Research Article

The Arens Algebras of Vector-Valued Functions

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A class of Arens algebras of vector Banach algebra valued functions is considered. It is shown that Arens algebra of vector-valued functions is complete locally convex metrizable algebra.

1. Introduction

Let (Ω, Σ, μ) be a measure space with finite measure μ and let $L^p = L^p(\Omega, \Sigma, \mu)$ be the Banach space of all measurable functions f on (Ω, Σ, μ) such that $\|f\|_p = (\int_{\Omega} |f|^p d\mu)^{1/p} < \infty$, $(1 \leq p < \infty)$. In [1] the set

$$L^{\omega} = \bigcap_{p \geq 1} L^p \quad (1)$$

is introduced and it is shown that L^{ω} is locally convex algebra in topology τ generated by the norms $\|\cdot\|_p$. Such algebras are called Arens algebras. In [2] it is shown that (L^{ω}, τ) is GB^* algebra. Properties of algebra L^{ω} are considered in [3, 4]. Moreover, it is shown that dual of L^{ω} is $\bigcup_{1 < q \leq \infty} L^q$ and L^{ω} is reflexive. In [5] a class of noncommutative analogue of Arens algebra $L^{\omega}(M, \varphi)$ associated with semifinite von Neumann algebra and semifinite trace φ is introduced and in [6] it is shown that $L^{\omega}(M, \varphi)$ is EW^* algebra. In [6–8], properties of noncommutative Arens algebras are considered. Noncommutative Arens algebras $L^{\omega}(M, \Phi)$ associated with semifinite von Neumann algebra and center-valued trace Φ are introduced in [9] and in [10] they are generalized for the case when Φ is the Maharam trace.

Let $L^p(\Omega, X)$ be of Bochner maps from Ω into a unital Banach algebra $(X, \|\cdot\|_X)$, integrable with degree p , $1 \leq p < \infty$, that is, L^p space of vector-valued functions, and

$$L^{\omega}(\Omega, X) = \bigcap_{p \geq 1} L^p(\Omega, X). \quad (2)$$

In the present paper we prove that $L^{\omega}(\Omega, X)$ is complete locally convex metrizable algebra, which we call Arens algebra of vector-valued functions. It is shown that if dual of X has Radon-Nikodym property, then dual space of $L^{\omega}(\Omega, X)$ is $\bigcup_{1 < q \leq \infty} L^q(\Omega, X^*)$; if X is reflexive, then $L^{\omega}(\Omega, X)$ also is reflexive.

2. Preliminaries

Let (Ω, Σ, μ) be a measure space with finite measure; let $L^0(\Omega)$ be the classes of all measurable functions on Ω .

Throughout this paper, X stands for a fixed initial Banach algebra with the norm $\|\cdot\|_X$. By $L^0(\Omega, X)$ we denote the classes of all measurable by Bochner mappings from Ω to X ; that is,

$$L^0(\Omega, X) = \{f : \Omega \longrightarrow X : \|f(\omega)\|_X \in L^0(\Omega)\}. \quad (3)$$

It is known [11] that $L^0(\Omega, X)$ is algebra with respect to multiplication operation $(fg)(\omega) = f(\omega)g(\omega)$ for any

$f, g \in L^0(\Omega, X)$. By $L^p(\Omega, X)$, $1 \leq p < \infty$, we denote the Banach space of X valued measurable functions f on (Ω, Σ, μ) with the norm defined as

$$\|f\|_p = \left(\int_{\Omega} \|f(\omega)\|_X^p d\mu \right)^{1/p} = \|f(\omega)\|_X|_p; \quad (4)$$

that is, $L^p(\Omega, X) = \{f \in L^0(\Omega, X) : \|f(\omega)\|_X \in L^p\}$ and $L^\infty(\Omega, X) = \{f \in L^0(\Omega, X) : \exists \lambda > 0, \|f(\omega)\|_X \leq \lambda \text{ a.e.}\} = \{f \in L^0(\Omega, X) : \|f(\omega)\|_X \in L^\infty\}$ with the norm defined as

$$\|f\|_\infty = \inf \{\lambda > 0 : \|f(\omega)\|_X \leq \lambda \text{ a.e.}\}. \quad (5)$$

It is known [11] that the pair $(L^\infty(\Omega, X), \|\cdot\|_\infty)$ is a Banach algebra.

The following well known properties hold true.

(a) For $f, g \in L^p(\Omega, X)$ the Minkowski inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (6)$$

is valid.

(b) For $h \in L^p$ and $f \in L^q(\Omega, X)$, where $q := p/(p-1)$ for $p \neq 1$ and $q := \infty$ for $p = 1$, the Hölder inequality

$$\|hf\|_1 \leq \|h\|_p \|f\|_q \quad (7)$$

is valid.

Proposition 1. (i) If $f \in L^\infty(\Omega, X)$ and $g \in L^p(\Omega, X)$ then $fg \in L^p(\Omega, X)$ and

$$\|fg\|_p \leq \|f\|_\infty \|g\|_p. \quad (8)$$

(ii) If $p, q, r \geq 1$, $1/r = (1/p) + (1/q)$, $f \in L^p(\Omega, X)$, and $g \in L^q(\Omega, X)$, then $fg \in L^r(\Omega, X)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$.

(iii) If $\|f(\omega)\|_X \leq \|g(\omega)\|_X$ a.e., $f \in L^0(\Omega, X)$, and $g \in L^p(\Omega, X)$, then $f \in L^p(\Omega, X)$ and $\|f\|_p \leq \|g\|_p$.

Proof. (i) Let $f \in L^\infty(\Omega, X)$ and $g \in L^p(\Omega, X)$. Since $\|f(\omega)g(\omega)\|_X \leq \|f(\omega)\|_X \|g(\omega)\|_X \leq \|f\|_\infty \|g(\omega)\|_X$ and $\|g(\omega)\|_X \in L^p$ we have that $fg \in L^p(\Omega, X)$ and

$$\begin{aligned} \|fg\|_p &= \left(\int_{\Omega} \|f(\omega)g(\omega)\|_X^p d\mu \right)^{1/p} \\ &\leq \|f\|_\infty \left(\int_{\Omega} \|g(\omega)\|_X^p d\mu \right)^{1/p} = \|f\|_\infty \|g\|_p. \end{aligned} \quad (9)$$

(ii) Let $p, q, r \geq 1$, $1/r = (1/p) + (1/q)$, $f \in L^p(\Omega, X)$, and $g \in L^q(\Omega, X)$. As $\|f(\omega)\|_X \in L^p$ and $\|g(\omega)\|_X \in L^q$ from inequality $\|f(\omega)g(\omega)\|_X \leq \|f(\omega)\|_X \|g(\omega)\|_X$ we get that $fg \in L^r(\Omega, X)$ and $\|fg\|_r = \left(\int_{\Omega} \|f(\omega)g(\omega)\|_X^r d\mu \right)^{1/r} \leq \left(\int_{\Omega} (\|f(\omega)\|_X \|g(\omega)\|_X)^r d\mu \right)^{1/r} \leq \left(\int_{\Omega} \|f(\omega)\|_X^p d\mu \right)^{1/p} \left(\int_{\Omega} \|g(\omega)\|_X^q d\mu \right)^{1/q} = \|f\|_p \|g\|_q$.

(iii) Let $\|f(\omega)\|_X \leq \|g(\omega)\|_X$ a.e., $f \in L^0(\Omega, X)$, and $g \in L^p(\Omega, X)$. Then $\int_{\Omega} \|f(\omega)\|_X^p d\mu \leq \int_{\Omega} \|g(\omega)\|_X^p d\mu < \infty$. This means that $f \in L^p(\Omega, X)$ and $\|f\|_p \leq \|g\|_p$. $\square$

3. The Arens Algebras $L^\omega(\Omega, X)$ of Vector-Valued Functions

In this section we introduce notion of the Arens algebras of vector-valued functions. Put

$$L^\omega(\Omega, X) = \bigcap_{p \geq 1} L^p(\Omega, X); \quad (10)$$

that is, $L^\omega(\Omega, X) = \{f \in L^0(\Omega, X) : \|f\|_1 < \infty, \|f\|_2 < \infty, \dots, \|f\|_p < \infty, \dots\}$, and will consider in $L^\omega(\Omega, X)$ local convex topology τ_X is generated by system of norms $\{\|\cdot\|_p\}_{p \geq 1}$. From (ii) Proposition 1 we get that $\|f\|_1 \leq \|f\|_2 \leq \dots \leq \|f\|_n \leq \dots$, that is, the topology τ_X generated by countable system of norms $\{\|\cdot\|_n\}_{n=1}^\infty$. By Theorem III.2.2 [12] it means that topological vector space $(L^\omega(\Omega, X), \tau_X)$ is metrizable space with respect to the metric

$$\rho(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f - g\|_k}{1 + \|f - g\|_k}. \quad (11)$$

Theorem 2. $(L^\omega(\Omega, X), \tau_X)$ is a complete topological algebra.

Proof. Let $\{f_n\}$ be a fundamental sequence in $(L^\omega(\Omega, X), \tau_X)$. Then $\{f_n\}$ is fundamental in $L^p(\Omega, X)$ for any $1 \leq p < \infty$. Since $L^p(\Omega, X)$ is complete there exist $f_p \in L^p(\Omega, X)$ such that $\|f_n - f_p\|_p \rightarrow 0$ as $n \rightarrow \infty$, as

$$\|f_n - f_p\|_q \leq \mu(\Omega)^{pq/p+q} \|f_n - f_p\|_p, \quad (12)$$

for all $1 \leq q < p < \infty$, implies that $\|f_n - f_p\|_q \rightarrow 0$. This means that $f_p = f_q$ if $1 \leq q < p < \infty$. Therefore $f = f_p$ is limit of sequence $\{f_n\}$ and f is an element of space $L^\omega(\Omega, X)$. Since the inequality

$$\|fg\|_p \leq \|f\|_{2p} \|g\|_{2p} \quad (13)$$

is valid the multiplication is continuous on $L^\omega(\Omega, X)$. Hence $(L^\omega(\Omega, X), \tau_X)$ is a topological algebra. $\square$

Theorem 3. (i) If $f \in L^0(\Omega, X)$, $g \in L^\omega(\Omega, X)$, and $\|f(\omega)\|_X \leq \|g(\omega)\|_X$ a.e. then $f \in L^\omega(\Omega, X)$ and $\rho(f, 0) \leq \rho(g, 0)$.

(ii) If $g \in L^\infty(\Omega, X)$ and $f \in L^\omega(\Omega, X)$ then $gf \in L^\omega(\Omega, X)$ and there exist $m \in \mathbb{N}$

$$\rho(gf, 0) \leq 2^m \rho(f, 0). \quad (14)$$

Proof. (i) As $g \in L^\omega(\Omega, X)$ we have that $g \in L^p(\Omega, X)$ for any $p \geq 1$. Then from $\|f(\omega)\|_X \leq \|g(\omega)\|_X$ a.e. by Proposition 1(iii) we get that $f \in L^p(\Omega, X)$ for any $p \geq 1$. This means that $f \in L^\omega(\Omega, X)$. Since $\|f\|_p \leq \|g\|_p$ for any $p \geq 1$ we have

$$\rho(f, 0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f\|_k}{1 + \|f\|_k} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|g\|_k}{1 + \|g\|_k} = \rho(g, 0). \quad (15)$$

(ii) Let $g \in L^\infty(\Omega, X)$ and $f \in L^\omega(\Omega, X)$; then $f \in L^p(\Omega, X)$ for all $p \geq 1$. By Proposition 1(i) $gf \in L^p(\Omega, X)$

for all $p \geq 1$. Therefore $gf \in L^\omega(\Omega, X)$. We chose $m \in \mathbb{N}$ that $\|g\|_\infty \leq 2^m$. Then using Proposition 1(i) we get

$$\begin{aligned} \rho(gf, 0) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|gf\|_k}{1 + \|gf\|_k} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|g\|_\infty \|f\|_k}{1 + \|g\|_\infty \|f\|_k} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{2^m \|f\|_k}{1 + 2^m \|f\|_k} \leq 2^m \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f\|_k}{1 + \|f\|_k} \\ &= 2^m \rho(f, 0). \end{aligned} \quad (16)$$

□

Theorem 4. $L^\infty(\Omega, X)$ is dense in $L^\omega(\Omega, X)$ and $L^\omega(\Omega, X)$ is dense in $L^p(\Omega, X)$.

Proof. Let $f \in L^\omega(\Omega, X)$. There exists sequence of idempotents e_n from $L^0(\Omega)$ such that $e_n \uparrow 1$, where 1 is unit element on $L^0(\Omega)$. Then $\|f(\omega)e_n(\omega)\|_X = \|f(\omega)\|_X e_n(\omega) \in L^\infty(\Omega)$; that is, $f(\omega)e_n(\omega) \in L^\infty(\Omega, X)$. Let $f_n = fe_n$. Then $f_n \in L^\infty(\Omega, X)$ and $\|f(\omega) - f_n(\omega)\|_X = \|f(\omega) - f(\omega)e_n(\omega)\|_X = \|f(\omega)(1(\omega) - e_n(\omega))\|_X = \|f(\omega)\|_X (1(\omega) - e_n(\omega)) \downarrow 0$. As norm $|\cdot|_p$ is order continuous in L^p space for all $p \geq 1$ we get that $\|f - f_n\|_p = \|f(\omega)\|_X (1(\omega) - e_n(\omega))|_p \rightarrow 0$ for all $p \geq 1$ as $n \rightarrow \infty$. Therefore

$$\rho(f_n, f) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f_n - f\|_k}{1 + \|f_n - f\|_k} \rightarrow 0, \quad (17)$$

as $n \rightarrow \infty$. The density $L^\omega(\Omega, X)$ in $L^p(\Omega, X)$ follows from density $L^\infty(\Omega, X)$ in $L^\omega(\Omega, X)$.

Let $\varphi_1, \varphi_2, \dots, \varphi_n \in L^\omega$ and $x_1, x_2, \dots, x_n \in X$. Then we can define function $f: \Omega \rightarrow X$ by setting

$$f(\omega) = \sum_{i=1}^n \varphi_i(\omega) x_i, \quad (18)$$

for almost all $\omega \in \Omega$.

We will denote $L^\omega \otimes X = \{f \in L^\omega(\Omega, X) : f(\omega) = \sum_{i=1}^n \varphi_i(\omega)x_i, \varphi_i \in L^\omega, x_i \in X, i = \overline{1, n}, n \in \mathbb{N}\}$. $L^\omega \otimes X$ is subspace of $L^\omega(\Omega, X)$, because $\int_\Omega \|f(\omega)\|_X^p d\mu \leq \sum_{i=1}^n \int_\Omega |\varphi_i(\omega)|^p \|x_i\|_X^p d\mu = \sum_{i=1}^n \|x_i\|_X^p \int_\Omega |\varphi_i(\omega)|^p d\mu < \infty$ for all $p \geq 1$. □

Theorem 5. $L^\omega \otimes X$ is dense in $L^\omega(\Omega, X)$.

Proof. Let $f \in L^\omega(\Omega, X)$. Then $f \in L^p(\Omega, X)$ for all $p \geq 1$.

Let $s_n(\omega)$ be the sequence simple X valued functions; that is, every $s_n(\omega)$ has form $\{\sum_{i=1}^m \chi_{B_i}(\omega)x_i : B_i \in \Sigma, B_i \cap B_j = \emptyset, x_i \in X\}$, such that $s_n(\omega) \rightarrow f(\omega)$ a.e. on Ω . Then $\|s_n(\omega)\|_X \rightarrow \|f(\omega)\|_X$ almost everywhere on Ω .

Let $A_n = \{\omega : \|s_n(\omega)\|_X \leq 2\|f(\omega)\|_X\}$. If we set $g_n(\omega) = s_n(\omega)\chi_{A_n}(\omega)$ we have that $g_n(\omega) \rightarrow f(\omega)$ a.e. on Ω and

$$\begin{aligned} \sup_n \|g_n(\omega) - f(\omega)\|_X &\leq \sup_n \|g_n(\omega)\|_X + \|f(\omega)\|_X \\ &\leq 3\|f(\omega)\|_X. \end{aligned} \quad (19)$$

By dominated convergence we have that

$$\int_\Omega \|g_n(\omega) - f(\omega)\|_X^p d\mu \rightarrow 0, \quad (20)$$

for all $p \geq 1$ and of course $g_n \in L^\omega \otimes X$. □

4. Dual of Arens Algebras $L^\omega(\Omega, X)$

In this section we study dual of Arens algebras of vector-valued functions. It is shown that if dual of X has Radon-Nikodym property, then dual space of $L^\omega(\Omega, X)$ is $\bigcup_{1 < q \leq \infty} L^q(\Omega, X^*)$; if X is reflexive, then $L^\omega(\Omega, X)$ also is reflexive.

Definition 6 (see [13]). Recall that a Banach space Y is said to possess the Radon-Nikodym property, with respect to μ , if for any μ -continuous vector measure $m: \Sigma \rightarrow Y$ of bounded total variation, there exists a Bochner integrable function $f: \Omega \rightarrow Y$ such that

$$m(A) = \int_A f d\mu, \quad (21)$$

for all $A \in \Sigma$. It is known [13] that dual of Banach algebra c_0 , $c_0^* = l_1$ has the Radon-Nikodym property.

A class of Banach algebras X whose duals X^* have the Radon-Nikodym property is studied in [14].

Theorem 7. If the dual space X^* of Banach algebra X has Radon-Nikodym property with respect to μ , then the dual of $L^\omega(\Omega, X)$ is the space

$$E = \bigcup_{1 < q \leq \infty} L^q(\Omega, X^*), \quad (22)$$

and for any $\ell \in (L^\omega(\Omega, X))^*$ there exists $g \in E$ such that

$$\ell(f) = \int_\Omega gf d\mu \quad (f \in L^\omega(\Omega, X)). \quad (23)$$

Proof. Since $L^\omega(\Omega, X)$ is countably normed space by [15] dual of $L^\omega(\Omega, X)$ is the space $\bigcup_{1 < q \leq \infty} (L^p(\Omega, X))^*$, where $(1/p) + (1/q) = 1$. As X^* has the Radon-Nikodym property with respect to μ by Theorem IV.1.1 [13] $(L^p(\Omega, X))^* = L^q(\Omega, X^*)$, where $(1/p) + (1/q) = 1$. Hence, the dual of $L^\omega(\Omega, X)$ is the space

$$E = \bigcup_{1 < q \leq \infty} L^q(\Omega, X^*), \quad (24)$$

where $(1/p) + (1/q) = 1$.

Let ℓ be a continuous linear functional on $L^\omega(\Omega, X)$. Then by [15] there exists natural number p ($1 \leq p < \infty$) and $C > 0$ such that

$$|\ell(f)| \leq C\|f\|_p. \quad (25)$$

It means $\ell \in (L^p(\Omega, X))^*$. Using the general form of continuous linear functionals in $L^p(\Omega, X)$ (see [13], IV.1) we get that there exists $g \in L^q(\Omega, X^*) \subset E$ such that

$$\ell(f) = \int_\Omega gf d\mu. \quad (26)$$

□

Theorem 8. *If X is a reflexive Banach algebra then $L^\omega(\Omega, X)$ is reflexive.*

Proof. Let X be a reflexive Banach algebra. Since every reflexive space has Radon-Nikodym property with respect to μ , by Theorem 7,

$$\begin{aligned} L^\omega(\Omega, X)^{**} &= \left(\bigcup_{1 < q \leq \infty} L^q(\Omega, X^*) \right)^* = \bigcap_{p \geq 1} L^p(\Omega, X^{**}) \\ &= \bigcap_{p \geq 1} L^p(\Omega, X) = L^\omega(\Omega, X). \end{aligned} \quad (27)$$

This means that $L^\omega(\Omega, X)$ is semireflexive space. By Theorem 2 $L^\omega(\Omega, X)$ is (F) -space. Since by Corollary II.7.1 [16] every (F) -space is barreled we have that $L^\omega(\Omega, X)$ is barreled. Hence by IV.5. [16] $L^\omega(\Omega, X)$ is reflexive. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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