Research Article

Existence and Uniqueness of Periodic Solutions for Second Order Differential Equations

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We study some second order ordinary differential equations. We establish the existence and uniqueness in some appropriate function space. By using Schauder's fixed-point theorem, new results on the existence and uniqueness of periodic solutions are obtained.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of periodic solution for the nonlinear equation

$$x'' = f(t, x, x'), \quad (1)$$

where $f: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, periodic in t with period T , and C^1 with respect to (x, x') .

Because of wide interests in physics and engineering, periodic solutions of second order differential equations have been investigated by many authors. We refer the reader to [1–7] and the references cited therein.

The purpose of this paper is to study the existence and uniqueness of periodic solution in some appropriate function space C_T^1 . To be precise, we first derive a result of existence and uniqueness, when the nonlinearity is a C^1 function with respect to (x, x') . Then, a similar result concerning the existence of periodic solutions is obtained, when the nonlinearity is not a C^1 function.

Throughout this paper we use the following assumptions.

- (A1) There exist two continuous functions $a(t)$ and $b(t)$ such that, for all $t \in [0, T]$,

$$0 \leq a(t) \leq f_x(t, x, x') \leq b(t), \quad (2)$$

where $0 \leq a(t)$ means $0 \leq a(t)$ and $0 \neq a(t)$.

- (A2) There exists a positive constant M such that, for all $t \in [0, T]$,

$$|f_{x'}(t, x, x')| \leq M. \quad (3)$$

We now present our main results of this paper.

Theorem 1. *Let assumptions (A1) and (A2) hold. Then (1) has a unique T -periodic solution.*

Remark 2. We point out that the condition $0 \leq a(t) \leq f_x$ in (A1) is necessary. Let us take the following equation, for example:

$$x'' = C, \quad (4)$$

where C is a constant and $C \neq 0$. The equation has no periodic solution.

Remark 3. Consider the following example:

$$x'' = \sin t \cdot \sin x \cdot \sin x' + x. \quad (5)$$

We can easily check that assumptions (A1) and (A2) hold. By Theorem 1, the equation has a unique 2π -periodic solution.

We also consider the case when the right-hand side of the equation is only continuous. In this case, we establish the existence of periodic solutions for the differential equation

$$x'' = f(t, x, x')x + g(t, x, x'), \quad (6)$$

where $f, g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and periodic in t with period T . We assume the following.

(A3) There exist continuous functions $a(t)$ and $b(t)$ and a positive constant M such that

$$\begin{aligned} 0 \leq a(t) \leq f(t, x, x') \leq b(t), \\ |g(t, x, x')| \leq M, \end{aligned} \quad (7)$$

for all $t \in [0, T]$.

Then, we have the following result.

Theorem 4. *Let (A3) hold. Then (6) has at least one T -periodic solution.*

Remark 5. Consider the following example:

$$x'' = 2x + x \sin x' + \cos t \cdot \cos x. \quad (8)$$

By Theorem 4, the equation has at least one 2π -periodic solution.

To establish the main results, we introduce some appropriate function space. By using Schauder's fixed-point theorem, the existence and uniqueness of periodic solutions are obtained. We know from the anonymous referee that the proof of this paper can be simplified, by using the results in [8]. The proof of this paper can be seen as an application of Schauder's fixed-point theorem.

2. Preliminary

In this section, We first introduce the function space, in which we will obtain the periodic solution for the problem. Then, some preliminary lemma is introduced, which is valuable for the proof of our main results.

Let C_T^1 be the space of continuously differentiable T -periodic functions, with norm $\|\cdot\|$ given by

$$\|x\| = \max_{t \in [0, T]} |x(t)| + \max_{t \in [0, T]} |x'(t)|. \quad (9)$$

It is well known that C_T^1 is a Banach space.

We now introduce the following lemma.

Lemma 6. *$u(t)$ and $v(t)$ are T -periodic functions. If $0 \leq v(t)$, for all $t \in \mathbb{R}$, then the equation*

$$x'' = u(t)x' + v(t)x \quad (10)$$

has a unique T -periodic solution $x(t) \equiv 0$.

Proof. Assume there exists a T -periodic solution $x(t)$, $x(t) \not\equiv 0$. Since $0 \leq v(t)$, $x(t)$ is not a constant. Otherwise we will get a contradiction by substituting $x = C \neq 0$ into (10). Then we claim that there exist t_0 and t_1 , $t_0 < t_1$, such that

$$\begin{aligned} x(t) > 0, \quad \text{for } t \in (t_0, t_1); \\ x'(t_0) > 0, \quad x'(t_1) = 0. \end{aligned} \quad (11)$$

Now we prove it.

Case 1. $x(t)$ has zero points. Assume that $x(\theta) = 0$; then, we have $x'(\theta) \neq 0$ because the initial value problem of (10) has a unique solution. If $x'(\theta) > 0$, then we let $t_0 = \theta$; if $x'(\theta) < 0$, by the periodic condition, there exists $t_0, t_0 > \theta$, such that

$$x(t) < 0, \quad t \in (\theta, t_0); \quad x(t_0) = 0. \quad (12)$$

So $x'(t_0) > 0$. Also by the periodic condition there exists η , $\eta > t_0$, such that

$$x(t) > 0, \quad t \in (t_0, \eta); \quad x(\eta) = 0. \quad (13)$$

So we can find a $t_1 \in (t_0, \eta)$ such that $x'(t_1) = 0$. Thus, under Case 1 we verify (11) holds.

Case 2. $x(t)$ has no zero point. Then $x(t)$ has a constant sign. Suppose $x(t) > 0$ (if not we can consider the T -periodic solution $\bar{x}(t) = -x(t)$, $\bar{x}(t) > 0$). Since $x(t)$ is periodic and is not a constant, there exists t_0 such that $x'(t_0) > 0$, $x(t_0) = x(t_0 + T)$. So we can find a t_1 , $t_1 > t_0$, $x'(t_1) = 0$. Then we prove that (11) holds.

Multiplying both sides of (10) by $\exp\{-\int_{t_0}^{t_1} u(s)ds\}$ and integrating between t_0 and t_1 we get

$$0 > -x'(t_0) = \int_{t_0}^{t_1} v(t)x(t) \exp\left\{-\int_{t_0}^{t_1} u(s)ds\right\} dt \geq 0, \quad (14)$$

which leads to a contradiction. This completes the proof of Lemma 6. $\square$

Remark 7. We can also prove the following. If $u(0) = u(T)$, $v(0) = v(T)$, $0 \leq v(t)$, a.e. $t \in [0, T]$, then the following periodic boundary value problem,

$$\begin{aligned} x'' &= u(t)x' + v(t)x, \quad t \in [0, T], \\ x(0) &= x(T), \quad x'(0) = x'(T), \end{aligned} \quad (15)$$

also has a unique solution $x(t) \equiv 0$.

3. Proof of the Main Results

Proof of Theorem 1. First prove the uniqueness. Assume that $x_1(t)$ and $x_2(t)$ are two T -periodic solutions of (1). Setting

$$x(t) = x_1(t) - x_2(t), \quad (16)$$

we get

$$\begin{aligned} x(t+T) &= x(t), \\ x'' &= f(t, x_1, x_1') - f(t, x_2, x_2') \\ &= f_x(t, x_1 + \theta_1(x_1 - x_2), x_1')x \\ &\quad + f_{x'}(t, x_2, x_2 + \theta_2(x_1 - x_2))x'. \end{aligned} \quad (17)$$

From assumption (A1) we know

$$0 \leq a(t) \leq f_x(t, x_1 + \theta_1(x_1 - x_2), x_1'). \quad (18)$$

Hence, by Lemma 6, $x(t) \equiv 0$.

We next prove the existence. Rewrite (1) in the following form:

$$\begin{aligned} x'' &= f(t, x, x') \\ &= (f(t, x, x') - f(t, x, 0)) + f(t, x, 0) \\ &\quad + (f(t, x, 0) - f(t, 0, 0)) + f(t, 0, 0) \\ &= \int_0^1 f_{x'}(t, x, \theta_1 x') d\theta_1 x' \\ &\quad + \int_0^1 f_x(t, \theta_2 x, 0) d\theta_2 x + f(t, 0, 0). \end{aligned} \quad (19)$$

From Lemma 6, for each $x \in C_T^1$, the equation

$$\begin{aligned} y'' &= \int_0^1 f_{x'}(t, x, \theta_1 x') d\theta_1 y' \\ &\quad + \int_0^1 f_x(t, \theta_2 x, 0) d\theta_2 y + f(t, 0, 0) \end{aligned} \quad (20)$$

has a unique T -periodic solution because the corresponding homogeneous equation only has trivial T -periodic solution $x(t) \equiv 0$.

We define operator

$$P : C_T^1 \longrightarrow C_T^1, \quad (21)$$

for given $x \in C_T^1$, $y(t) = P[x](t)$ is the unique T -periodic solution of (20). Then the existence of the T -periodic solution is equivalent to the existence of fixed point of P in the space C_T^1 . We will prove that P is continuous and compact, and $P(C_T^1)$ is a bounded subset of C_T^1 .

Proof of Continuity. For any convergent sequence $\{x_k\} \subset C_T^1$, satisfying $x_k \rightarrow x_0$ ($k \rightarrow \infty$), let $y_k = Px_k$. Then

$$\begin{aligned} y_k'' &= \int_0^1 f_{x'}(t, x_k, \theta_1 x_k') d\theta_1 y_k' \\ &\quad + \int_0^1 f_x(t, \theta_2 x_k, 0) d\theta_2 y_k + f(t, 0, 0). \end{aligned} \quad (22)$$

We claim that $\{y_k\}$ is a bounded sequence in C_T^1 . If not, we can find a subsequence of $\{y_k\}$ (for convenience we also use the same notations) such that $\|y_k\| \rightarrow \infty$ ($k \rightarrow \infty$). Let $\omega_k = y_k / \|y_k\|$. Then $\{\omega_k\} \subset C_T^1$, $\|\omega_k\| = 1$,

$$\begin{aligned} \omega_k'' &= \int_0^1 f_{x'}(t, x_k, \theta_1 x_k') d\theta_1 \omega_k' \\ &\quad + \int_0^1 f_x(t, \theta_2 x_k, 0) d\theta_2 \omega_k + \frac{f(t, 0, 0)}{\|y_k\|}. \end{aligned} \quad (23)$$

So $\|\omega_k''\| \leq M + \max_{t \in [0, T]} b(t) + 1 < \infty$ ($k \rightarrow \infty$). Since $\{\omega_k''\}$ is bounded and

$$\omega_k'(t) = \omega_k'(0) + \int_0^t \omega_k''(s) ds, \quad (24)$$

$\{\omega_k'\}$ is bounded and equicontinuous sequence of functions. Furthermore, by

$$\omega_k(t) = \omega_k(0) + \int_0^t \omega_k'(s) ds, \quad (25)$$

$\{\omega_k\}$ is bounded and equicontinuous sequence of functions. By Ascoli-Arzelà Theorem, $\{\omega_k'\}$ and $\{\omega_k\}$ contain a uniformly convergent subsequence, respectively, (for convenience we also use the same notations) such that

$$\omega_k \xrightarrow{1} \omega_0, \quad \omega_k' \xrightarrow{1} \nu_0, \quad (26)$$

where the notation " $\xrightarrow{1}$ " means uniform convergence. Obviously, $\omega_0, \nu_0 \in C_T^1$. From (23) and (24), we obtain

$$\begin{aligned} \omega_k'(t) &= \omega_k'(0) \\ &\quad + \int_0^t \left(\int_0^1 f_{x'}(s, x_k, \theta_1 x_k') d\theta_1 \omega_k' \right. \\ &\quad \left. + \int_0^1 f_x(s, \theta_2 x_k, 0) d\theta_2 \omega_k + \frac{f(s, 0, 0)}{\|y_k\|} \right) ds. \end{aligned} \quad (27)$$

Let $k \rightarrow \infty$. From (25) and (27), we get

$$\begin{aligned} \omega_0(t) &= \omega_0(0) + \int_0^t \nu_0(s) ds, \\ \nu_0(t) &= \nu_0(0) + \int_0^t \left(\int_0^1 f_{x'}(s, x_k, \theta_1 x_k') d\theta_1 \nu_0(s) \right. \\ &\quad \left. + \int_0^1 f_x(s, \theta_2 x_k, 0) d\theta_2 \omega_0(s) \right) ds. \end{aligned} \quad (28)$$

Hence

$$\omega_0'' = \int_0^1 f_{x'}(t, x_0, \theta_1 x_0') d\theta_1 \omega_0' + \int_0^1 f_x(t, \theta_2 x_0, 0) d\theta_2 \omega_0. \quad (29)$$

By Lemma 6, we have $\omega_0 \equiv 0$, which is in contradiction with $\|\omega_0\| = 1$, so $\{y_k\}$ is a bounded sequence. Then, by (22), we know $\{y_k''\}$ is bounded, so $\{y_k'\}$ and $\{y_k\}$ are bounded and equicontinuous sequences of functions. By Ascoli-Arzelà Theorem, $\{y_k'\}$ and $\{y_k\}$ contain a uniformly convergent subsequence, respectively, (for convenience we also use the same notations) such that

$$y_k \xrightarrow{1} y_0, \quad y_k' \xrightarrow{1} \bar{\nu}_0. \quad (30)$$

We know

$$\begin{aligned} y'_k(t) &= y'_k(0) + \int_0^t y''_k(s) ds \\ &= y'_k(0) + \int_0^t \left(\int_0^1 f_{x'}(s, x_k, \theta_1 x'_k) d\theta_1 y'_k \right. \\ &\quad \left. + \int_0^1 f_x(s, \theta_2 x_k, 0) d\theta_2 y_k \right. \\ &\quad \left. + f(s, 0, 0) \right) ds, \end{aligned} \quad (31)$$

$$y_k(t) = y_k(0) + \int_0^t y'_k(s) ds.$$

When $k \rightarrow \infty$, from (31), we obtain

$$\begin{aligned} y''_0 &= \int_0^1 f_{x'}(t, x_0, \theta_1 x'_0) d\theta_1 y'_0 \\ &\quad + \int_0^1 f_x(t, \theta_2 x_0, 0) d\theta_2 y_0 + f(t, 0, 0). \end{aligned} \quad (32)$$

By the uniqueness we know $y_0 = Px_0$; thus, operator P is continuous.

Proof of Compactness. For each bounded set $S \subset C_T^1$, we claim that $P(S)$ is bounded in C_T^1 . If not, by an analogous manner as above we will reach a contradiction. For every $x \in S$, $y = Px$ is defined by (20). Since $\|y'\|$, $\|y\|$, $\|f_x\|$, and $\|f_{x'}\|$ are all bounded, then $\|y''\| < \infty$. Proceeding as proof of continuity we conclude that $\{y'\}$ and $\{y\}$ are bounded and equicontinuous sequences of functions. By the Ascoli-Arzelà Theorem, P is a compact operator.

We claim that $P(C_T^1)$ is bounded in C_T^1 . If not, there exist $\{x_k\}$, $k = 1, 2, \dots$, such that $\|Px_k\| \rightarrow \infty$ ($k \rightarrow \infty$). Let $y_k = Px_k$. Then (22) holds. Take $\omega_k = y_k/\|y_k\|$. Then $\{\omega_k\} \subset C_T^1$, $\|\omega_k\| = 1$, and (23), (24), (25), and (27) hold. By the above proof we know $\{\omega'_k\}$ and $\{\omega_k\}$ are bounded and equicontinuous sequences of functions and contain a uniformly convergent subsequence, respectively, (also use the same notations) such that

$$\omega_k \xrightarrow{1} \omega_0, \quad \omega'_k \xrightarrow{1} v_0, \quad \|\omega_0\| = 1. \quad (33)$$

Sequences $\{\int_0^1 f_{x'}(t, x_k, \theta_1 x'_k) d\theta_1\}_{k=1}^\infty$ and $\{\int_0^1 f_x(t, \theta_2 x_k, 0) d\theta_2\}_{k=1}^\infty$ are bounded sequences in $L^2[0, T]$. By the weakly sequential compactness of L^2 space, both of them have a weakly convergent subsequence (also use the same notations) such that

$$\begin{aligned} \int_0^1 f_{x'}(t, x_k, \theta_1 x'_k) d\theta_1 &\xrightarrow{w} f_1(t), \\ \int_0^1 f_x(t, \theta_2 x_k, 0) d\theta_2 &\xrightarrow{w} f_2(t), \end{aligned} \quad (34)$$

in $L^2[0, T]$, where “ $\xrightarrow{w}$ ” means weak convergence. Obviously,

$$\begin{aligned} |f_1(t)| &\leq M, \quad 0 \leq a(t) \leq f_2(t) \leq b(t), \\ &\text{a.e. } t \in [0, T]. \end{aligned} \quad (35)$$

When $k \rightarrow \infty$, from (25) and (27), for a.e. $t \in [0, T]$, we have

$$v'_0(t) = f_1(t) v_0(t) + f_2(t) \omega_0(t), \quad \omega'_0(t) = v_0(t). \quad (36)$$

Then

$$\begin{aligned} \omega''_0(t) &= f_1(t) \omega'_0(t) + f_2(t) \omega_0(t), \\ \omega_0(0) &= \omega_0(T), \quad \omega'_0(0) = \omega'_0(T). \end{aligned} \quad (37)$$

By Remark 3, we obtain $\omega_0 \equiv 0$, which is in contradiction with $\|\omega_0\| = 1$. Then there exists a constant $K > 0$ such that $\|Px\| \leq K$, $x \in C_T^1$.

Assume $D = \{x \in C_T^1, \|x\| \leq K + 1\}$. By Schauder's fixed-point theorem, $P : D \rightarrow D$ has at least one fixed point. This completes the proof of Theorem 1. $\square$

Proof of Theorem 4. For each $x \in C_T^1$, from Lemma 6, we know the equation

$$y'' = f(t, x, x') y + g(t, x, x') \quad (38)$$

has a unique T -periodic solution because the corresponding homogeneous equation only has trivial T -periodic solution $x(t) \equiv 0$.

We define $\bar{P} : C_T^1 \rightarrow C_T^1$, for each given $x \in C_T^1$ $y(t) = \bar{P}[x](t)$ is the unique T -periodic solution of (38). Hence the existence of the periodic solutions is equivalent to the existence of fixed points of $\bar{P}$ in the Banach space C_T^1 .

Proceeding as the proof of Theorem 1 we can prove that $\bar{P}$ is a compact continuous operator and $\bar{P}(C_T^1)$ is a bounded subset of C_T^1 .

Then there exists a constant $K_1 > 0$ such that $\|\bar{P}x\| \leq K_1$, for all $x \in C_T^1$. Let $\bar{D} = \{x \in C_T^1, \|x\| \leq K_1 + 1\}$. By Schauder's fixed-point theorem, $\bar{P} : \bar{D} \rightarrow \bar{D}$ has at least one fixed point. This completes the proof of Theorem 4. $\square$

4. Another Simple Proof

Actually, by the anonymous referee, we know that the proof of the theorem can be much simplified if we use the theorem in [8]. In fact, for equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}), \quad (39)$$

where $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is continuous and T -periodic in t , Theorem 2.2 of [8] implies the following propositions.

Lemma 8. Let there exist continuous and T -periodic in the first argument functions $f_k : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ($k = 1, 2, \dots, n$) such that

$$\left| f(t, x_1, \dots, x_n) - \sum_{k=1}^n f_k(t, x_1, \dots, x_n) x_k \right| \leq r, \quad (40)$$

$$p_{1k}(t) \leq f_k(t, x_1, \dots, x_n) \leq p_{2k}(t) \quad (k = 1, 2, \dots, n)$$

are satisfied on $\mathbb{R}^{n+1}$, where $r = \text{const} > 0$ and $p_{ik} : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2; k = 1, 2, \dots, n$) are continuous T -periodic functions. Let, moreover, for any continuous T -periodic functions $p_k : \mathbb{R} \rightarrow \mathbb{R}$ ($k = 1, 2, \dots, n$), satisfying

$$p_{1k}(t) \leq p_k(t) \leq p_{2k}(t) \quad (k = 1, 2, \dots, n), \quad (41)$$

the equation

$$u^{(n)} = \sum_{k=1}^n p_k(t) u^{(k-1)} \quad (42)$$

have no nontrivial T -periodic solution. Then equation has at least one T -periodic solution.

Lemma 9. Let the function f in the last n arguments have continuous partial derivatives, satisfying

$$p_{1k}(t) \leq \frac{\partial f_k(t, x_1, \dots, x_n)}{\partial x_k} \leq p_{2k}(t) \quad (k = 1, 2, \dots, n), \quad (43)$$

where $p_{ik} : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2; k = 1, 2, \dots, n$) are continuous T -periodic functions. Let, moreover, for any T -periodic $p_k : \mathbb{R} \rightarrow \mathbb{R}$ ($k = 1, 2, \dots, n$), satisfying (41), (42) has no nontrivial T -periodic solution. Then, the equation has a unique T -periodic solution.

On the basis of these theorems, we can prove the theorem by checking the conditions in the previous lemma. By Lemma 6, the conditions above can be easily proved.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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