in \Lambda_\beta BV$.

(a) For $1 \leq p \leq \beta$,

$$\omega(f; t)_p \leq c \left(\frac{V_{\Lambda_\beta}(f; [0, 2\pi])}{\int_0^{1/t} (ds/\lambda(s))} \right)^{1/\beta}, \quad (21)$$

and this estimate is sharp in the sense of order provided that Λ satisfies the condition Δ_2 .

(b) For $1 < p < \infty$,

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f; [0, 2\pi]) \right)^{1/\beta} \left(t \int_0^{1/t} \frac{dz}{\phi(z)^{p/\beta}} \right)^{1/p}. \quad (22)$$

(c) If, for some $\beta < p < \infty$, $\phi(z)^{(\beta-p)/\beta} \lambda(z)$ is bounded on $[1, \infty)$, then

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f; [0, 2\pi]) \right)^{1/\beta} t^{1/p}. \quad (23)$$

Unfortunately we cannot assert the sharpness of our estimates in (b) and (c) of Theorem 6 for the case $\beta < p < \infty$. Our next theorems concern some important special case of the class $\Lambda_\beta BV$ and elaborate on the estimates in Theorem 6. Among the classes considered here are $\{n^\alpha\}_\beta BV$ ($0 \leq \alpha \leq 1, \beta \geq 1$) and $\{(n+1)^\alpha \ln^\gamma(n+1)\}_\beta BV$ ($0 \leq \alpha \leq 1, \beta \geq 1, \gamma \in \mathbb{R}$).

Theorem 7. (a) Let $\{n^\alpha\}_\beta BV$ ($0 \leq \alpha \leq 1, \beta \geq 1$) be the class of functions of $\{n^\alpha\}_\beta$ -bounded variation. Then, for $f \in \{n^\alpha\}_\beta BV$ and $1 \leq p < \infty$, one has

$$(i) \quad \omega(f; t)_p \leq c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{\min\{(1-\alpha)/\beta, 1/p\}}, \quad 0 \leq \alpha < 1;$$

$$(ii) \quad \omega(f; t)_p \leq c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} (1/\ln t^{-1})^{1/\beta}, \quad \alpha = 1.$$

(b) Let BV_β ($\beta \geq 1$) be the class of β -bounded variation, that is, the Wiener class. Then, for $f \in BV_\beta$ and $1 \leq p < \infty$,

$$\omega(f; t)_p \leq c(V_\beta(f; [0, 2\pi]))^{1/\beta} t^{\min\{1/\beta, 1/p\}}, \quad (24)$$

and both estimates in (a) and (b) are sharp in the sense of order.

Li and Wang's results in [9] are extended in the Theorems 6 and 7, which can be treated as the case $\beta = 1$ of our theorems.

Theorem 8. Let $f \in \Lambda_\beta BV$ ($\beta \geq 1$). The following assertions are true.

(a) For $\Lambda = \{\ln^\gamma(n+1)\}$ ($\gamma > 0$),

$$\begin{aligned} \omega(f; t)_p &\leq \begin{cases} c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} \left(t \ln^\gamma \left(\frac{1}{t}\right)\right)^{1/\beta}, & 1 \leq p \leq \beta, \\ c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{1/p}, & \beta < p < \infty. \end{cases} \end{aligned} \quad (25)$$

(b) For $\Lambda = \{(n+1) \ln(n+1)\}$,

$$\omega(f; t)_p \leq c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} \left(\ln \ln \frac{1}{t}\right)^{-1/\beta}, \quad 1 \leq p < \infty. \quad (26)$$

(c) For $\Lambda = \{(n+1) \ln^\gamma(n+1)\}$ ($-\infty < \gamma < 1$),

$$\omega(f; t)_p \leq c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} \left(\ln \frac{1}{t}\right)^{-(1-\gamma)/\beta}, \quad 1 \leq p < \infty. \quad (27)$$

(d) For $\Lambda = \{(n+1)^\alpha / \ln^\gamma(n+1)\}$ ($0 < \alpha < 1, \gamma > 0$),

$$\begin{aligned} \omega(f; t)_p &\leq \begin{cases} c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{(1-\alpha)/\beta} \left(\ln \frac{1}{t}\right)^{-\gamma/\beta}, & 1 \leq p < \frac{\beta}{1-\alpha}, \\ c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{1/p}, & p \geq \frac{\beta}{1-\alpha}. \end{cases} \end{aligned} \quad (28)$$

(e) For $\Lambda = \{(n+1)^\alpha \ln^\gamma(n+1)\}$ ($0 < \alpha < 1, \gamma > 0$)

$$\begin{aligned} \omega(f; t)_p &\leq \begin{cases} c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{(1-\alpha)/\beta} \left(\ln \frac{1}{t}\right)^{\gamma/\beta}, & 1 \leq p < \frac{\beta}{1-\alpha}, \\ c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{1/p}, & p > \frac{\beta}{1-\alpha}. \end{cases} \end{aligned} \quad (29)$$

And all above estimates are sharp in the sense of order.

Remark 9. In (e), for $p = \beta/(1-\alpha)$, we only have

$$\omega(f; t)_p \leq c(V_{\Lambda_\beta}(f; [0, 2\pi]))^{1/\beta} t^{1/p} \left(\ln \frac{1}{t}\right)^{1/p + \gamma/\beta}. \quad (30)$$

We do not know whether this estimate is sharp in the sense of order. However there exists $f_n \in \Lambda_\beta BV$ such that

$$\begin{aligned} V_{\Lambda_\beta}(f_n; [0, 2\pi]) &\asymp 1, \quad n \longrightarrow +\infty, \\ \omega\left(f_n; \frac{\pi}{n}\right)_p &\geq c\left(\frac{\pi}{n}\right)^{1/p}, \quad n \geq 10. \end{aligned} \quad (31)$$

This exception indicates that the estimates of the L_p modulus of continuity $\omega(f; t)_p$ ($\beta < p < +\infty$) of classes of Λ_β -bounded variation are complicated, and Theorem 6 cannot cover all cases of the class $\Lambda_\beta BV$ ($\beta \geq 1$).

As direct applications of the above theorems, we characterize some sufficient and necessary conditions for the embedding relations between the generalized bounded variation classes and the class H_p^ω (or H_p^δ).

Corollary 10. On the embedding relations between the generalized bounded variation classes and the class H_p^ω or H_p^δ , the following assertions are true.

(a) Let $\Lambda_\varphi BV$ be the class of functions of Λ_φ -bounded variation and let φ be convex; let $\lambda(s)$ generate Λ and Λ satisfy the condition Δ_2 . Then

$$\Lambda_\varphi BV \subset H_1^\omega \quad \text{iff} \quad \omega(t) = \mathbf{O}\left(\varphi^{-1}\left(\left(\int_0^{1/t} \frac{ds}{\lambda(s)}\right)^{-1}\right)\right), \quad t \longrightarrow 0. \quad (32)$$

(b) Let φBV be the class of functions of φ -bounded variation and let φ be convex. Then

$$\varphi BV \subset H_1^\omega \quad \text{iff} \quad \omega(t) = \mathbf{O}\left(\varphi^{-1}(t)\right), \quad t \longrightarrow 0. \quad (33)$$

(c) Let $\Lambda_\beta BV$ be the class of functions of Λ_β -bounded variation and let $\lambda(s)$ generate Λ .

(i) If, for $1 \leq p \leq \beta$, Λ satisfies the condition Δ_2 , then

$$\Lambda_\beta BV \subset H_p^\omega \quad \text{iff} \quad \omega(t) = \mathbf{O}\left(\left(\int_0^{1/t} \frac{ds}{\lambda(s)}\right)^{-1/\beta}\right), \quad t \longrightarrow 0. \quad (34)$$

(ii) If, for $\beta < p < \infty$, $\phi(z)^{(\beta-p)/\beta} \lambda(z)$ is bounded on $[1, \infty)$ and $0 < \delta \leq 1/p$, then $\Lambda_\beta BV \subset H_p^\delta$.

(d) Let $\{n^\alpha\}_\beta BV$ ($0 \leq \alpha \leq 1, \beta \geq 1$) be the class of functions of $\{n^\alpha\}_\beta$ -bounded variation. Then, for $1 \leq p < \infty$,

(i) $\{n^\alpha\}_\beta BV \subset H_p^\delta$ ($0 \leq \alpha < 1$) if and only if $\delta \leq \min\{(1-\alpha)/\beta, 1/p\}$.

(ii) $\{n\}_\beta BV \subset H_p^\omega$ if and only if $\omega(t) = \mathbf{O}((\ln(1/t))^{-1/\beta})$, $t \rightarrow 0$.

(e) Let BV_β ($\beta \geq 1$) be the class of functions of β -bounded variation, that is, Wiener class; then, for $1 \leq p < \infty$, $BV_\beta \subset H_p^\delta$ if and only if $\delta \leq \min\{1/\beta, 1/p\}$.

This paper is organized as follows. In Section 2, we first state three lemmas, and then by them we prove Theorem 2. Lemma 12 provides our proofs crucial upper estimates and Lemma 13 will be used repeatedly in proving the sharpness of our estimates. In Section 3, we prove Theorem 6, and the same estimate technique in [9] is partly used in our proof. Theorems 7 and 8 are proved in Section 4. For the case $\beta < p < \infty$, the difficulty in our proofs of Theorems 7 and 8 is to prove the sharpness of our estimates and the key is to construct extreme functions by Lemma 13.

2. Proof of Theorem 2

Before we start our proof of Theorem 2, we prove three lemmas. Lemmas 11 and 12 will also be used in the proof of Theorem 6. Lemma 13 will be employed repeatedly in the proof of the sharpness of our estimates. Lemmas 12 and 13 are of independent interest for functions of Λ_φ -bounded variation.

Lemma 11. Let $f \in L_p[a, b]$ ($p \geq 1$). Then

$$F(h) = \left(\int_a^b |f(x+h) - f(x)|^p dx \right)^{1/p} \quad (35)$$

is continuous on $[0, \infty)$.

Proof. Using triangle inequality of L_p norm, for any $h_1, h_2 > 0$, we have

$$\begin{aligned} |F(h_1) - F(h_2)| &\leq \|f(\cdot + h_1) - f(\cdot + h_2)\|_p \\ &\leq \omega(f; |h_1 - h_2|)_p. \end{aligned} \quad (36)$$

The right continuity of $\omega(f; t)_p$ at $t = 0$ implies the continuity of $F(h)$. $\square$

Lemma 12. Let $\Delta = \{[a_k, b_k] : 1 \leq k \leq n, n \in \mathbf{Z}^+\} \in \Gamma$ be an arbitrary sequence of nonoverlapping intervals in $[a, b]$, $\Lambda = \{\lambda_k\}$, φ convex, and φ^{-1} the function of φ . Then, for $f \in \Lambda_\varphi BV$,

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq n\varphi^{-1} \left(\frac{V_{\Lambda_\varphi}(f; [a, b])}{\sum_{k=1}^n \lambda_k^{-1}} \right). \quad (37)$$

Specifically, if $f \in \varphi BV$, then $\sum_{k=1}^n |f(b_k) - f(a_k)| \leq n\varphi^{-1}(V_\varphi(f; [a, b])/n)$.

Proof. By the definition of Λ_φ -total variation and letting $A_i = 1/\lambda_i$ and $B_k = \varphi(|f(b_k) - f(a_k)|)$ in summation transform

$$\begin{aligned} &\left(\sum_{i=1}^n A_i \right) \left(\sum_{k=1}^n B_k \right) \\ &= \sum_{k=0}^{n-1} \left\{ \sum_{i=1}^{n-k} A_i B_{i+k} + \sum_{i=n-k+1}^n A_i B_{i+k-n} \right\}, \end{aligned} \quad (38)$$

we have

$$\sum_{k=1}^n \varphi(|f(b_k) - f(a_k)|) \leq \frac{n}{\sum_{k=1}^n \lambda_k^{-1}} V_{\Lambda_\varphi}(f; [a, b]). \quad (39)$$

Note that if φ is an increasing convex function on $[0, \infty)$, then φ^{-1} is increasing and concave on $[0, \infty)$. The concavity of φ^{-1} implies that $\varphi^{-1}(ax) \geq a\varphi^{-1}(x)$, $0 < a \leq 1$. Therefore, by Jensen's inequality and the above inequality, we finally get

$$\begin{aligned} &\sum_{k=1}^n |f(b_k) - f(a_k)| \\ &= n \sum_{k=1}^n \frac{\varphi^{-1}(\varphi(|f(b_k) - f(a_k)|))}{n} \\ &\leq n\varphi^{-1} \left(\frac{1}{n} \sum_{k=1}^n \varphi(|f(b_k) - f(a_k)|) \right) \\ &\leq n\varphi^{-1} \left(\frac{V_{\Lambda_\varphi}(f; [a, b])}{\sum_{k=1}^n \lambda_k^{-1}} \right). \end{aligned} \quad (40)$$

This completes the proof of Lemma 12. $\square$

Lemma 13. Let $\mathbf{a}_n = \{a_k\}_{k=1}^n$ be a given set of nonnegative and nonincreasing numbers. Define a function $f(x)$ on $[0, 2\pi]$ as

$$f_{\mathbf{a}_n}(x) = \begin{cases} a_i, & \frac{(2i-1)\pi}{n} \leq x < \frac{2i\pi}{n}, \quad i = 1, 2, \dots, n, \\ 0, & \text{other points of } [0, 2\pi], \end{cases} \quad (41)$$

and extend it to $\mathbb{R}$ with period 2π . Then

- (a) $f \in \Lambda_\varphi BV[0, 2\pi]$ and $V_{\Lambda_\varphi}(f_{\mathbf{a}_n}; [0, 2\pi]) = \sum_{i=1}^n (1/\lambda_{2i-1} + 1/\lambda_{2i})\varphi(a_i)$;
- (b) $\omega(f_{\mathbf{a}_n}; \pi/n)_p = ((2\pi/n) \sum_{k=1}^n a_k^p)^{1/p}$, $p \geq 1$;
- (c) $\sum_{i=1}^n (\varphi(a_i)/\lambda_i) \leq V_{\Lambda_\varphi}(f_{\mathbf{a}_n}; [0, 2\pi]) \leq 2 \sum_{i=1}^n (\varphi(a_i)/\lambda_i)$.

Proof. By the definition of Λ_φ -total variation and the non-negative and nonincreasing properties of $\mathbf{a}_n$, direct computation proves (a). Since $\Lambda = \{\lambda_k\}$ is nondecreasing, it is obvious that (a) implies (c). For (b), computation shows that

$$\begin{aligned} &\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p \\ &= \sum_{k=1}^n \int_{(2(k-1)\pi)/n}^{2k\pi/n} \left| f_{\mathbf{a}_n}\left(x + \frac{\pi}{n}\right) - f_{\mathbf{a}_n}(x) \right|^p dx \\ &= \frac{2\pi}{n} \sum_{k=1}^n a_k^p. \end{aligned} \quad (42)$$

$\square$

Now we prove Theorem 2.

Proof of Theorem 2. We write $V_{\Lambda_\varphi}(f) = V_{\Lambda_\varphi}(f; [0, 2\pi])$ for simplicity. By Lemma 11, there exists an $h_t \in (0, t]$ such that

$$\omega(f; t)_1 = \int_0^{2\pi} |f(x + h_t) - f(x)| dx. \quad (43)$$

If we set $N = [2\pi/h_t]$ and consider the periodicity of $f(x)$, then

$$\omega(f; t)_1 = \frac{1}{N} \int_0^{2\pi} F_t(x) dx, \quad (44)$$

where

$$F_t(x) = \sum_{k=1}^N |f(x + kh_t) - f(x + (k-1)h_t)|. \quad (45)$$

From Jensen's inequality and Lemma 12, for all $x \in [0, 2\pi]$, the concavity of φ^{-1} implies

$$F_t(x) \leq N\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f)}{\sum_{k=1}^N \lambda_k^{-1}}\right) \leq 2N\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f)}{\int_0^N (ds/\lambda(s))}\right). \quad (46)$$

Substituting (46) into (44), we get

$$\omega(f; t)_1 \leq 4\pi\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f)}{\int_0^N (ds/\lambda(s))}\right). \quad (47)$$

Since the right of (47) is decreasing with respect to N and $N = [2\pi/h_t] \geq [2\pi/t] \geq 1/t$, the estimate in Theorem 2 is obtained from (47) directly.

Now we show that our estimate is sharp in the sense of order under the assumption that Λ satisfies the condition Δ_2 .

We choose

$$a_k = \varphi^{-1}\left(\frac{1}{2\pi \int_0^{n/\pi} (ds/\lambda(s))}\right), \quad k = 1, \dots, n, \quad (48)$$

and consider function $f_{a_n}(x)$ defined in Lemma 13.

We have

$$\begin{aligned} \frac{\int_0^n (ds/\lambda(s))}{4 \int_0^n (ds/\lambda(s/\pi))} &= \frac{\int_0^n (ds/\lambda(s))}{4\pi \int_0^{n/\pi} (ds/\lambda(s))} \\ &\leq V_{\Lambda_\varphi}(f_{a_n}) \\ &\leq \frac{\int_0^n (ds/\lambda(s))}{\pi \int_0^{n/\pi} (ds/\lambda(s))} \\ &= \frac{\int_0^n (ds/\lambda(s))}{\int_0^n (ds/\lambda(s/\pi))}, \quad n \geq 10, \end{aligned} \quad (49)$$

$$\omega\left(f_{a_n}; \frac{\pi}{n}\right)_1 = 2\pi\varphi^{-1}\left(\frac{1}{2\pi \int_0^{n/\pi} (ds/\lambda(s))}\right), \quad n \geq 10. \quad (50)$$

If Λ satisfies the condition Δ_2 , that is, there exists $a > 0$ such that $\lambda(2x) \leq a\lambda(x)$, $x > 0$, then

$$\lambda\left(\frac{s}{\pi}\right) \leq \lambda(s) = \lambda\left(\pi \cdot \frac{s}{\pi}\right) \leq \lambda\left(4 \cdot \frac{s}{\pi}\right) \leq a^2\lambda\left(\frac{s}{\pi}\right), \quad (51)$$

and this yields

$$\int_0^n \frac{ds}{\lambda(s)} \leq \int_0^n \frac{ds}{\lambda(s/\pi)} \leq a^2 \int_0^n \frac{ds}{\lambda(s)}. \quad (52)$$

From (49) and (52), we have

$$0 < \frac{1}{4a^2} \leq V_{\Lambda_\varphi}(f_{a_n}) \leq 1, \quad n \geq 10. \quad (53)$$

On the other hand, (50), the concavity and the monotonicity of φ^{-1} imply that

$$\begin{aligned} \omega\left(f_{a_n}; \frac{\pi}{n}\right)_1 &\geq \varphi^{-1}\left(\frac{1}{\int_0^{n/\pi} (ds/\lambda(s))}\right) \\ &\geq \varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f_{a_n})}{\int_0^{n/\pi} (ds/\lambda(s))}\right), \quad n \geq 10. \end{aligned} \quad (54)$$

Obviously, (53) and (54) mean the sharpness of our estimate in Theorem 2. $\square$

Proof of Corollary 4. Obviously the Λ 's in Corollary 4 satisfy the condition Δ_2 . The proofs of (a), (b), (c), and (e) in Corollary 4 are obvious. In (d), we have

$$\begin{aligned} \lambda(s) &= \ln^\gamma(s+1), \quad s \geq 1, \\ \int_0^z \frac{ds}{\lambda(s)} &\asymp \frac{z}{\ln^\gamma z}, \quad z \longrightarrow +\infty, \end{aligned} \quad (55)$$

and in (f), we have

$$\begin{aligned} \lambda(s) &= (s+1)^\alpha \ln^\gamma(s+1), \quad s \geq 1, \\ \int_0^z \frac{ds}{\lambda(s)} &\asymp \frac{z^{1-\alpha}}{\ln^\gamma z}, \quad z \longrightarrow +\infty, \end{aligned} \quad (56)$$

which complete the proofs of (d) and (f) of Corollary 4. $\square$

3. Proof of Theorem 6

In this section we prove Theorem 6. The proof of Theorem 6(a) is based on Hölder's inequality and Lemma 12. We use techniques used by Li and Wang [9] in the proofs of (b) and (c) of Theorem 6. Lemma 12 plays a crucial role in the whole proof of Theorem 6.

Proof of Theorem 6. As in the proof of Theorem 2, it is easily seen from Lemma 11 that, for $1 \leq p < \infty$, there exists an $h_t \in (0, t]$ such that

$$\omega(f; t)_p^p = \frac{1}{N} \int_0^{2\pi} F_t(x) dx, \quad (57)$$

where

$$\begin{aligned} F_t(x) &= \sum_{k=1}^N |f(x + kh_t) - f(x + (k-1)h_t)|^p, \\ N &= \left\lceil \frac{2\pi}{h_t} \right\rceil. \end{aligned} \quad (58)$$

We first prove (a). We note that $r = \beta/p > 1$ for $1 \leq p < \beta$. Let $s > 0$ satisfy $1/r + 1/s = 1$. By Hölder's inequality and Lemma 12, we have

$$\begin{aligned} F_t(x) &\leq \left(\sum_{k=1}^N |f(x + kh_t) - f(x + (k-1)h_t)|^\beta \right)^{1/r} \left(\sum_{k=1}^N 1^s \right)^{1/s} \\ &\leq \left(\frac{NV_{\Lambda_\beta}(f)}{\sum_{k=1}^N \lambda_k^{-1}} \right)^{1/r} N^{1/s} = N \left(\frac{V_{\Lambda_\beta}(f)}{\sum_{k=1}^N \lambda_k^{-1}} \right)^{p/\beta}. \end{aligned} \quad (59)$$

For $p = \beta$, Lemma 12 directly yields

$$F_t(x) \leq N \left(\frac{V_{\Lambda_\beta}(f)}{\sum_{k=1}^N \lambda_k^{-1}} \right). \quad (60)$$

Thus, for $1 \leq p \leq \beta$, we obtain

$$F_t(x) \leq N \left(\frac{V_{\Lambda_\beta}(f)}{\sum_{k=1}^N \lambda_k^{-1}} \right)^{p/\beta}. \quad (61)$$

Substituting (61) into (57) and noting that $N = \lfloor 2\pi/h_t \rfloor \geq \lfloor 2\pi/t \rfloor \geq 1/t$, we prove the desired estimate in Theorem 6(a).

If Λ satisfies the condition Δ_2 , then from the proof of Theorem 2, we know that

$$\int_0^n \frac{ds}{\lambda(s)} \asymp \int_0^{n/\pi} \frac{ds}{\lambda(s)}, \quad n \rightarrow \infty. \quad (62)$$

If we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\frac{1}{\int_0^{n/\pi} (ds/\lambda(s))} \right)^{1/\beta}, \quad k = 1, 2, \dots, n, \quad (63)$$

$$\text{where } \varepsilon_n = \frac{\int_0^{n/\pi} (ds/\lambda(s))}{2 \int_0^n (ds/\lambda(s))} \asymp 1, \quad n \rightarrow +\infty,$$

and consider the functions $f_{a_n}(x)$ defined in Lemma 13, then we have

$$\frac{1}{4} \leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \quad (64)$$

$$\begin{aligned} \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\pi \varepsilon_n^{p/\beta} \left(\frac{1}{\int_0^{\pi/n} (ds/\lambda(s))} \right)^{p/\beta} \\ &\geq c \left(\frac{V_{\Lambda_\beta}(f_{a_n})}{\int_0^{\pi/n} (ds/\lambda(s))} \right)^{p/\beta}, \quad n \geq 10. \end{aligned} \quad (65)$$

Equations (64) and (65) show that the estimate in Theorem 6(a) is sharp.

Now we prove (b). Without loss of generality, we assume that $\lambda_1 = 1$ and denote $\psi(x) = x^\beta$ ($\beta \geq 1$) and $\psi^{-1}(x) = x^{1/\beta}$. From the definition of $V_{\Lambda_\beta}(f)$, we first have

$$|f(x + kh_t) - f(x + (k-1)h_t)| \leq \psi^{-1}(V_{\Lambda_\beta}(f)). \quad (66)$$

Denote by σ_m ($m \geq 0$) the set of integers k ($1 \leq k \leq N$) for which

$$\begin{aligned} 2^{-m-1} \psi^{-1}(V_{\Lambda_\beta}(f)) &< |f(x + kh_t) - f(x + (k-1)h_t)| \\ &\leq 2^{-m} \psi^{-1}(V_{\Lambda_\beta}(f)). \end{aligned} \quad (67)$$

Then $\sum_{m=0}^\infty |\sigma_m| = N$, and there are at most N nonempty σ_m , where $|\sigma_m|$ denotes the number of the elements in σ_m . Obviously $|\sigma_m| \leq N$, $m \geq 0$.

Hence we have

$$\begin{aligned} F_t(x) &= \sum_{m=0}^\infty \sum_{k \in \sigma_m} |f(x + kh_t) - f(x + (k-1)h_t)|^p \\ &\leq \left(\psi^{-1}(V_{\Lambda_\beta}(f)) \right)^p \sum_{m=0}^\infty 2^{-mp} |\sigma_m| \\ &= \left(V_{\Lambda_\beta}(f) \right)^{p/\beta} \left\{ \sum_{m \leq M} 2^{-mp} |\sigma_m| + \sum_{m > M} 2^{-mp} |\sigma_m| \right\} \\ &=: \left(V_{\Lambda_\beta}(f) \right)^{p/\beta} (A_1 + A_2), \end{aligned} \quad (68)$$

with $M > 0$ to be determined.

From (67) and Lemma 12, we have, for $|\sigma_m| \neq 0$,

$$\begin{aligned} 2^{-m-1} \psi^{-1}(V_{\Lambda_\beta}(f)) |\sigma_m| &\leq \sum_{k \in \sigma_m} |f(x + kh_t) - f(x + (k-1)h_t)| \\ &\leq |\sigma_m| \psi^{-1} \left(\frac{V_{\Lambda_\beta}(f)}{\sum_{i=1}^{|\sigma_m|} \lambda_i^{-1}} \right), \end{aligned} \quad (69)$$

and thus

$$\sum_{i=1}^{|\sigma_m|} \frac{1}{\lambda_i} \leq \frac{V_{\Lambda_\beta}(f)}{\psi(2^{-m-1} \psi^{-1}(V_{\Lambda_\beta}(f)))} = 2^{\beta(m+1)}. \quad (70)$$

Therefore,

$$\phi(|\sigma_m|) = \int_0^{|\sigma_m|} \frac{ds}{\lambda(s)} \leq 2 \sum_{i=1}^{|\sigma_m|} \frac{1}{\lambda_i} \leq 2^{\beta(m+1)+1}. \quad (71)$$

If we set $z_m = \phi^{-1}(2^{\beta(m+1)+1})$, where ϕ^{-1} is the inverse function of ϕ , then

$$|\sigma_m| \leq z_m, \quad 2^{-m} = \frac{2^{1+1/\beta}}{\phi(z_m)^{1/\beta}}. \quad (72)$$

From the monotonicity of $\lambda(x)$, it is easily verified that $\phi(2z) \leq 2\phi(z)$. And from this we also have

$$\phi(2z_{m-1}) \leq 2\phi(z_{m-1}) = 2^{1-\beta}\phi(z_m) \leq \phi(z_m), \quad m \geq 1. \quad (73)$$

This yields

$$2z_{m-1} \leq z_m, \quad z_m \leq 2(z_m - z_{m-1}), \quad m \geq 1. \quad (74)$$

Now we estimate A_1 and A_2 in (68). By means of (72) and (74), we first have

$$\begin{aligned} A_1 &\leq \sum_{m \leq M} 2^{(2+1/\beta)p} \frac{z_m}{\phi(z_m)^{p/\beta}} \\ &\leq 2^{(2+1/\beta)p+1} \sum_{m \leq M} \frac{z_m - z_{m-1}}{\phi(z_m)^{p/\beta}} \\ &\leq c_p \int_0^{z_M} \frac{dz}{\phi(z)^{p/\beta}}. \end{aligned} \quad (75)$$

If we choose $M = [\log_2^{\phi(N)/2}/\beta] - 1$, then $z_M \leq N \leq z_{M+1}$.

From (72) and the monotonicity of $\phi(z)$, we have

$$\begin{aligned} A_2 &\leq N \sum_{m > M} 2^{-pm} \\ &= N \cdot \frac{2^{-p(M+1)}}{1 - 2^{-p}} \\ &\leq 2^{(1+1/\beta)p+1} \frac{N}{\phi(z_{M+1})^{p/\beta}} \\ &\leq c_p \frac{N}{\phi(N)^{p/\beta}}. \end{aligned} \quad (76)$$

Inserting (75) and (76) into (68) and noting that $\int_0^N (dz/\phi(z)^{p/\beta}) \geq (N/\phi(N)^{p/\beta})$, for $x \in [0, 2\pi]$, we get

$$\begin{aligned} F_t(x) &\leq c_3 \left(V_{\Lambda_\beta}(f) \right)^{p/\beta} \left[\int_0^N \frac{dz}{\phi(z)^{p/\beta}} + \frac{N}{\phi(N)^{p/\beta}} \right] \\ &\leq 2c_p \left(V_{\Lambda_\beta}(f) \right)^{p/\beta} \int_0^N \frac{dz}{\phi(z)^{p/\beta}}. \end{aligned} \quad (77)$$

Combining (77) and (57), we finally obtain

$$\omega(f; t)_p^p \leq c_p \left(V_{\Lambda_\beta}(f) \right)^{p/\beta} \frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}}. \quad (78)$$

Since $N = [2\pi/h_t] \geq [2\pi/t] \geq 1/t$ and the right of (78) is decreasing with respect to N , the proof of Theorem 6(b) is complete.

Finally we prove Theorem 6(c). Notice that Λ is nondecreasing. It follows from (67) that

$$\begin{aligned} F_t(x) &= \sum_{m=0}^{\infty} \sum_{k \in \sigma_m} |f(x + kh_t) - f(x + (k-1)h_t)|^{p-\beta} \\ &\quad \times |f(x + kh_t) - f(x + (k-1)h_t)|^\beta \\ &\leq \left(V_{\Lambda_\beta}(f) \right)^{(p-\beta)/\beta} \\ &\quad \times \sum_{m=0}^{\infty} 2^{-m(p-\beta)} \sum_{k \in \sigma_m} |f(x + kh_t) - f(x + (k-1)h_t)|^\beta \\ &\leq \left(V_{\Lambda_\beta}(f) \right)^{(p-\beta)/\beta} \sum_{m=0}^{\infty} 2^{-m(p-\beta)} \lambda_{k_m} \Omega_m, \end{aligned} \quad (79)$$

where

$$\begin{aligned} k_m &= |\sigma_0| + |\sigma_1| + \cdots + |\sigma_m|, \\ \Omega_m &= \sum_{j=k_{m-1}+1}^{k_m} \frac{|f(x + \sigma_m(j)h_t) - f(x + (\sigma_m(j)-1)h_t)|^\beta}{\lambda_j}, \end{aligned} \quad (80)$$

and $\sigma_m(j) \in \sigma_m$, $k_{-1} = 0$. It is obvious that $\sum_{m=0}^{\infty} \Omega_m \leq V_{\Lambda_\beta}(f)$.

By (72) and (74), we know that

$$k_m \leq \sum_{j=0}^m z_j \leq \sum_{j=1}^m 2(z_j - z_{j-1}) + z_0 \leq 2z_m. \quad (81)$$

Assume that $\phi(z)^{(\beta-p)/\beta} \lambda(z)$ is bounded on $[1, \infty)$. From (72) and the fact $\phi(2z_m) \leq 2\phi(z_m)$, $m \geq 0$, we obtain

$$\begin{aligned} 2^{-m(p-\beta)} \lambda_{k_m} &\leq \frac{2^{(1+1/\beta)(p-\beta)} \lambda(2z_m)}{\phi(z_m)^{(p-\beta)/\beta}} \\ &\leq c \phi(2z_m)^{(\beta-p)/\beta} \lambda(2z_m) \leq c, \quad m \geq 0. \end{aligned} \quad (82)$$

Finally, (79) yields

$$\begin{aligned} F_t(x) &\leq c \left(V_{\Lambda_\beta}(f) \right)^{(p-\beta)/\beta} \sum_{m=0}^{\infty} \Omega_m \\ &\leq c \left(V_{\Lambda_\beta}(f) \right)^{p/\beta}, \quad x \in [0, 2\pi]. \end{aligned} \quad (83)$$

Since $N \geq 1/t$, we prove Theorem 6(c) from (57). $\square$

4. Proofs of Theorems 7 and 8

In this section, we prove Theorems 7 and 8.

Proof of Theorem 7. Theorem 7(a) implies Theorem 7(b) by letting $\alpha = 0$. We only need to prove Theorem 7(a). Obviously $\Lambda = \{n^\alpha\}$ ($0 \leq \alpha < 1$) satisfies the condition Δ_2 and we have

$$\begin{aligned}\lambda(z) &= z^\alpha, \quad z \geq 1, \\ \phi(z) &= \int_0^z \frac{ds}{\lambda(s)} \asymp z^{1-\alpha}, \quad z \geq 1.\end{aligned}\quad (84)$$

For $1 \leq p \leq \beta$, it follows from Theorem 6(a) that $\omega(f; t)_p \leq c(V_{\Lambda_\beta}(f))^{1/\beta} t^{(1-\alpha)/\beta}$, and the order of this estimate is sharp.

For $\beta < p < \beta/(1-\alpha)$, $1 - (p/\beta)(1-\alpha) > 0$, we have

$$\begin{aligned}\int_0^N \frac{dz}{\phi(z)^{p/\beta}} &\asymp \int_1^N \frac{dz}{z^{p(1-\alpha)/\beta}} \asymp N^{1-(p/\beta)(1-\alpha)}, \\ N &\longrightarrow \infty.\end{aligned}\quad (85)$$

Substituting into Theorem 6(b), we get $\omega(f; t)_p \leq c(V_{\Lambda_\beta}(f))^{1/\beta} t^{(1-\alpha)/\beta}$.

For $\beta/(1-\alpha) \leq p < \infty$, $\alpha + (1-\alpha)(\beta-p)/\beta \leq 0$, we have

$$\phi(z)^{(\beta-p)/\beta} \lambda(z) \leq c_1 z^{\alpha+(1-\alpha)(\beta-p)/\beta} \leq C < \infty, \quad z \geq 1. \quad (86)$$

By Theorem 6(c), we get the estimate

$$\omega(f; t)_p \leq (V_{\Lambda_\beta}(f))^{1/\beta} t^{1/p}. \quad (87)$$

If $\Lambda = \{n\}$, then Λ satisfies the condition Δ_2 , and

$$\begin{aligned}\lambda(z) &= z, \quad z \geq 1; \\ \phi(z) &= \int_0^z \frac{ds}{\lambda(s)} \asymp \ln(z+1) \asymp \ln z, \quad z \geq 1.\end{aligned}\quad (88)$$

For $1 \leq p \leq \beta$, (ii) follows from Theorem 6(a) and the order in the estimate (ii) is also sharp.

For $\beta < p < \infty$, we have

$$\begin{aligned}\frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N} \int_1^N \frac{dz}{\ln(z+1)^{p/\beta}} \asymp \frac{1}{(\ln z)^{p/\beta}}, \\ N &\longrightarrow \infty.\end{aligned}\quad (89)$$

Substituting into Theorem 6(b), we get (ii).

Now we show the sharpness of the estimates in (i) and (ii) for $\beta < p < \infty$.

For this purpose, we consider $f_{a_n}(x)$ defined in Lemma 13.

In (i), for $\beta < p < \beta/(1-\alpha)$, we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\frac{\pi}{n}\right)^{(1-\alpha)/\beta}, \quad k = 1, 2, \dots, n, \quad (90)$$

$$\text{where } \varepsilon_n = \frac{n^{1-\alpha}}{2\pi^{1-\alpha} \sum_{k=1}^n k^{-\alpha}} \asymp 1, \quad n \longrightarrow \infty.$$

From Lemma 13, we know that

$$\begin{aligned}\frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\pi \varepsilon_n^{p/\beta} \left(\frac{\pi}{n}\right)^{p(1-\alpha)/\beta} \\ &\geq c(V_{\Lambda_\beta}(f_{a_n}))^{p/\beta} \left(\frac{\pi}{n}\right)^{p(1-\alpha)/\beta}, \quad n \geq 10.\end{aligned}\quad (91)$$

For $\beta/(1-\alpha) \leq p < \infty$, $p(1-\alpha)/\beta \geq 1$, we choose

$$\begin{aligned}a_k &= \varepsilon_n^{1/\beta} k^{-(1-\alpha)/\beta} (\ln(k+1))^{-(1+\sigma)/\beta} \quad (\sigma > 0), \\ k &= 1, 2, \dots, n,\end{aligned}\quad (92)$$

where

$$\varepsilon_n = \left(2 \sum_{k=1}^n k^{-1} (\ln(k+1))^{-(1+\sigma)}\right)^{-1} \asymp 1, \quad n \longrightarrow +\infty. \quad (93)$$

From Lemma 13, we have

$$\begin{aligned}\frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= \frac{2\pi}{n} \varepsilon_n^{p/\beta} \sum_{k=1}^n k^{-(1-\alpha)p/\beta} (\ln(1+k))^{-(1+\sigma)p/\beta} \\ &\geq c(V_{\Lambda_\beta}(f_{a_n}))^{p/\beta} \frac{\pi}{n}, \quad n \geq 10.\end{aligned}\quad (94)$$

In (ii), for $\beta < p < \infty$, we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\frac{n}{\pi}\right)^{1/\beta}, \quad k = 1, 2, \dots, n, \quad (95)$$

where

$$\varepsilon_n = \frac{\ln(n/\pi)}{2 \sum_{k=1}^n k^{-1}} \asymp 1, \quad n \longrightarrow +\infty. \quad (96)$$

From Lemma 13, we know that

$$\begin{aligned}\frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\pi \varepsilon_n^{p/\beta} \left(\ln\left(\frac{n}{\pi}\right)\right)^{-p/\beta} \\ &\geq c(V_{\Lambda_\beta}(f_{a_n}))^{-p/\beta} \left(\ln\frac{n}{\pi}\right)^{-p/\beta}, \quad n \geq 10.\end{aligned}\quad (97)$$

Obviously the above functions f_{a_n} chosen prove the sharpness of our estimates in Theorem 7 for $\beta < p < \infty$. $\square$

The proof of Theorem 8 is similar to that of Theorem 7, but computations are more complicated.

Proof of Theorem 8. Let $\lambda(s)$ generate Λ and $\phi(z) = \int_0^z (ds/\lambda(s))$. It is readily seen that Λ 's in (a), (b), (c), (d), and (e) satisfy the condition Δ_2 .

In (a), we have

$$\begin{aligned}\lambda(s) &= \ln^\gamma(s+1) \quad (\gamma > 0), \quad s \geq 1, \\ \phi(z) &\asymp \frac{z+1}{\ln^\gamma(z+1)} \asymp \frac{z}{\ln^\gamma z}, \quad z \geq 2.\end{aligned}\quad (98)$$

In (b), we have

$$\begin{aligned}\lambda(s) &= (s+1) \ln(s+1), \quad s \geq 1, \\ \phi(z) &\asymp \ln \ln(z+1) \asymp \ln \ln z, \quad z \geq 10.\end{aligned}\quad (99)$$

In (c), we have

$$\begin{aligned}\lambda(s) &= (s+1) \ln^\gamma(s+1) \quad (-\infty < \gamma < 1), \quad s \geq 1, \\ \phi(z) &\asymp (\ln(z+1))^{1-\gamma} \asymp (\ln z)^{1-\gamma}, \quad z \geq 2.\end{aligned}\quad (100)$$

In (d), we have

$$\begin{aligned}\lambda(s) &= \frac{(s+1)^\alpha}{\ln^\gamma(s+1)} \quad (0 < \alpha < 1, \gamma > 0), \quad s \geq 1, \\ \phi(z) &\asymp (z+1)^{1-\alpha} (\ln^\gamma(z+1)) \asymp z^{1-\alpha} \ln^\gamma z, \quad z \geq 2.\end{aligned}\quad (101)$$

In (e), we have

$$\begin{aligned}\lambda(s) &= (s+1)^\alpha \ln^\gamma(s+1) \quad (0 < \alpha < 1, \gamma > 0), \quad s \geq 2, \\ \phi(z) &\asymp \frac{(z+1)^{1-\alpha}}{\ln^\gamma(z+1)} \asymp \frac{z^{1-\alpha}}{\ln^\gamma z}, \quad z \geq 2.\end{aligned}\quad (102)$$

From Theorem 6(a) we obtain sharp estimates in (a), (b), (c), (d), and (e) for $1 \leq p \leq \beta$.

Now we prove the estimates in Theorem 8 for $\beta < p < \infty$. We apply (b) and (c) in Theorem 6 for upper estimates. The technique used here to show the sharpness is the same as that of the proof of Theorem 7 and the key is to choose $\mathbf{a}_n$ to construct extreme functions $f_{\mathbf{a}_n}(x)$ defined in Lemma 13.

In (a), for $\beta < p < \infty$, we have

$$\phi(z)^{(\beta-p)/\beta} \lambda(z) \asymp \frac{(\ln(z+1))^{p\gamma/\beta}}{(z+1)^{(p-\beta)/\beta}} < +\infty, \quad z \geq 1. \quad (103)$$

From Theorem 6(b), we obtain

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} t^{1/p}. \quad (104)$$

If we choose

$$\begin{aligned}a_k &= \varepsilon_n^{1/\beta} (k+1)^{-1/\beta} (\ln(k+1))^{-(1+\gamma+\sigma)/\beta} \\ &\quad (\sigma > 0), \quad k = 1, 2, \dots, n,\end{aligned}\quad (105)$$

where

$$\varepsilon_n = \left(2 \sum_{k=1}^n (k+1)^{-1} (\ln(k+1))^{-(1+\sigma)} \right)^{-1} \asymp 1, \quad n \longrightarrow +\infty, \quad (106)$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}(f_{\mathbf{a}_n}) \leq 1, \quad n \geq 10, \quad (107)$$

$$\begin{aligned}\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= 2\varepsilon_n^{p/\beta} \sum_{k=1}^n ((k+1)^{-p/\beta} (\ln(k+1))^{-(p(1+\sigma)/\beta)}) \\ &\quad \cdot \frac{\pi}{n} \geq c \left(V_{\Lambda_\beta}(f_{\mathbf{a}_n}) \right)^{p/\beta} \cdot \frac{\pi}{n}, \quad n \geq 10.\end{aligned}\quad (108)$$

Equations (107) and (108) imply the sharpness of (104).

In (b), we have

$$\begin{aligned}\frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N} \int_1^N \frac{dz}{(\ln \ln(z+1))^{p/\beta}} \\ &\asymp \frac{1}{(\ln \ln N)^{p/\beta}}, \quad N \longrightarrow +\infty.\end{aligned}\quad (109)$$

Again from Theorem 6(b), we have

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} \left(\ln \ln \frac{1}{t} \right)^{-1/\beta}. \quad (110)$$

If we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\ln \ln \frac{n}{\pi} \right)^{-1/\beta}, \quad k = 1, 2, \dots, n, \quad (111)$$

where

$$\varepsilon_n = \frac{\ln \ln(n/\pi)}{2 \sum_{k=1}^n [(k+1) (\ln(k+1))]^{-1}} \asymp 1, \quad n \longrightarrow +\infty, \quad (112)$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}(f_{\mathbf{a}_n}) \leq 1, \quad n \geq 10, \quad (113)$$

$$\begin{aligned}\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= \frac{2\pi}{n} \varepsilon_n^{p/\beta} \sum_{k=1}^n \left(\ln \ln \frac{n}{\pi} \right)^{-p/\beta} \\ &\geq c \left(V_{\Lambda_\beta}(f_{\mathbf{a}_n}) \right)^{p/\beta} \left(\ln \ln \frac{n}{\pi} \right)^{-p/\beta}, \quad n \geq 10.\end{aligned}\quad (114)$$

Equations (113) and (114) imply the sharpness of (110).

In (c), for $\beta < p < \infty$, we have

$$\begin{aligned}\frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N} \int_1^N \frac{dz}{(\ln(z+1))^{p(1-\gamma)/\beta}} \\ &\asymp \frac{1}{(\ln N)^{p(1-\gamma)/\beta}}, \quad N \longrightarrow \infty.\end{aligned}\quad (115)$$

From Theorem 6(b), we get

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} \left(\ln \frac{1}{t} \right)^{-(1-\gamma)/\beta}. \quad (116)$$

If we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\ln \frac{n}{\pi} \right)^{-(1-\gamma)/\beta}, \quad k = 1, 2, \dots, n, \quad (117)$$

where

$$\varepsilon_n = \frac{(\ln(n/\pi))^{1-\gamma}}{2 \sum_{k=1}^n (k+1)^{-1} (\ln(k+1))^{-\gamma}} \asymp 1, \quad n \rightarrow +\infty, \quad (118)$$

then

$$\begin{aligned} \frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \quad (119) \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\pi \varepsilon_n^{p/\beta} \left(\ln \frac{n}{\pi} \right)^{-p(1-\gamma)/\beta} \\ &\geq c \left(V_{\Lambda_\beta}(f_{a_n}) \right)^{p/\beta} \left(\ln \frac{n}{\pi} \right)^{-p(1-\gamma)/\beta}, \quad n \geq 10. \end{aligned} \quad (120)$$

Equations (119) and (120) imply the sharpness of (116).

In (d), for $\beta < p < \beta/(1-\alpha)$, we have

$$\begin{aligned} \frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N} \int_1^N \frac{dz}{(z+1)^{p(1-\alpha)/\beta} (\ln(z+1))^{p\gamma/\beta}} \\ &\asymp \frac{1}{N^{p(1-\alpha)/\beta} (\ln N)^{p\gamma/\beta}}, \quad N \rightarrow \infty. \end{aligned} \quad (121)$$

From Theorem 6(b), we get

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} t^{(1-\alpha)/\beta} \left(\ln \frac{1}{t} \right)^{-\gamma/\beta}. \quad (122)$$

If we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\frac{\pi}{n} \right)^{(1-\alpha)/\beta} \left(\ln \frac{n}{\pi} \right)^{-\gamma/\beta}, \quad k = 1, 2, \dots, n, \quad (123)$$

where

$$\varepsilon_n = \frac{(n/\pi)^{1-\alpha} (\ln(n/\pi))^\gamma}{2 \sum_{k=1}^n (k+1)^{-\alpha} \ln^\gamma(k+1)} \asymp 1, \quad n \rightarrow +\infty, \quad (124)$$

then

$$\begin{aligned} \frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \quad (125) \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\pi \varepsilon_n^{p/\beta} \left(\frac{\pi}{n} \right)^{p(1-\alpha)/\beta} \left(\ln \frac{n}{\pi} \right)^{-p\gamma/\beta} \\ &\geq c \left(V_{\Lambda_\beta}(f_{a_n}) \right)^{p/\beta} \left(\frac{\pi}{n} \right)^{p(1-\alpha)/\beta} \left(\ln \frac{n}{\pi} \right)^{-p\gamma/\beta}, \\ &\quad n \geq 10. \end{aligned} \quad (126)$$

Equations (125) and (126) imply the sharpness of (122).

For $\beta/(1-\alpha) \leq p < \infty$, $\beta - p(1-\alpha) \leq 0$, we have

$$\phi(z)^{(\beta-p)/\beta} \lambda(z) \asymp \frac{(z+1)^{\beta-p(1-\alpha)/\beta}}{(\ln(z+1))^{p\gamma/\beta}} < +\infty, \quad z \geq 1. \quad (127)$$

From Theorem 6(c), we get

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} t^{1/p}. \quad (128)$$

If we choose

$$\begin{aligned} a_k &= \varepsilon_n^{1/\beta} (k+1)^{-(1-\alpha+\sigma)/\beta} (\ln(k+1))^{-\gamma/\beta} \\ &\quad (\sigma > 0), \quad k = 1, 2, \dots, n, \end{aligned} \quad (129)$$

where

$$\varepsilon_n = \left(2 \sum_{k=1}^n (k+1)^{-(1+\sigma)} \right)^{-1} \asymp 1, \quad n \rightarrow +\infty, \quad (130)$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \quad (131)$$

$$\begin{aligned} \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\varepsilon_n^{p/\beta} \sum_{k=1}^n \left((k+1)^{-(p(1-\alpha+\sigma)/\beta)} (\ln(k+1))^{-p\gamma/\beta} \right) \\ &\quad \cdot \frac{\pi}{n} \geq c \left(V_{\Lambda_\beta}(f_{a_n}) \right)^{p/\beta} \cdot \frac{\pi}{n}, \quad n \geq 10. \end{aligned} \quad (132)$$

Equations (131) and (132) imply the sharpness of (128).

In (e), for $\beta < p < \beta/(1-\alpha)$, we have

$$\begin{aligned} \frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N} \int_1^N \frac{(\ln(z+1))^{p\gamma/\beta}}{(z+1)^{p(1-\alpha)/\beta}} dz \\ &\asymp \frac{(\ln N)^{p\gamma/\beta}}{N^{p(1-\alpha)/\beta}}, \quad N \rightarrow \infty. \end{aligned} \quad (133)$$

From Theorem 6(b), we get

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} t^{(1-\alpha)/\beta} \left(\ln \frac{1}{t} \right)^{\gamma/\beta}. \quad (134)$$

If we choose

$$a_k = \varepsilon_n^{1/\beta} \left(\frac{\pi}{n} \right)^{(1-\alpha)/\beta} \left(\ln \frac{n}{\pi} \right)^{\gamma/\beta}, \quad k = 1, 2, \dots, n, \quad (135)$$

where

$$\varepsilon_n = \frac{(n/\pi)^{1-\alpha} (\ln(n/\pi))^{-\gamma}}{2 \sum_{k=1}^n (k+1)^{-\alpha} (\ln(k+1))^{-\gamma}} \asymp 1, \quad n \rightarrow +\infty, \quad (136)$$

then

$$\begin{aligned} \frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \quad (137) \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\pi \varepsilon_n^{p/\beta} \left(\frac{n}{\pi} \right)^{p(1-\gamma)/\beta} \left(\ln \frac{n}{\pi} \right)^{p\gamma/\beta} \\ &\geq c \left(V_{\Lambda_\beta}(f_{a_n}) \right)^{p/\beta} \left(\frac{n}{\pi} \right)^{p(1-\gamma)/\beta} \left(\ln \frac{n}{\pi} \right)^{p\gamma/\beta}, \\ &\quad n \geq 10. \end{aligned} \quad (138)$$

Equations (137) and (138) imply the sharpness of (134).

For $\beta/(1-\alpha) < p < \infty$, $\beta - p(1-\alpha) < 0$, we have

$$\phi(z)^{(\beta-p)/\beta} \lambda(z) \asymp \frac{(\ln(z+1))^{p\gamma/\beta}}{(z+1)^{(p(1-\alpha)-\beta)/\beta}} < +\infty, \quad z \geq 1. \quad (139)$$

From Theorem 6(c), we get

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f) \right)^{1/\beta} t^{1/p}. \quad (140)$$

If we choose

$$a_k = \varepsilon_n^{1/\beta} (k+1)^{-(1-\alpha+\sigma)/\beta} (\ln(k+1))^{\gamma/\beta} \quad (\sigma > 0), \quad k = 1, 2, \dots, n, \quad (141)$$

where

$$\varepsilon_n = \left(2 \sum_{k=1}^n (k+1)^{-(1+\sigma)} \right)^{-1} \asymp 1, \quad n \longrightarrow +\infty, \quad (142)$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \quad (143)$$

$$\begin{aligned} \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\varepsilon_n^{p/\beta} \sum_{k=1}^n (k+1)^{-p(1-\alpha+\sigma)/\beta} (\ln(k+1))^{p\gamma/\beta} \\ &\cdot \frac{\pi}{n} \geq c \left(V_{\Lambda_\beta}(f_{a_n}) \right)^{p/\beta} \cdot \frac{\pi}{n}, \quad n \geq 10. \end{aligned} \quad (144)$$

Equations (143) and (144) imply the sharpness of (140).

But, for $p = \beta/(1-\alpha)$, we have

$$\phi(z)^{(\beta-p)/\beta} \lambda(z) \asymp (\ln(z+1))^{p\gamma/\beta} \longrightarrow +\infty, \quad \text{as } z \longrightarrow +\infty,$$

$$\frac{1}{N} \int_0^N \frac{dz}{\phi(z)^{p/\beta}} \asymp \frac{1}{N} \int_1^N \frac{(\ln(z+1))^{p\gamma/\beta}}{z+1} dz \asymp \frac{(\ln N)^{1+p\gamma/\beta}}{N}, \quad N \longrightarrow +\infty. \quad (145)$$

Theorem 6(c) is not applicable for this case. From Theorem 6(b), we obtain

$$\omega(f; t)_p \leq c \left(V_{\Lambda_\beta}(f; [0, 2\pi]) \right)^{1/\beta} t^{1/p} \left(\ln \frac{1}{t} \right)^{1/p+\gamma/\beta}. \quad (146)$$

Unfortunately this estimate is not sharp in the sense of order. However, if we choose

$$a_k = \varepsilon_n^{1/\beta} (k+1)^{-(1-\alpha)/\beta} (\ln(k+1))^{-(1-\gamma+\sigma)/\beta} \quad (\sigma > \max\{0, \gamma-\alpha\}), \quad k = 1, 2, \dots, n, \quad (147)$$

where

$$\varepsilon_n = \left[\sum_{k=1}^n (k+1)^{-1} (\ln(k+1))^{-(1+\sigma)} \right]^{-1} \asymp 1, \quad n \longrightarrow +\infty, \quad (148)$$

then we have

$$\begin{aligned} \frac{1}{2} &\leq V_{\Lambda_\beta}(f_{a_n}) \leq 1, \quad n \geq 10, \\ \omega\left(f_{a_n}; \frac{\pi}{n}\right)_p^p &= 2\varepsilon_n^{p/\beta} \sum_{k=1}^n (k+1)^{-1} (\ln(k+1))^{-(1-\gamma+\sigma)/(1-\alpha)} \\ &\cdot \frac{\pi}{n} \geq c \cdot \frac{\pi}{n}, \quad n \geq 10. \end{aligned} \quad (149)$$

In other words, there exists $f_n \in \Lambda_\beta BV$ such that

$$\begin{aligned} V_{\Lambda_\beta}(f) &\asymp 1, \quad n \longrightarrow +\infty, \\ \omega\left(f_n; \frac{\pi}{n}\right)_p &\geq c \left(\frac{\pi}{n} \right)^{1/p}, \quad n \geq 10. \end{aligned} \quad (150)$$

This exception indicates that our methods used in this paper cannot cover all cases of estimates of L_p ($1 \leq p < \infty$) modulus of continuity of classes of functions of Λ_β -bounded variation. $\square$

Proof of Corollary 10. Obviously Theorem 2 implies (a) and (b). (c) follows from Theorem 6. Finally, (d) and (e) are obtained from Theorem 7 directly. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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