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Research Article

Q_K Spaces on the Unit Circle

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We introduce a new space $Q_K(\partial\mathbb{D})$ of Lebesgue measurable functions on the unit circle connecting closely with the Sobolev space. We obtain a necessary and sufficient condition on K such that $Q_K(\partial\mathbb{D}) = \mathrm{BMO}(\partial\mathbb{D})$, as well as a general criterion on weight functions K_1 and K_2, $K_1 \le K_2$, such that $Q_{K_1}(\partial\mathbb{D}) \subsetneqq Q_{K_2}(\partial\mathbb{D})$. We also prove that a measurable function belongs to $Q_K(\partial\mathbb{D})$ if and only if it is Möbius bounded in the Sobolev space $L^2_K(\partial\mathbb{D})$. Finally, we obtain a dyadic characterization of functions in $Q_K(\partial\mathbb{D})$ spaces in terms of dyadic arcs on the unit circle.

1. Introduction

In recent years a new class of Möbius invariant function spaces, called Q spaces, has attracted a lot of attention. These spaces were originally defined in [1] as spaces of analytic functions in the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Later on, some further generalizations such as $F(p,q,s)$ and Q_K appeared; see [2, 3], for example. Let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For $p \in (-\infty,\infty)$, Xiao studied the space $Q_p(\partial\mathbb{D})$ in paper [4], consisting of all Lebesgue measurable functions $f:\partial\mathbb{D} \to \mathbb{C}$ with

$$\|f\|_{Q_p(\partial\mathbb{D})} = \sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{|I|^p}\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,|du|\,|dv|\right)^{1/2} < \infty, \tag{1}$$

where the supremum is taken over all subarcs $I \subset \partial\mathbb{D}$ and $|I|$ is the arc length of I. A series of results of $Q_p(\partial\mathbb{D})$ can be found in [4–6]. Note that if $p = 2$, then $Q_p(\partial\mathbb{D})$ coincides with $\mathrm{BMO}(\partial\mathbb{D})$, the space of measurable functions of bounded mean oscillation on $\partial\mathbb{D}$ introduced by John and Nirenberg in [7]. For any given arc $I \subset \partial\mathbb{D}$ and $L^2(\partial\mathbb{D})$ function f, the square mean oscillation of f on I is defined by

$$\Phi_f(I) = \frac{1}{|I|}\int_I |f(u)-f_I|^2\,|du|, \tag{2}$$

where

$$f_I = \frac{1}{|I|}\int_I f(u)\,|du|. \tag{3}$$

Then a function $f \in L^2(\partial\mathbb{D})$ is said to belong to the space $\mathrm{BMO}(\partial\mathbb{D})$ if and only if $\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} = \sup_{I\subset\partial\mathbb{D}}\Phi_f(I) < \infty$.

In paper [2], Essén and Wulan studied Q_K spaces of holomorphic functions on the unit disc $\mathbb{D}$ and developed their general theory. Later on, Wulan and Zhou gave a decomposition theorem on Q_K spaces and built a relationship between Q_K spaces of analytic functions and the Morrey type space; see [8, 9], for example. Our aim in this paper will be to extend these ideas to the real Q_K spaces so that we may obtain related results on the "real Q_K spaces" by using known results on real Hardy spaces. Historically, the "real variable" theory of Hardy spaces has proved to be important in the development of harmonic analysis. We feel that these spaces are intrinsically interesting and that understanding them better will help inform our study of spaces of holomorphic functions.

As a continuation of [2], Essén et al. described the boundary values behavior of analytic functions in Q_K spaces [10] as follows.

Theorem EWX. *Let $K : [0,\infty) \to [0,\infty)$ be nondecreasing and satisfy the conditions*

$$\int_0^1 \varphi_K(t)\frac{dt}{t} < \infty, \tag{4}$$

$$\int_1^\infty \varphi_K(t)\frac{dt}{t^2} < \infty, \tag{5}$$

where

$$\varphi_K(s) = \sup_{0<t\le 1}\frac{K(st)}{K(t)}, \quad 0 < s < \infty. \tag{6}$$

Then $f \in H^2$ belongs to the space Q_K if and only if

$$\sup_{I\subset\partial\mathbb{D}}\left(\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2}K\left(\frac{|u-v|}{|I|}\right)|du||dv|\right)^{1/2} < \infty. \tag{7}$$

The above theorem suggests the following definition of $Q_K(\partial\mathbb{D})$ spaces on the unit circle. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. The space $Q_K(\partial\mathbb{D})$ consists of all Lebesgue measurable functions f on $\partial\mathbb{D}$ for which (7) holds. If $K(t) = t^p$, $0 \le p < \infty$, $Q_K(\partial\mathbb{D})$ coincides with $Q_p(\partial\mathbb{D})$. The space $Q_K(\partial\mathbb{D})$ first appeared in [11], where Pau gave that the Szegö projection from $Q_K(\partial\mathbb{D})$ to Q_K is bounded and surjective. By [10] and [11] we know that $Q_K = H^2 \cap Q_K(\partial\mathbb{D})$ if the weight function K satisfies conditions (4) and (5).

In addition, $f \lesssim g$ (for two functions f and g) means that there is a constant $C > 0$ (independent of f and g) such that $f \le Cg$. We say that $f \approx g$ (i.e., f is comparable with g) whenever $f \lesssim g \lesssim f$. In the whole paper we assume that K is doubling; that is, $K(2t) \approx K(t)$.

2. BMO and $Q_K(\partial\mathbb{D})$ Spaces

In this section, we investigate the relationship between spaces $Q_K(\partial\mathbb{D})$ and BMO$(\partial\mathbb{D})$ and study how $Q_K(\partial\mathbb{D})$ depends on the weight function K.

The following identity is easily verified:

$$\frac{1}{|I|^2}\iint_I |f(u)-f(v)|^2|du||dv| = 2\Phi_f(I). \tag{8}$$

Proposition 1. *$Q_K(\partial\mathbb{D})$ is a subset of BMO$(\partial\mathbb{D})$ for all K.*

Proof. For $I \subset \partial\mathbb{D}$, it is easy to see that

$$\begin{aligned} &I\times I \\ &\quad = \{(z,w) : 0 < |z-w| < |I|, z,w \in I\} \bigcup \{(z,z), z \in I\}. \end{aligned} \tag{9}$$

Note that the area measure of $\{(z,z), z \in I\}$ is zero. For $z, w \in I$, we have

$$\begin{aligned} &\{(z,w) : 0 < |z-w| < |I|, z,w \in I\} \\ &\quad \subset \bigcup_{k=1}^\infty \left\{(z,w) : \frac{|I|}{2^k} < |z-w| \le \frac{|I|}{2^{k-1}}\right\}. \end{aligned} \tag{10}$$

Suppose that $f \in Q_K(\partial\mathbb{D})$. For $I \subset \partial\mathbb{D}$ and integer k, denote by $2^{-k}I$ the subarcs of I with arc length $2^{-k}|I|$. Then

$$\begin{aligned} &\iint_I |f(u)-f(v)|^2|du||dv| \\ &\quad \le \sum_{k=1}^\infty \iint_{|I|/2^k<|u-v|\le|I|/2^{k-1}} |f(u)-f(v)|^2|du||dv| \\ &\quad \lesssim \frac{1}{K(1)}\sum_{k=1}^\infty\left(\frac{|I|}{2^k}\right)^2 \\ &\qquad \times \iint_{|u-v|\le|I|/2^{k-1}} \frac{|f(u)-f(v)|^2}{|u-v|^2}K\left(\frac{|u-v|}{2^{-k}|I|}\right) \\ &\qquad \times |du||dv| \\ &\quad \lesssim \frac{1}{K(1)}\sum_{k=1}^\infty\left(\frac{|I|}{2^k}\right)^2 \\ &\qquad \times 2^k \iint_{I/2^{k-1}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\qquad \times K\left(\frac{|u-v|}{2^{1-k}|I|}\right)|du||dv| \\ &\quad \lesssim |I|^2\|f\|^2_{Q_K(\partial\mathbb{D})}. \end{aligned} \tag{11}$$

We have $f \in$ BMO$(\partial\mathbb{D})$ by (8). □

Corollary 2. *The space $Q_K(\partial\mathbb{D})$ is Banach with the norm of $\|f\| = |f(0)| + \|f\|_{Q_K(\partial\mathbb{D})}$, where $\|f\|_{Q_K(\partial\mathbb{D})}$ is the supremum of* (7).

Proof. Let $\{f_n\}$ be a Cauchy sequence in $Q_K(\partial\mathbb{D})$. By Proposition 1 we know that $Q_K(\partial\mathbb{D})$ is subset of BMO$(\partial\mathbb{D})$. Hence $\{f_n\}$ is a Cauchy sequence in BMO$(\partial\mathbb{D})$ as well and $f_n \to f$ in BMO$(\partial\mathbb{D})$ for some f. It follows from Fatou's lemma that, for every integer $n \ge 1$,

$$\|f-f_n\|_{Q_K(\partial\mathbb{D})} \le \limsup_{j\to\infty}\|f_j-f_n\|_{Q_K(\partial\mathbb{D})}. \tag{12}$$

This gives $f_n \to f$ in $Q_K(\partial\mathbb{D})$. □

Theorem 3. *$Q_K(\partial\mathbb{D}) =$ BMO$(\partial\mathbb{D})$ if and only if*

$$\int_0^1 \frac{K(s)}{s^2}ds < \infty. \tag{13}$$

Proof. Assume that $f \in$ BMO$(\partial\mathbb{D})$ and (13) holds. We use nI for the arc in $\partial\mathbb{D}$ which has the same center as I and length

$n|I|$ for a nonnegative integer n. For any given $I \subset \partial\mathbb{D}$ and $|t| \le |I|$, then

$$\begin{aligned}
&\int_I \left|f\left(e^{i(\theta+t)}\right) - f\left(e^{i\theta}\right)\right|^2 d\theta \\
&\quad \lesssim \int_I \left\{\left|f\left(e^{i(\theta+t)}\right) - f_I\right|^2 + \left|f\left(e^{i\theta}\right) - f_I\right|^2\right\} d\theta \\
&\quad \lesssim |I| \left\|f\right\|^2_{\mathrm{BMO}(\partial\mathbb{D})} + \int_I \left|f\left(e^{i(\theta+t)}\right) - f_{3I}\right|^2 d\theta \\
&\qquad + |I| \left|f_{3I} - f_I\right|^2 \\
&\quad \lesssim |I| \left\|f\right\|^2_{\mathrm{BMO}(\partial\mathbb{D})}.
\end{aligned} \tag{14}$$

By the inequality $2x/\pi < \sin x < x$ for $0 < x < \pi/2$ and the above estimate, we have

$$\begin{aligned}
&\iint_I \frac{\left|f\left(e^{i\theta}\right) - f\left(e^{i\varphi}\right)\right|^2}{\left|e^{i\theta} - e^{i\varphi}\right|^2} K\left(\frac{\left|e^{i\theta} - e^{i\varphi}\right|}{|I|}\right) d\theta\, d\varphi \\
&\quad \lesssim \int_{|t|\le|I|} \int_I \frac{\left|f\left(e^{i(\theta+t)}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i(\theta+t)} - e^{i\theta}\right|^2} \\
&\qquad\qquad \times K\left(\frac{\left|e^{i(\theta+t)} - e^{i\theta}\right|}{|I|}\right) d\theta\, dt \\
&\quad \lesssim \left\|f\right\|^2_{\mathrm{BMO}(\partial\mathbb{D})} |I| \int_{|t|\le|I|} \frac{K\left(|\sin(t/2)| / |I|\right)}{(\sin(t/2))^2} dt \\
&\quad \lesssim \left\|f\right\|^2_{\mathrm{BMO}(\partial\mathbb{D})} |I| \int_0^{|I|} \frac{K(t/|I|)}{(t/2)^2} dt \\
&\quad \lesssim \left\|f\right\|^2_{\mathrm{BMO}(\partial\mathbb{D})} \int_0^1 \frac{K(s)}{s^2} ds.
\end{aligned} \tag{15}$$

The above estimate shows that $\mathrm{BMO}(\partial\mathbb{D}) \subset Q_K(\partial\mathbb{D})$. This and Proposition 1 imply $\mathrm{BMO}(\partial\mathbb{D}) = Q_K(\partial\mathbb{D})$.

Conversely, suppose that $Q_K(\partial\mathbb{D}) = \mathrm{BMO}(\partial\mathbb{D})$. If

$$\int_0^1 \frac{K(s)}{s^2} ds = \infty, \tag{16}$$

we can choose an integer sequence $\{\lambda_j\}_{j=1}^{\infty}$ such that

$$\int_{2\pi 2^{-\lambda_j}}^1 \frac{K(s)}{s^2} ds \ge j, \quad j = 1, 2, 3, \ldots. \tag{17}$$

Define a function f as follows:

$$f\left(e^{i\theta}\right) = \sum_{j=1}^{\infty} \frac{1}{j} e^{i2^{\lambda_j}\theta}. \tag{18}$$

Then $f \in \mathrm{BMO}(\partial\mathbb{D})$ ([12], page 178). By assumption we have $f \in Q_K(\partial\mathbb{D})$. It is easy to see that

$$\int_0^{2\pi} \left|f\left(e^{i(\theta+t)}\right) - f\left(e^{i\theta}\right)\right|^2 d\theta = \sum_{j=1}^{\infty} \frac{1}{j^2} \left|e^{i2^{\lambda_j}t} - 1\right|^2. \tag{19}$$

We give the following estimate which will be proved later:

$$\int_0^{\pi} \sin^2 \frac{nt}{2} \frac{K(t)}{t^2} dt \gtrsim \int_{2\pi/n}^{\pi} \frac{K(t)}{t^2} dt, \quad n = 1, 2, \ldots. \tag{20}$$

By (17), (19), and (20), we have

$$\begin{aligned}
&\iint_0^{2\pi} \frac{\left|f\left(e^{i\theta}\right) - f\left(e^{i\varphi}\right)\right|^2}{\left|e^{i\theta} - e^{i\varphi}\right|^2} K\left(\frac{\left|e^{i\theta} - e^{i\varphi}\right|}{2\pi}\right) d\theta\, d\varphi \\
&\quad = \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^{2\pi} \frac{\left|e^{i2^{\lambda_j}t} - 1\right|^2}{\left|e^{it} - 1\right|^2} K\left(\frac{\left|e^{it} - 1\right|}{2\pi}\right) dt \\
&\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^{\pi} \frac{\sin^2\left(2^{\lambda_j}t/2\right)}{\sin^2(t/2)} K(t)\, dt \\
&\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^{\pi} \sin^2 \frac{2^{\lambda_j}t}{2} \frac{K(t)}{t^2} dt \\
&\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j^2} \int_{2\pi 2^{-\lambda_j}}^{\pi} \frac{K(t)}{t^2} dt \\
&\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j} = \infty.
\end{aligned} \tag{21}$$

We now prove (20). Note that $j/n \ge (j+1)/2n$ is valid for all $j, n = 1, 2, \ldots$ and $K(2t) \approx K(t)$. Then

$$\begin{aligned}
&\int_0^{\pi} \sin^2 \frac{nt}{2} \frac{K(t)}{t^2} dt \\
&\quad = \sum_{j=0}^{n-1} \int_{j\pi/n}^{((j+1)/n)\pi} \frac{1 - \cos nt}{2} \frac{K(t)}{t^2} dt \\
&\quad \ge \sum_{j=0}^{n-1} \frac{K(j\pi/n)}{((j+1)\pi/n)^2} \int_{j\pi/n}^{((j+1)/n)\pi} \frac{1 - \cos nt}{2} dt \\
&\quad \gtrsim \sum_{j=1}^{n-1} \frac{K((j+1)\pi/n)}{((j+1)\pi/n)^2} \frac{\pi}{2n} \\
&\quad \ge \frac{1}{2} \sum_{j=1}^{n-1} \left(\frac{j}{j+1}\right)^2 \int_{j\pi/n}^{((j+1)/n)\pi} \frac{K(t)}{t^2} dt \\
&\quad \gtrsim \int_{2\pi/n}^{\pi} \frac{K(t)}{t^2} dt.
\end{aligned} \tag{22}$$

The proof is complete. □

It is reasonable to assume that $\lim_{r\to 0^+} K(r) = 0$ for otherwise weight function K basically dose not play any role. Moreover, the function f must be at least locally L^2 on the boundary when f belongs to the Q_K spaces. Therefore the weight function K plays a role only if t is small. Then the following result is obvious.

Theorem 4. *Let $r_0 \in (0,1)$ such that $K(r_0) > 0$, and set $K_1(r) = \inf(K(r), K(r_0))$. Then $Q_{K_1}(\partial\mathbb{D}) = Q_K(\partial\mathbb{D})$.*

Proof. Since $K_1 \le K$ and K_1 is nondecreasing, it is easy to see that $Q_K(\partial\mathbb{D}) \subset Q_{K_1}(\partial\mathbb{D})$. We now prove $Q_{K_1}(\partial\mathbb{D}) \subset Q_K(\partial\mathbb{D})$. Note that there exists an integer $m \in \mathbb{N}$ such that $m^{-1} \le r_0/2$. If $f \in Q_{K_1}(\partial\mathbb{D})$, then $f \in \mathrm{BMO}(\partial\mathbb{D})$ by Proposition 1. For any $I \subset \partial\mathbb{D}$, divide I into the m subarcs of length $|I|/m$. For $1 \le j \le m$, denote I_j the jth subarcs, arranged in the natural order. Let $I_{j,k}$ be the smallest subarcs containing I_j and I_k. Then we have

$$\begin{aligned}
A &= \sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&= \sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K_1\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\le \iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K_1\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\le \|f\|_{Q_{K_1}(\partial\mathbb{D})}, \\
B &= \sum_{\substack{j,k=1\\ j>k+1,\, j<k-1}}^{m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\le K(1) \sum_{\substack{j,k=1\\ j>k+1,\, j<k-1}}^{m} \left(\frac{m}{I}\right)^2 \int_{I_j}\int_{I_k} |f(u)-f(v)|^2\,|du|\,|dv| \\
&\lesssim \sum_{\substack{j,k=1\\ j>k+1,\, j<k-1}}^{m} \frac{m}{|I|}\left(\int_{I_j}\left|f(u)-f_{I_{j,k}}\right|^2|du| \right. \\
&\qquad \left. + \int_{I_k}\left|f(v)-f_{I_{j,k}}\right|^2|dv|\right) \\
&\lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} \lesssim \|f\|^2_{Q_{K_1}(\partial\mathbb{D})}.
\end{aligned} \tag{23}$$

The above estimate gives

$$\begin{aligned}
&\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\quad = A + B \lesssim \|f\|^2_{Q_{K_1}(\partial\mathbb{D})}.
\end{aligned} \tag{24}$$

Hence $f \in Q_K(\partial\mathbb{D})$. So we have $Q_{K_1}(\partial\mathbb{D}) \subset Q_K(\partial\mathbb{D})$. The proof is complete. □

The following result is natural in view of Proposition 1 and Theorem 3.

Theorem 5. *Let $K_1 \le K_2$ and assume that $K_1(r)/K_2(r) \to 0$ as $r \to 0$. If*

$$\int_0^1 \frac{K_2(s)}{s^2}\,ds = \infty, \tag{25}$$

then $Q_{K_2}(\partial\mathbb{D}) \subsetneq Q_{K_1}(\partial\mathbb{D})$.

Proof. Obviously, we have $Q_{K_2}(\partial\mathbb{D}) \subset Q_{K_1}(\partial\mathbb{D})$. We assume that $Q_{K_1}(\partial\mathbb{D}) = Q_{K_2}(\partial\mathbb{D})$. The open mapping theorem tells us that the identity map from one of those spaces into the other one is continuous. Therefore there exists a constant C such that $\|\cdot\|_{Q_{K_2}(\partial\mathbb{D})} \le C\|\cdot\|_{Q_{K_1}(\partial\mathbb{D})}$. By the assumption, there exists an integer m such that $K_1(t) \le (2C)^{-1}K_2(t)$ for $t \le m^{-1}$. For any $I \subset \partial\mathbb{D}$, divide I into the $2m$ subarcs of length $|I|/(2m)$. For $1 \le j \le 2m$, denote by I_j the jth subarcs, arranged in the natural order. Applying the same manner in handing A and B in the proof of Theorem 4, we can deduce that if $f \in Q_{K_2}(\partial\mathbb{D})$, then

$$\begin{aligned}
&\|f\|^2_{Q_{K_2}(\partial\mathbb{D})} \\
&\quad \le C\sup_{I\subset\partial\mathbb{D}} \iint_I \frac{\left|f\left(e^{i\theta}\right)-f\left(e^{i\varphi}\right)\right|^2}{\left|e^{i\theta}-e^{i\varphi}\right|^2} \\
&\qquad\qquad \times K_1\left(\frac{\left|e^{i\theta}-e^{i\varphi}\right|}{|I|}\right)d\theta\,d\varphi \\
&\quad = C\sup_{I\subset\partial\mathbb{D}}\left(\sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{2m} + \sum_{\substack{j,k=1\\ j<k-1,\, j>k+1}}^{2m}\right) \\
&\qquad \times \int_{I_j}\int_{I_k} \frac{\left|f\left(e^{i\theta}\right)-f\left(e^{i\varphi}\right)\right|^2}{\left|e^{i\theta}-e^{i\varphi}\right|^2} K_1\left(\frac{\left|e^{i\theta}-e^{i\varphi}\right|}{|I|}\right)d\theta\,d\varphi \\
&\quad \le C\sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{2C}\sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{2m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K_2 \right. \\
&\qquad\qquad \left. \times\left(\frac{|u-v|}{|I|}\right)|du|\,|dv|\right) \\
&\qquad + C\sup_{I\subset\partial\mathbb{D}}\left(\sum_{\substack{j,k=1\\ j<k-1,\, j>k+1}}^{2m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K_1 \right. \\
&\qquad\qquad \left. \times\left(\frac{|u-v|}{|I|}\right)|du|\,|dv|\right) \\
&\quad = \frac{1}{2}\|f\|^2_{Q_{K_2}(\partial\mathbb{D})} + M\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})},
\end{aligned} \tag{26}$$

where M is a constant which is dependent on C. Consequently, for any $f \in Q_{K_2}(\partial\mathbb{D})$ and $I \subset \partial\mathbb{D}$, we have

$$\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K_2\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}. \tag{27}$$

A simple computation shows that $z^n \in Q_{K_2}(\partial\mathbb{D})$ for $n = \pm1, \pm2, \ldots$. So all polynomials belong to $Q_{K_2}(\partial\mathbb{D})$ spaces. For any given $g(u) = \sum_{j=-\infty}^{\infty} a_j u^j \in \mathrm{BMO}(\partial\mathbb{D})$, denote by $g_n(u) = \sum_{j=-n}^{n} a_j u^j$ the truncation of the function g. Then $g_n \in Q_{K_2}(\partial\mathbb{D})$ and $\|g_n\|_{\mathrm{BMO}(\partial\mathbb{D})} \le \|g\|_{\mathrm{BMO}(\partial\mathbb{D})}$. Applying Fatou's lemma, we deduce that

$$\begin{aligned}\sup_{I\subset\partial\mathbb{D}} &\iint_I \frac{|g(u)-g(v)|^2}{|u-v|^2} K_2\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\ &\lesssim \|g\|^2_{\mathrm{BMO}(\partial\mathbb{D})}.\end{aligned} \tag{28}$$

Equation (28) and Proposition 1 show $\mathrm{BMO}(\partial\mathbb{D}) = Q_{K_2}(\partial\mathbb{D})$. It follows from Theorem 3 that the integral (13) with $K = K_2$ must be convergent, which contradicts our assumption. We conclude that we must have $Q_{K_2}(\partial\mathbb{D}) \subsetneq Q_{K_1}(\partial\mathbb{D})$. □

3. Möbius Invariant $Q_K(\partial\mathbb{D})$ Spaces

Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. The Sobolev type space $L^2_K(\partial\mathbb{D})$ consists of those Lebesgue measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$ satisfying

$$\begin{aligned}\|f\|&_{L^2_K(\partial\mathbb{D})} \\ &= \left(\iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} K(|u-v|)\,|du|\,|dv|\right)^{1/2} \\ &< \infty.\end{aligned} \tag{29}$$

If $K(t) = t^p, 0 \le p < \infty$, then $L^2_K(\partial\mathbb{D}) = L^2_p(\partial\mathbb{D})$ are sobolev spaces and are introduced in [4]. See [13] about the theory of Sobolev spaces. If $K(t) = t^p$, $p > 1$, then $L^2(\partial\mathbb{D})$ is a subspace of $L^2_K(\partial\mathbb{D})$. From Section 2 it turns out that $Q_K(\partial\mathbb{D})$ is closely related to the Sobolev type space $L^2_K(\partial\mathbb{D})$ on the unit circle. By (7) and (29) it follows that $Q_K(\partial\mathbb{D})$ is a subspace of $L^2_K(\partial\mathbb{D})$. As a matter of fact, we have the following result.

Theorem 6. *Let K satisfy condition* (4). *Then $f \in Q_K(\partial\mathbb{D})$, if and only if*

$$|||f|||_{Q_K(\partial\mathbb{D})} = \sup_{a\in\mathbb{D}} \|f\circ\phi_a\|_{L^2_K(\partial\mathbb{D})} < \infty, \tag{30}$$

where $\phi_a(z) = (a-z)/(1-\bar{a}z)$ is a Möbius transformation of the unit disk for $a \in \mathbb{D}$.

Proof. We acknowledge that this proof is suggested by the technique of [4]. Firstly, we give the following equality for $u = \phi_a(z)$ and $v = \phi_a(w)$:

$$\begin{aligned}&\iint_{\partial\mathbb{D}} \frac{|f\circ\phi_a(z) - f\circ\phi_a(w)|^2}{|z-w|^2} K(|z-w|)\,|dz|\,|dw| \\ &\quad= \iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\qquad\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|.\end{aligned} \tag{31}$$

Sufficiency. Suppose that (30) holds. Choose an arc $I \subset \partial\mathbb{D}$. Without loss of generality, we assume $|I| < 1/4$. We choose a point of $a \in \mathbb{D}$ $(a \ne 0)$ such that $a/|a|$ and $2\pi(1-|a|)$ are the center and arc length of I, respectively. We have the following estimate:

$$\frac{1}{|1-\bar{a}u|} \approx \frac{1}{|I|}, \quad u \in I. \tag{32}$$

Then

$$\begin{aligned}&\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\ &\quad\lesssim \iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\qquad\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|.\end{aligned} \tag{33}$$

By (31) we complete the proof of sufficiency.

Necessity. We assume that $f \subset Q_K(\partial\mathbb{D})$. For any $a \in \mathbb{D}$, let I_a be the arc in $\partial\mathbb{D}$ with the midpoint of $a/|a|$ and the arc length of $2\pi(1-|a|)$. If $a = 0$, we set $I_a = \partial\mathbb{D}$. Also, define

$$I_n = 2^n I_a, \quad n = 0, 1, \ldots, N-1, \tag{34}$$

where N is the smallest integer such that $2^N|I_a| \ge 2\pi$; that is, $I_N = \partial\mathbb{D}$. Then

$$\begin{aligned}&\iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\qquad\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv| \\ &\quad= \iint_{I_0} + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m} \{\cdots\} \\ &\quad\le \|f\|^2_{Q_K(\partial\mathbb{D})} + A + B,\end{aligned} \tag{35}$$

where

$$A = \sum_{n=0}^{N-1}\sum_{m\le n}^{N-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{|f(u)-f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|,$$

$$B = \sum_{n=0}^{N-1}\sum_{m>n}^{N-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{|f(u)-f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|. \tag{36}$$

For any given $u \in I_{n+1} \setminus I_n, n = 1, 2, \ldots$, we have

$$\frac{1}{|1-\bar{a}u|} \lesssim \frac{1}{2^n(1-|a|)}. \tag{37}$$

By (32) and (37), we obtain that

$$\begin{aligned}
&\sum_{n=0}^{N-1}\iint_{I_{n+1}\setminus I_n}\frac{|f(u)-f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv| \\
&\lesssim \sum_{n=0}^{N-1}\iint_{I_{n+1}\setminus I_n}\frac{|f(u)-f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|}{2^{2n}|I_0|}\right)|du|\,|dv| \\
&\lesssim \sum_{n=0}^{N-1}\varphi_K\left(\frac{1}{2^n}\right)\iint_{I_{n+1}}\frac{|f(u)-f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|}{2^{n+1}|I_0|}\right)|du|\,|dv| \\
&\lesssim \|f\|^2_{Q_K(\partial\mathbb{D})}\int_0^1\frac{\varphi_K(s)}{s}ds.
\end{aligned} \tag{38}$$

On the other hand, by Lemma 2.1 of [10], condition (4) implies that $K(t) \lesssim t^c$ for small enough $c > 0$. Then

$$\begin{aligned}
&\sum_{n=1}^{N-1}\sum_{m<n-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{|f(u)-f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv| \\
&\lesssim \sum_{n=1}^{N-1}\sum_{m<n-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{|f(u)-f(v)|^2}{(2^n-2^m)^2|I_0|^2} \times K\left(\frac{(2^n-2^m)|I_0|^2}{2^{m+n}|I_0|^2}\right)|du|\,|dv| \\
&\lesssim \sum_{n=1}^{N-1}\sum_{m<n-1}K\left(\frac{1}{2^m}\right) \times \int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\left(\left|f(u)-f_{I_m}\right|^2+\left|f(v)-f_{I_m}\right|^2\right) \times\left(2^{2n}|I_0|^2\right)^{-1}|du|\,|dv| \\
&\lesssim \sum_{n=1}^{N-1}\sum_{m<n-1}\frac{2^{m(1-c)}}{2^n} \times\left(\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}+\frac{1}{2^n|I_0|} \times\int_{I_{n+1}\setminus I_n}\left|f(u)-f_{I_m}\right|^2|du|\right) \\
&\lesssim \sum_{n=1}^{N-1}\sum_{m<n-1}\frac{2^{m(1-c)}}{2^n}\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}\left(1+(n-m)^2\right) \\
&\lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}\sum_{n=1}^{\infty}\frac{n^2}{2^{nc}}.
\end{aligned} \tag{39}$$

Here we apply the following estimate:

$$\begin{aligned}
&\frac{1}{|I_{n+1}|}\int_{I_{n+1}}\left|f(u)-f_{I_m}\right|^2|du| \\
&\le \frac{1}{|I_{n+1}|}\int_{I_{n+1}}\left|f(u)-f_{I_{n+1}}\right|^2|du|+\left|f_{I_m}-f_{I_{n+1}}\right|^2 \\
&\le \frac{1}{|I_{n+1}|}\int_{I_{n+1}}\left|f(u)-f_{I_{n+1}}\right|^2|du| \\
&\quad+\left(\sum_{j=m+1}^{n+1}\left|f_{I_{j-1}}-f_{I_j}\right|\right)^2 \\
&\lesssim (n-m)^2\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}.
\end{aligned} \tag{40}$$

The above estimate gives that

$$\begin{aligned} A &= \left(\sum_{n=0}^{N-1} \iint_{I_{n+1}\setminus I_n} + \sum_{n=1}^{N-1} \int_{I_{n+1}\setminus I_n} \int_{I_n\setminus I_{n-1}} \right) \{\cdots\} \\ &\quad + \sum_{n=1}^{N-1} \sum_{m<n-1} \int_{I_{n+1}\setminus I_n} \int_{I_{m+1}\setminus I_m} \{\cdots\} \\ &\lesssim \|f\|^2_{Q_K(\partial\mathbb{D})}. \end{aligned} \tag{41}$$

Applying the same manner in handing A, we have $B \lesssim \|f\|^2_{Q_K(\partial\mathbb{D})}$. Therefore, we obtain

$$\begin{aligned} &\iint_{\partial\mathbb{D}} \frac{|f(u) - f(v)|^2}{|u-v|^2} \\ &\quad \times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|} \right) |du|\,|dv| \lesssim \|f\|^2_{Q_K}. \end{aligned} \tag{42}$$

The proof is complete. □

Corollary 7. *$Q_K(\partial\mathbb{D})$ is a Möbius invariant space in the sense that $\|\|f\|\|_{Q_K(\partial\mathbb{D})} = \|\|f\circ\phi_a\|\|_{Q_K(\partial\mathbb{D})}$ for any $f \in Q_K(\partial\mathbb{D})$ and $a \in \mathbb{D}$.*

Proof. Corollary 7 is obvious by Theorem 6. □

4. Dyadic Characterization

For given arc $I \subset \partial\mathbb{D}$, denote by I_n the set of the 2^n arcs of length $2^{-n}|I|$ obtained by n successive bipartition of I. The discrete characterization of $Q_p(\partial\mathbb{D})$ space is given in [5]. We will prove a discrete characterization of $Q_K(\partial\mathbb{D})$ spaces. The following is the principle result of this section.

Theorem 8. *Let K satisfy condition* (4). *Then $f \in L^2(\partial\mathbb{D})$ belongs to the space $Q_K(\partial\mathbb{D})$, if and only if*

$$\sup_{I\subset\partial\mathbb{D}} \sum_{n=0}^{\infty} \sum_{J\in I_n} K\left(\frac{1}{2^n}\right) \Phi_f(J) < \infty. \tag{43}$$

We first acknowledge that this proof is suggested by the technique of [5]. To prove Theorem 8, we need the following lemmas.

Lemma 9. *Let $I \subset \partial\mathbb{D}$ be an arc. If $f \in L^2(\partial\mathbb{D})$, then*

$$\Psi_{f,K}(I) \approx \sum_{J\in I_1} \Psi_{f,K}(J) + \sum_{J\in I_1} |f_J - f_I|^2, \tag{44}$$

where

$$\Psi_{f,K}(I) = \sum_{n=0}^{\infty} \sum_{J\in I_n} K\left(\frac{1}{2^n}\right) \Phi_f(J). \tag{45}$$

Proof. The following result can be found in [5]:

$$\Phi_f(I) = \frac{1}{2} \sum_{J\in I_1} \Phi_f(J) + \frac{1}{2^2} \sum_{J\in I_1} |f_J - f_I|^2. \tag{46}$$

Note that $I_k = \cup_{J\in I_1} J_{k-1}$ and $K(2^{-k}) \approx K(2^{-k-1}), k = 0, 1, 2\ldots$. By (46), we have

$$\begin{aligned} \Psi_{f,K}(I) &= \Phi_f(I) + \sum_{k=1}^{\infty} \sum_{J\in I_1} \sum_{U\in I_{k-1}} K\left(\frac{1}{2^k}\right) \Phi_f(U) \\ &\approx \sum_{J\in I_1} \left(\Psi_{f,K}(J) + \Phi_f(J) + |f_J - f_I|^2 \right) \\ &\approx \sum_{J\in I_1} \left(\Psi_{f,K}(J) + |f_J - f_I|^2 \right). \end{aligned} \tag{47}$$

The proof is complete. □

Lemma 10. *Let K satisfy condition* (4). *Let I, I', I'' be three arcs of equal length: $|I| = |I'| = |I''|$, such that I' and I'' are adjacent and $I \subset I' \cup I''$. Then for any $f \in L^2(\partial\mathbb{D})$, we have*

$$\Psi_{f,K}(I) \le \Psi_{f,K}(I') + \Psi_{f,K}(I'') + |f_{I'} - f_{I''}|. \tag{48}$$

Proof. See [5] about the proof of the following inequality:

$$\Phi_f(I) \le \Phi_f(I') + \Phi_f(I'') + |f_{I'} - f_{I''}|^2. \tag{49}$$

Without loss of generality, we assume that $I' = [0, 1)$ and $I'' = [1, 2)$. For each integer $j \ge 0$, let $\{I_{j,k}\}_{k=1}^{2^j+1}$ be the set of the 2^{j+1} dyadic arcs of length 2^{-j} contained in $I' \cup I''$, arranged in the natural order. If $J \in I_j$, then $J \in I_{j,k} \cup I_{j,k+1}$ for some k; by (48) we have

$$\Phi_f(I_j) \le \Phi_f(I_{j,k}) + \Phi_f(I_{j,k+1}) + \left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2. \tag{50}$$

The different choices of $J \in I_j$ yield different k. Summing over all j and J, we have

$$\begin{aligned} \Psi_{f,K}(I) &= \sum_{j=0}^{\infty} \sum_{J\in I_j} K\left(\frac{1}{2^j}\right) \Phi_f(J) \\ &\le 2\sum_{j=0}^{\infty} \sum_{k=1}^{2^{j+1}} K\left(\frac{1}{2^j}\right) \Phi_f(I_{j,k}) \\ &\quad + \sum_{j=0}^{\infty} \sum_{k=1}^{2^{j+1}-1} K\left(\frac{1}{2^j}\right) \left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2. \end{aligned} \tag{51}$$

It is easy to see that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{2^{j+1}} K\left(\frac{1}{2^j}\right) \Phi_f(I_{j,k}) \lesssim \Psi_{f,K}(I') + \Psi_{f,K}(I''). \tag{52}$$

The following estimate about the final double sum first appeared in Lemma 1 of [5]. Consider

$$\sum_{k=1}^{2^{j+1}-1} \left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2 \lesssim \sum_{l=1}^{j} \sum_{J\in I'_{j-l}\cup I''_{j-l}} l^2 \Phi_f(J) + |f_{I'} - f_{I''}|^2. \tag{53}$$

If K satisfies condition (4), Lemma 2.1 in [10] implies that there exists some small enough $c > 0$ such that $t^{-c}K(t)$ is nondecreasing. Substituting $j = m + l$ and summing over j, we finally obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{k=1}^{2^{j+1}-1} K\left(\frac{1}{2^j}\right)\left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2 \\
&\quad\lesssim \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{J\in I'_m\cup I''_m} K\left(\frac{1}{2^{m+l}}\right) l^2\Phi_f(J) \\
&\qquad + \sum_{j=0}^{\infty} K\left(\frac{1}{2^j}\right)\left|f_{I'} - f_{I''}\right|^2 \\
&\quad\lesssim \sum_{l=1}^{\infty}\frac{l^2}{2^{lc}}\sum_{m=0}^{\infty}\sum_{J\in I'_m\cup I''_m} K\left(\frac{1}{2^m}\right)\Phi_f(J) \\
&\qquad + \left|f_{I'} - f_{I''}\right|^2\int_0^1 \varphi_K(t)\frac{dt}{t} \\
&\quad\lesssim \Psi_{f,K}\left(I'\right) + \Psi_{f,K}\left(I''\right) + \left|f_{I'} - f_{I''}\right|^2.
\end{aligned} \tag{54}$$

Thus we have proved (49) and hence the proof is complete. □

Lemma 11. *If K satisfies condition* (4)*, then there exists a $p \in (0,\infty)$ such that $K(t)/t^p$ is nonincreasing. Furthermore, $K(t) \approx K(2t)$ for any $0 < t < \infty$.*

Proof. Lemma 11 can be found in [14]. □

Proof of Theorem 8. We now prove the necessity. It is easy to see that

$$\frac{1}{|I|^2}\iint_I \left|f(u) - f(v)\right|^2 |du|\,|dv| = 2\Phi_f(I). \tag{55}$$

By (55), we have

$$\begin{aligned}
\Psi_{f,K}(I) &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{J\in I_k} K\left(\frac{1}{2^k}\right)\frac{2^{2k}}{|I|^2}\iint_J \left|f(u) - f(v)\right|^2 |du|\,|dv| \\
&= \iint_{\partial\mathbb{D}} \alpha_I(u,v)\left|f(u) - f(v)\right|^2 |du|\,|dv|,
\end{aligned} \tag{56}$$

where

$$\alpha_I(u,v) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{J\in I_k} K\left(\frac{1}{2^k}\right)\frac{2^{2k}}{|I|^2}\chi_J(u)\,\chi_J(v) \tag{57}$$

and $\chi_J(u) = 1$, for $u \in J$, and $\chi_J(u) = 0$, for $u \in \partial\mathbb{D} \setminus J$.

Note that $|u - v| \le 2^{-k}|I|$ because of $u, v \in J \in I_k$. Since K satisfies condition (4), by Lemma 11 we may assume that $t^{-2}K(t)$ is nonincreasing. In fact, if $p \ge 2$, we can replace $K(t)$ with t^2 by Theorem 3. Then

$$\begin{aligned}
\alpha_I(u,v) &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{J\in I_k} K\left(\frac{|I|}{2^k|I|}\right)\frac{2^{2k}}{|I|^2}\chi_J(u)\,\chi_J(v) \\
&\lesssim \sum_{2^k|u-v|\le|I|} K\left(\frac{|I|}{2^k|I|}\right)\frac{2^{2k}}{|I|^2} \\
&\le \frac{K(|u-v|/|I|)}{|u-v|^2}.
\end{aligned} \tag{58}$$

This gives

$$\Psi_{f,K}(I) \lesssim \iint_I \frac{\left|f(u) - f(v)\right|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv|. \tag{59}$$

For sufficiency, we claim that

$$\begin{aligned}
&\iint_I \frac{\left|f(u) - f(v)\right|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\quad\lesssim \frac{1}{|I|}\int_{-|I|}^{|I|}\Psi_{f,K}(\{I+t\})\,dt + \Psi_{f,K}(I),
\end{aligned} \tag{60}$$

where $\{I + t\} = \{z + e^{it}, z \in I\}$ for $I \subset \partial\mathbb{D}$.

In fact, by (56) and Fubini's theorem, we have

$$\begin{aligned}
&\frac{1}{|I|}\int_{-|I|}^{|I|}\Psi_{f,K}(\{I+t\})\,dt \\
&\quad= \iint_{\partial\mathbb{D}}\frac{1}{|I|}\int_{-|I|}^{|I|}\alpha_{\{I+t\}}(u,v)\,dt \\
&\qquad\times\left|f(u) - f(v)\right|^2 |du|\,|dv|.
\end{aligned} \tag{61}$$

This and (56) show that it suffices to verify

$$\frac{K(|u-v|/|I|)}{|u-v|^2} \lesssim \frac{1}{|I|}\int_{-|I|}^{|I|}\alpha_{\{I+t\}}(u,v)\,dt + \alpha_I(u,v). \tag{62}$$

First, suppose that $u, v \in I$ with $|u-v| \le |I|/2$ and let $l \in \mathbb{N}\cup 0$ be such that $2^{-l-2}|I| < |u-v| \le 2^{-l-1}|I|$. Noting that $u \notin \{I+t\}$ and thus $\alpha_{\{I+t\}}(u,v) = 0$ when $|t| > |I|$,

$$\begin{aligned}
&\frac{1}{|I|}\int_{-|I|}^{|I|}\alpha_{\{I+t\}}(u,v)\,dt \\
&\quad\ge \frac{1}{2|I|}\int_{\partial\mathbb{D}}\sum_{J\in\{I+t\}_l} K\left(\frac{1}{2^l}\right)\frac{1}{\left|2^{-l}I\right|^2}\chi_J(u)\,\chi_J(v)\,dt \\
&\quad= \frac{2^{2l}}{|I|^3}K\left(\frac{1}{2^l}\right)\sum_{J\in I_l}\int_{\partial\mathbb{D}}\chi_{\{J+t\}}(u)\,\chi_{\{J+t\}}(y)\,dt \\
&\quad\gtrsim \frac{1}{|I|}\frac{K(|u-v|/|I|)}{|u-v|^2}\sum_{J\in I_l}\int_{\partial\mathbb{D}}\chi_{\{J+t\}}(u)\,\chi_{\{J+t\}}(v)\,dt.
\end{aligned} \tag{63}$$

For each J, the final integral equals $|J| - |u-v| \ge |J|/2$. Hence the sum over J is at least $|I|/2$ and (62) holds for $|u-v| \le |I|/2$.

If $u, v \in I$ with $|u - v| > |I|/2$, by (57) we have

$$\alpha_I(u, v) \geq K\left(\frac{1}{2}\right)\frac{1}{|I|^2} \gtrsim \frac{K(|u - v| / |I|)}{|u - v|^2}. \tag{64}$$

Hence (62) holds in this case.

We now assume that f is defined on $\mathbb{R}$ with constant f_I outside I. Let I_+ and I_- be the two arcs of the same length as I that are adjacent to I on the left and right, respectively. Note that $\Psi_{f,K}(I_-) = \Psi_{f,K}(I_+) = 0$ and $f_{I_+} = f_{I_-} = f_I$. Note that $\{I + t\} \subset I \cup I_+$ for $0 < t < |I|$ and $\{I + t\} \subset I \cup I_-$ for $-|I| < t < 0$. Lemma 10 and (60) give

$$\begin{aligned}
&\iint_I \frac{|f(u) - f(v)|}{|u - v|^2} K\left(\frac{|u - v|}{|I|}\right)|du|\,|dv| \\
&\lesssim \frac{1}{|I|}\int_{-|I|}^{|I|} \Psi_{f,K}(\{I + t\})\,dt + \Psi_{f,K}(I) \\
&\lesssim \frac{1}{|I|}\int_{-|I|}^{|I|} \Psi_{f,K}(I)\,dt + \Psi_{f,K}(I) \\
&\lesssim \Psi_{f,K}(I).
\end{aligned} \tag{65}$$

The proof is complete. □

Corollary 12. *Let $0 < p < \infty$. Then $f \in L^2(\partial\mathbb{D})$ belongs to $Q_p(\partial\mathbb{D})$ if and only if*

$$\sup_{I \subset \partial\mathbb{D}} \sum_{n=0}^{\infty} \sum_{J \in I_n} \left(\frac{1}{2^n}\right)^p \Phi_f(J) < \infty. \tag{66}$$

Proof. The nondecreasing function K satisfies condition (4) if $K(t) = t^p, 0 < p < \infty$. The desired result follows from Theorem 8. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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