Research Article

The Boundedness of Intrinsic Square Functions on the Weighted Herz Spaces

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We will obtain the strong type and weak type estimates of intrinsic square functions including the Lusin area integral, Littlewood-Paley $\mathcal{G}$ -function, and $\mathcal{G}_\lambda^*$ -function on the weighted Herz spaces $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ with general weights.

1. Introduction and Main Results

Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$. The classical square function (Lusin area integral) is a familiar object. If $u(x, t) = P_t * f(x)$ is the Poisson integral of f , where $P_t(x) = (c_n)(t/(t^2 + |x|^2))^{(n+1)/2}$ denotes the Poisson kernel in $\mathbb{R}_+^{n+1}$, then we define the classical square function (Lusin area integral) $S(f)$ by (see [1, 2])

$$S(f)(x) = \left(\iint_{\Gamma(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}, \quad (1)$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\},$$

$$|\nabla u(y, t)|^2 = \left| \frac{\partial u}{\partial t} \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial y_j} \right|^2. \quad (2)$$

Similarly, we can define a cone of aperture γ for any $\gamma > 0$:

$$\Gamma_\gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \gamma t\}, \quad (3)$$

and corresponding square function

$$S_\gamma(f)(x) = \left(\iint_{\Gamma_\gamma(x)} |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}. \quad (4)$$

The Littlewood-Paley g -function (could be viewed as a “zero-aperture” version of $S(f)$) and the g_λ^* -function (could be

viewed as an “infinite aperture” version of $S(f)$) are defined, respectively, by (see, e.g., [3, 4])

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x, t)|^2 t dt \right)^{1/2},$$

$$g_\lambda^*(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \times |\nabla u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}, \quad (5)$$

$$\lambda > 1.$$

The modern (real-variable) variant of $S_\gamma(f)$ can be defined in the following way (here we drop the subscript γ if $\gamma = 1$). Let $\psi \in C^\infty(\mathbb{R}^n)$ be real, radial and have support contained in $\{x : |x| \leq 1\}$, and let $\int_{\mathbb{R}^n} \psi(x) dx = 0$. The continuous square function $S_{\psi,\gamma}(f)$ is defined by (see, e.g., [5, 6])

$$S_{\psi,\gamma}(f)(x) = \left(\iint_{\Gamma_\gamma(x)} |f * \psi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}. \quad (6)$$

In 2007, Wilson [7] introduced a new square function called intrinsic square function which is universal in a sense (see also [8]). This function is independent of any particular kernel ψ , and it dominates pointwise all the above-defined

square functions. On the other hand, it is not essentially larger than any particular $S_{\psi, \gamma}(f)$. For $0 < \beta \leq 1$, let $\mathcal{C}_\beta$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \varphi(x) dx = 0$, and for all $x, x' \in \mathbb{R}^n$,

$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\beta. \quad (7)$$

For $(y, t) \in \mathbb{R}_+^{n+1}$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, we set

$$\begin{aligned} A_\beta(f)(y, t) &= \sup_{\varphi \in \mathcal{C}_\beta} |\varphi * \varphi_t(y)| \\ &= \sup_{\varphi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} \varphi_t(y - z) f(z) dz \right|. \end{aligned} \quad (8)$$

Then we define the intrinsic square function of f (of order β) by the following formula:

$$\mathcal{S}_\beta(f)(x) = \left(\iint_{\Gamma(x)} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}. \quad (9)$$

We can also define varying-aperture versions of $\mathcal{S}_\beta(f)$ by the formula

$$\mathcal{S}_{\beta, \gamma}(f)(x) = \left(\iint_{\Gamma_\gamma(x)} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}. \quad (10)$$

The intrinsic Littlewood-Paley $\mathcal{G}$ -function and the intrinsic $\mathcal{G}_\lambda^*$ -function will be given, respectively, by

$$\begin{aligned} \mathcal{G}_\beta(f)(x) &= \left(\int_0^\infty (A_\beta(f)(x, t))^2 \frac{dt}{t} \right)^{1/2}, \\ \mathcal{G}_{\lambda, \beta}^*(f)(x) &= \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \right. \\ &\quad \times (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \Big)^{1/2}, \quad \lambda > 1. \end{aligned} \quad (11)$$

In [8], Wilson showed the following weighted L^p boundedness of the intrinsic square functions.

Theorem A. *Let $0 < \beta \leq 1$, $1 < p < \infty$, and $w \in A_p$ (Muckenhoupt weight class). Then there exists a constant $C > 0$ independent of f such that*

$$\|\mathcal{S}_\beta(f)\|_{L_w^p} \leq C \|f\|_{L_w^p}. \quad (12)$$

Moreover, in [9], Lerner obtained sharp L_w^p norm inequalities for the intrinsic square functions in terms of the A_p characteristic constant of w for all $1 < p < \infty$. For further discussions about the boundedness of intrinsic square functions on various function spaces, we refer the readers to [10–17].

Before stating our main results, let us first recall some definitions about the weighted Herz and weak Herz spaces. For

more information about these spaces, one can see [18–22] and the references therein. Let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and let $C_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$, and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_E is the characteristic function of the set E . For any given weight function w on $\mathbb{R}^n$ and $0 < q < \infty$, we denote by $L_w^q(\mathbb{R}^n)$ the space of all functions f satisfying

$$\|f\|_{L_w^q} = \left(\int_{\mathbb{R}^n} |f(x)|^q w(x) dx \right)^{1/q} < \infty. \quad (13)$$

Definition 1 (see [19]). Let $\alpha \in \mathbb{R}$, $0 < p$, and $q < \infty$, and let w_1, w_2 be two weight functions on $\mathbb{R}^n$.

(a) The homogeneous weighted Herz space $\dot{K}_q^{\alpha, p}(w_1, w_2)$ is defined by

$$\begin{aligned} \dot{K}_q^{\alpha, p}(w_1, w_2) \\ = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}, w_2) : \|f\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)} < \infty \right\}, \end{aligned} \quad (14)$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)} = \left(\sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f \chi_k\|_{L_{w_2}^q}^p \right)^{1/p}. \quad (15)$$

(b) The nonhomogeneous weighted Herz space $K_q^{\alpha, p}(w_1, w_2)$ is defined by

$$\begin{aligned} K_q^{\alpha, p}(w_1, w_2) \\ = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n, w_2) : \|f\|_{K_q^{\alpha, p}(w_1, w_2)} < \infty \right\}, \end{aligned} \quad (16)$$

where

$$\|f\|_{K_q^{\alpha, p}(w_1, w_2)} = \left(\sum_{k=0}^{\infty} [w_1(B_k)]^{\alpha p/n} \|f \tilde{\chi}_k\|_{L_{w_2}^q}^p \right)^{1/p}. \quad (17)$$

For any $k \in \mathbb{Z}$, $\lambda > 0$, and any measurable function f on $\mathbb{R}^n$, we set $E_k(\lambda, f) = \{x \in C_k : |f(x)| > \lambda\}$. Let $\tilde{E}_k(\lambda, f) = E_k(\lambda, f)$ for $k \in \mathbb{N}$ and $\tilde{E}_0(\lambda, f) = \{x \in B(0, 1) : |f(x)| > \lambda\}$.

Definition 2 (see [21]). Let $\alpha \in \mathbb{R}$, $0 < p$, and $q < \infty$, and let w_1, w_2 be two weight functions on $\mathbb{R}^n$.

(c) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weighted weak Herz space $W\dot{K}_q^{\alpha, p}(w_1, w_2)$ if

$$\begin{aligned} \|f\|_{W\dot{K}_q^{\alpha, p}(w_1, w_2)} \\ = \sup_{\lambda > 0} \lambda \left(\sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} [w_2(E_k(\lambda, f))]^{p/q} \right)^{1/p} < \infty. \end{aligned} \quad (18)$$

- (d) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the nonhomogeneous weighted weak Herz space $WK_q^{\alpha,p}(w_1, w_2)$ if

$$\|f\|_{WK_q^{\alpha,p}(w_1, w_2)} = \sup_{\lambda > 0} \lambda \left(\sum_{k=0}^{\infty} [w_1(B_k)]^{\alpha p/n} [w_2(\tilde{E}_k(\lambda, f))]^{p/q} \right)^{1/p} < \infty. \quad (19)$$

Obviously, if $\alpha = 0$, then $\dot{K}_q^{0,q}(w_1, w_2) = K_q^{0,q}(w_1, w_2) = L_{w_2}^q(\mathbb{R}^n)$ for any $0 < q < \infty$. We also have $WK_q^{0,q}(w_1, w_2) = WL_{w_2}^q(\mathbb{R}^n)$ when $\alpha = 0$ and $0 < q < \infty$, where

$$\|f\|_{WL_{w_2}^q} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/q} < \infty. \quad (20)$$

Thus, weighted (weak) Herz spaces are generalizations of the weighted (weak) Lebesgue spaces. The main purpose of this paper is to consider the boundedness of intrinsic square functions on weighted Herz spaces with A_p weights. At the extreme case, we will also prove that these operators are bounded from the weighted Herz spaces to the weighted weak Herz spaces. Our main results in the paper are formulated as follows.

Theorem 3. Let $0 < \beta \leq 1$, $0 < p < \infty$, $1 < q < \infty$, $w_1 \in A_{q_1}$, and $w_2 \in A_{q_2}$. Then $\mathcal{S}_\beta$ is bounded on $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ provided that w_1 and w_2 satisfy either of the following:

- (i) $w_1 = w_2$, $1 \leq q_1 = q_2 \leq q$, and $-nq_1/q < \alpha q_1 < n(1 - q_2/q)$;
- (ii) $w_1 \neq w_2$, $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, and $0 < \alpha q_1 < n(1 - q_2/q)$.

Theorem 4. Let $0 < \beta \leq 1$, $0 < p \leq 1$, $1 < q < \infty$, $w_1 \in A_{q_1}$, and $w_2 \in A_{q_2}$. If $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, and $\alpha q_1 = n(1 - q_2/q)$, then $\mathcal{S}_\beta$ is bounded from $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ into $W\dot{K}_q^{\alpha,p}(w_1, w_2)(WK_q^{\alpha,p}(w_1, w_2))$.

Theorem 5. Let $0 < \beta \leq 1$, $0 < p < \infty$, $1 < q < \infty$, $w_1 \in A_{q_1}$, and $w_2 \in A_{q_2}$. If $\lambda > \max\{q_2, 3\}$, then $\mathcal{S}_{\lambda,\beta}^*$ is bounded on $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ provided that w_1 and w_2 satisfy either of the following:

- (i) $w_1 = w_2$, $1 \leq q_1 = q_2 \leq q$, and $-nq_1/q < \alpha q_1 < n(1 - q_2/q)$;
- (ii) $w_1 \neq w_2$, $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, and $0 < \alpha q_1 < n(1 - q_2/q)$.

Theorem 6. Let $0 < \beta \leq 1$, $0 < p \leq 1$, $1 < q < \infty$, $w_1 \in A_{q_1}$, and $w_2 \in A_{q_2}$. If $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, $\alpha q_1 = n(1 - q_2/q)$, and $\lambda > \max\{q_2, 3\}$, then $\mathcal{S}_{\lambda,\beta}^*$ is bounded from $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ into $W\dot{K}_q^{\alpha,p}(w_1, w_2)(WK_q^{\alpha,p}(w_1, w_2))$.

In [7], Wilson also showed that for any $0 < \beta \leq 1$, the functions $\mathcal{S}_\beta(f)(x)$ and $\mathcal{E}_\beta(f)(x)$ are pointwise comparable, with comparability constants depending only on β and n . Thus, as direct consequences of Theorems 3 and 4, we obtain the following.

Corollary 7. Let $0 < \beta \leq 1$, $0 < p < \infty$, $1 < q < \infty$, $w_1 \in A_{q_1}$, and $w_2 \in A_{q_2}$. Then $\mathcal{S}_\beta$ is bounded on $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ provided that w_1 and w_2 satisfy either of the following:

- (i) $w_1 = w_2$, $1 \leq q_1 = q_2 \leq q$, and $-nq_1/q < \alpha q_1 < n(1 - q_2/q)$;
- (ii) $w_1 \neq w_2$, $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, and $0 < \alpha q_1 < n(1 - q_2/q)$.

Corollary 8. Let $0 < \beta \leq 1$, $0 < p \leq 1$, $1 < q < \infty$, $w_1 \in A_{q_1}$, and $w_2 \in A_{q_2}$. If $1 \leq q_1 < \infty$, $1 \leq q_2 \leq q$, and $\alpha q_1 = n(1 - q_2/q)$, then $\mathcal{S}_\beta$ is bounded from $\dot{K}_q^{\alpha,p}(w_1, w_2)(K_q^{\alpha,p}(w_1, w_2))$ into $W\dot{K}_q^{\alpha,p}(w_1, w_2)(WK_q^{\alpha,p}(w_1, w_2))$.

2. A_p Weights

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [23]. A weight w is a nonnegative, locally integrable function on $\mathbb{R}^n$, $B = B(x_0, r_B)$ that denotes the ball with the center x_0 and radius r_B . For any ball B and $\lambda > 0$, λB denotes the ball concentric with B whose radius is λ times as long. For a given weight function w and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $w(E) = \int_E w(x) dx$. We say that w is in the Muckenhoupt class A_p with $1 < p < \infty$, if there exists a constant $C > 0$ such that for every ball $B \subseteq \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C. \quad (21)$$

For the endpoint case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \cdot \operatorname{ess\,inf}_{x \in B} w(x) \quad (22)$$

for every ball $B \subseteq \mathbb{R}^n$,

where C is a positive constant which is independent of the choice of B . The smallest value of C such that the above inequalities hold is called the A_p characteristic constant of w and denoted by $[w]_{A_p}$. If there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds:

$$\left(\frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq C \left(\frac{1}{|B|} \int_B w(x) dx \right) \quad (23)$$

for every ball $B \subseteq \mathbb{R}^n$,

then we say that w satisfies the reverse Hölder condition of order r and write $w \in RH_r$. It is well known that if $w \in A_p$

with $1 \leq p < \infty$, then $w \in A_q$ for all $q > p$. Moreover, if $w \in A_p$ with $1 \leq p < \infty$, then there exists $r > 1$ such that $w \in RH_r$.

The following properties for A_p weights will be repeatedly used in this paper.

Lemma 9 (see [24]). *Let $w \in A_p$ with $p \geq 1$. Then, for any ball B , there exists an absolute constant $C > 0$ such that*

$$w(2B) \leq Cw(B). \quad (24)$$

In general, for any $\lambda > 1$, one has

$$w(\lambda B) \leq C \cdot \lambda^{np} w(B), \quad (25)$$

where C does not depend on B nor on λ .

Lemma 10 (see [1, 24]). *Let $w \in A_p \cap RH_r$, $p \geq 1$, and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \left(\frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left(\frac{|E|}{|B|} \right)^{(r-1)/r} \quad (26)$$

for any measurable subset E of a ball B .

Throughout this paper, C always denotes a positive constant which is independent of the main parameters involved but may vary from line to line.

3. Proofs of Theorems 3 and 4

Proof of Theorem 3. We only need to show the theorem for the homogeneous case because the proof of the nonhomogeneous result is similar and so is omitted here. Let $f \in \dot{K}_q^{\alpha,p}(w_1, w_2)$. Following [25], for any $k \in \mathbb{Z}$, we decompose $f(x)$ as

$$\begin{aligned} f(x) &= f(x) \chi_{\{2^{k-2} < |x| \leq 2^{k+1}\}}(x) \\ &\quad + f(x) \chi_{\{|x| \leq 2^{k-2}\}}(x) + f(x) \chi_{\{|x| > 2^{k+1}\}}(x) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned} \quad (27)$$

Since $\mathcal{S}_\beta$ ($0 < \beta \leq 1$) is a sublinear operator, then we can write

$$\begin{aligned} \|\mathcal{S}_\beta(f)\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}^p &= \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|\mathcal{S}_\beta(f) \chi_k\|_{L_{w_2}^q}^p \\ &\leq C \sum_{i=1}^3 \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|\mathcal{S}_\beta(f_i) \chi_k\|_{L_{w_2}^q}^p \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (28)$$

Since $w_2 \in A_{q_2}$ and $1 \leq q_2 \leq q$, then $w_2 \in A_q$. By Theorem A and Lemma 9, we have

$$\begin{aligned} I_1 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f_1\|_{L_{w_2}^q}^p \\ &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f \chi_k\|_{L_{w_2}^q}^p \\ &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}^p. \end{aligned} \quad (29)$$

For the term I_2 , we first use Minkowski's inequality to derive

$$I_2 \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\sum_{\ell=-\infty}^{k-2} \|\mathcal{S}_\beta(f \chi_\ell) \chi_k\|_{L_{w_2}^q} \right)^p. \quad (30)$$

For any $\varphi \in \mathcal{C}_\beta$, $0 < \beta \leq 1$, and $(y, t) \in \Gamma(x)$, we have

$$\begin{aligned} |(f \chi_\ell) * \varphi_t(y)| &= \left| \int_{2^{\ell-1} < |z| \leq 2^\ell} \varphi_t(y-z) f(z) dz \right| \\ &\leq C \cdot t^{-n} \int_{\{2^{\ell-1} < |z| \leq 2^\ell\} \cap \{|z| \cdot |y-z| \leq t\}} |f(z)| dz. \end{aligned} \quad (31)$$

For any $x \in C_k$, $(y, t) \in \Gamma(x)$, and $z \in \{2^{\ell-1} < |z| \leq 2^\ell\} \cap B(y, t)$ with $\ell \leq k-2$, by a direct computation, we can easily see that

$$2t \geq |x-y| + |y-z| \geq |x-z| \geq |x| - |z| \geq \frac{|x|}{2}. \quad (32)$$

Thus, by using inequality (31) and Minkowski's inequality, we deduce

$$\begin{aligned} &|\mathcal{S}_\beta(f \chi_\ell)(x)| \\ &= \left(\iint_{\Gamma(x)} \left(\sup_{\varphi \in \mathcal{C}_\beta} |(f \chi_\ell) * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left(\int_{|x|/4}^\infty \int_{|x-y| < t} \left| t^{-n} \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \left(\int_{|x|/4}^\infty \frac{dt}{t^{2n+1}} \right)^{1/2} \\ &\leq C \cdot \frac{1}{|x|^n} \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right). \end{aligned} \quad (33)$$

Denote the conjugate exponent of $q > 1$ by $q' = q/(q-1)$. Applying Hölder's inequality and the A_q condition, we can deduce that

$$\begin{aligned} & \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \\ & \leq \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)|^q w_2(z) dz \right)^{1/q} \\ & \quad \times \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} w_2(z)^{-q'/q} dz \right)^{1/q'} \\ & \leq C \cdot |B_\ell| [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q}. \end{aligned} \quad (34)$$

Substituting inequality (34) into (33), we thus obtain

$$\begin{aligned} I_2 & \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\ & \quad \times \left(\sum_{\ell=-\infty}^{k-2} \left\{ \int_{2^{k-1} < |x| \leq 2^k} |\mathcal{S}_\beta(f\chi_\ell)(x)|^q w_2(x) dx \right\}^{1/q} \right)^p \\ & \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\ & \quad \times \left(\sum_{\ell=-\infty}^{k-2} |B_\ell| [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q} \right. \\ & \quad \times \left. \left\{ \int_{2^{k-1} < |x| \leq 2^k} \frac{w_2(x)}{|x|^{nq}} dx \right\}^{1/q} \right)^p \\ & \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\ & \quad \times \left(\sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p. \end{aligned} \quad (35)$$

Here, we will consider two cases. For the case of $0 < p \leq 1$, using the well-known inequality $(\sum_\ell |a_\ell|)^p \leq \sum_\ell |a_\ell|^p$ and changing the order of summation, we find that

$$\begin{aligned} I_2 & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\ & \quad \times \left(\sum_{k=\ell+2}^{\infty} \frac{|B_\ell|^p}{|B_k|^p} \cdot \frac{[w_2(B_k)]^{p/q}}{[w_2(B_\ell)]^{p/q}} \cdot \frac{[w_1(B_k)]^{\alpha p/n}}{[w_1(B_\ell)]^{\alpha p/n}} \right). \end{aligned} \quad (36)$$

Moreover, it follows immediately from Lemma 9 that

$$\begin{aligned} I_2 & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\ & \quad \times \left(\sum_{k=\ell+2}^{\infty} \frac{|B_\ell|^p}{|B_k|^p} \cdot \frac{[w_2(B_k)]^{p/q}}{[w_2(B_{\ell+2})]^{p/q}} \cdot \frac{[w_1(B_k)]^{\alpha p/n}}{[w_1(B_{\ell+2})]^{p/n}} \right). \end{aligned} \quad (37)$$

Since $B_k \supseteq B_{\ell+2}$ when $k \geq \ell+2$ and $w_i \in A_{q_i}$ for $i = 1, 2$, by Lemma 10, we can get

$$\frac{w_i(B_k)}{w_i(B_{\ell+2})} \leq C \left(\frac{|B_k|}{|B_{\ell+2}|} \right)^{q_i}, \quad \text{for } i = 1 \text{ and } 2. \quad (38)$$

Therefore

$$\begin{aligned} I_2 & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\ & \quad \times \left(\sum_{k=\ell+2}^{\infty} \left[\frac{|B_{\ell+2}|}{|B_k|} \right]^{p-\alpha q_1 p/n - q_2 p/q} \right) \\ & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\ & \quad \times \left(\sum_{k=0}^{\infty} 2^{-kn(p-\alpha q_1 p/n - q_2 p/q)} \right) \\ & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p, \end{aligned} \quad (39)$$

where the last inequality holds since $\alpha q_1 < n(1-q_2/q)$. On the other hand, for the case of $1 < p < \infty$, we will use Hölder's inequality to obtain

$$\begin{aligned} & \left(\sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \cdot [w_1(B_k)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \right)^p \\ & \leq \left(\sum_{\ell=-\infty}^{k-2} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \right. \\ & \quad \times \frac{|B_\ell|^{p/2}}{|B_k|^{p/2}} \cdot \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \Bigg) \\ & \quad \times \left(\sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|^{p'/2}}{|B_k|^{p'/2}} \cdot \frac{[w_2(B_k)]^{p'/2q}}{[w_2(B_\ell)]^{p'/2q}} \right. \\ & \quad \times \left. \frac{[w_1(B_k)]^{\alpha p'/2n}}{[w_1(B_\ell)]^{\alpha p'/2n}} \right)^{p/p'}. \end{aligned} \quad (40)$$

Using the same arguments as above, we can also prove the following estimates under the assumption that $\alpha q_1 < n(1-q_2/q)$:

$$\begin{aligned} & \sum_{k=\ell+2}^{\infty} \frac{|B_\ell|^{p/2}}{|B_k|^{p/2}} \cdot \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \leq C, \\ & \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|^{p'/2}}{|B_k|^{p'/2}} \cdot \frac{[w_2(B_k)]^{p'/2q}}{[w_2(B_\ell)]^{p'/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p'/2n}}{[w_1(B_\ell)]^{\alpha p'/2n}} \leq C. \end{aligned} \quad (41)$$

Hence

$$\begin{aligned}
I_2 &\leq C \sum_{k \in \mathbb{Z}} \left(\sum_{\ell=-\infty}^{k-2} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \right. \\
&\quad \times \frac{|B_\ell|^{p/2}}{|B_k|^{p/2}} \cdot \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \Big) \\
&\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\
&\quad \times \left(\sum_{k=\ell+2}^{\infty} \frac{|B_\ell|^{p/2}}{|B_k|^{p/2}} \cdot \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \right) \\
&\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p. \tag{42}
\end{aligned}$$

Summarizing the above estimates for the term I_2 , we obtain that for every $0 < p < \infty$,

$$I_2 \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \leq C \|f\|_{\dot{K}_{q,p}^{\alpha,p}(w_1,w_2)}^p. \tag{43}$$

Let us now turn to estimate the last term I_3 . In this case, for any $x \in C_k$, $(y, t) \in \Gamma(x)$, and $z \in \{2^{\ell-1} < |z| \leq 2^\ell\} \cap B(y, t)$ with $\ell \geq k+2$, it is easy to check that

$$2t \geq |x - y| + |y - z| \geq |x - z| \geq |z| - |x| \geq \frac{|z|}{2}. \tag{44}$$

Then it follows from inequality (31) and Minkowski's inequality that

$$\begin{aligned}
|\mathcal{S}_\beta(f\chi_\ell)(x)| &\leq C \left(\int_{|z|/4}^{\infty} \int_{|x-y|<t} |t^{-n} \int_{2^{\ell-1}<|z|\leq 2^\ell} |f(z)| dz|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
&\leq C \left(\int_{2^{\ell-1}<|z|\leq 2^\ell} |f(z)| dz \right) \left(\int_{|z|/4}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
&\leq C \left(\int_{2^{\ell-1}<|z|\leq 2^\ell} \frac{|f(z)|}{|z|^n} dz \right). \tag{45}
\end{aligned}$$

This estimate together with (34) implies that

$$\begin{aligned}
|\mathcal{S}_\beta(f\chi_\ell)(x)| &\leq C \cdot \frac{1}{|B_\ell|} \left(\int_{2^{\ell-1}<|z|\leq 2^\ell} |f(z)| dz \right) \\
&\leq C \cdot [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{46}
\end{aligned}$$

Hence

$$\begin{aligned}
I_3 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\
&\quad \times \left(\sum_{\ell=k+2}^{\infty} \left\{ \int_{2^{k-1}<|x|\leq 2^k} |\mathcal{S}_\beta(f\chi_\ell)(x)|^q w_2(x) dx \right\}^{1/q} \right)^p \\
&\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\sum_{\ell=k+2}^{\infty} [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q} \right. \\
&\quad \times \left. \left\{ \int_{2^{k-1}<|x|\leq 2^k} w_2(x) dx \right\}^{1/q} \right)^p \\
&\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\sum_{\ell=k+2}^{\infty} \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p. \tag{47}
\end{aligned}$$

Now we will consider the following two cases again. For the case of $0 < p \leq 1$, by using the inequality $(\sum_\ell |a_\ell|)^p \leq \sum_\ell |a_\ell|^p$ and changing the order of summation, we obtain

$$\begin{aligned}
I_3 &\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\
&\quad \times \left(\sum_{k=-\infty}^{\ell-2} \frac{[w_2(B_k)]^{p/q}}{[w_2(B_\ell)]^{p/q}} \cdot \frac{[w_1(B_k)]^{\alpha p/n}}{[w_1(B_\ell)]^{\alpha p/n}} \right) \\
&\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\
&\quad \times \left(\sum_{k=-\infty}^{\ell-2} \frac{[w_2(B_k)]^{p/q}}{[w_2(B_{\ell-2})]^{p/q}} \cdot \frac{[w_1(B_k)]^{\alpha p/n}}{[w_1(B_{\ell-2})]^{p/q}} \right). \tag{48}
\end{aligned}$$

Since $w_i \in A_{q_i}$, then there exists $r_i > 1$ such that $w_i \in RH_{r_i}$ for $i = 1, 2$. Thus, by Lemma 10 again, we can get

$$\frac{w_i(B_k)}{w_i(B_{\ell-2})} \leq C \left(\frac{|B_k|}{|B_{\ell-2}|} \right)^{\delta_i}, \quad \text{for } i = 1 \text{ and } 2, \tag{49}$$

where $\delta_i = (r_i - 1)/r_i > 0$. Therefore, we have

$$\begin{aligned}
I_3 &\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\
&\quad \times \left(\sum_{k=-\infty}^{\ell-2} \left[\frac{|B_k|}{|B_{\ell-2}|} \right]^{\alpha \delta_1 p/n + \delta_2 p/q} \right) \\
&\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\
&\quad \times \left(\sum_{k=-\infty}^0 2^{kn(\alpha \delta_1 p/n + \delta_2 p/q)} \right) \tag{50}
\end{aligned}$$

$$\leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p,$$

where in the last inequality we have used the fact that $\alpha \delta_1 p/n + \delta_2 p/q > 0$ under our assumption (i) or (ii).

On the other hand, for the case of $1 < p < \infty$, an application of Hölder's inequality gives us that

$$\begin{aligned} & \left(\sum_{\ell=k+2}^{\infty} \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \cdot [w_1(B_k)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p \\ & \leq \left(\sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \right. \\ & \quad \cdot \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \Bigg) \\ & \quad \times \left(\sum_{\ell=k+2}^{\infty} \frac{[w_2(B_k)]^{p'/2q}}{[w_2(B_\ell)]^{p'/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p'/2n}}{[w_1(B_\ell)]^{\alpha p'/2n}} \right)^{p/p'}. \end{aligned} \quad (51)$$

By using the same arguments as for I_3 , we are able to prove that the following two series are convergent by an absolute constant under the assumption (i) or (ii):

$$\begin{aligned} & \sum_{k=-\infty}^{\ell-2} \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \leq C, \\ & \sum_{\ell=k+2}^{\infty} \frac{[w_2(B_k)]^{p'/2q}}{[w_2(B_\ell)]^{p'/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p'/2n}}{[w_1(B_\ell)]^{\alpha p'/2n}} \leq C. \end{aligned} \quad (52)$$

Consequently

$$\begin{aligned} I_3 & \leq C \sum_{k \in \mathbb{Z}} \left(\sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \right. \\ & \quad \cdot \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \Bigg) \\ & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \\ & \quad \times \left(\sum_{k=-\infty}^{\ell-2} \frac{[w_2(B_k)]^{p/2q}}{[w_2(B_\ell)]^{p/2q}} \cdot \frac{[w_1(B_k)]^{\alpha p/2n}}{[w_1(B_\ell)]^{\alpha p/2n}} \right) \\ & \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p. \end{aligned} \quad (53)$$

From the above discussions for the term I_3 , we know that, for any $0 < p < \infty$,

$$I_3 \leq C \sum_{\ell \in \mathbb{Z}} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p. \quad (54)$$

Summing up the above estimates for I_1 , I_2 , and I_3 , we complete the proof of Theorem 3. $\square$

Proof of Theorem 4. Let $f \in \dot{K}_q^{\alpha,p}(w_1, w_2)$. For any $k \in \mathbb{Z}$, as in the proof of Theorem 3, we will split $f(x)$ into three parts:

$$\begin{aligned} f(x) &= f(x) \chi_{\{2^{k-2} < |x| \leq 2^{k+1}\}}(x) + f(x) \chi_{\{|x| \leq 2^{k-2}\}}(x) \\ & \quad + f(x) \chi_{\{|x| > 2^{k+1}\}}(x) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned} \quad (55)$$

Then, for any given $\lambda > 0$, we have

$$\begin{aligned} & \lambda^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} w_2(\{x \in C_k : |\mathcal{S}_\beta(f)(x)| > \lambda\})^{p/q} \\ & \leq \sum_{i=1}^3 \lambda^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\ & \quad \times w_2\left(\left\{x \in C_k : |\mathcal{S}_\beta(f_i)(x)| > \frac{\lambda}{3}\right\}\right)^{p/q} \\ &= I'_1 + I'_2 + I'_3. \end{aligned} \quad (56)$$

Applying Chebyshev's inequality, Theorem A, and Lemma 9, we obtain

$$\begin{aligned} I'_1 & \leq \lambda^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\frac{3^q}{\lambda^q} \|\mathcal{S}_\beta(f_1)\|_{L_{w_2}^q}^q \right)^{p/q} \\ & \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f_1\|_{L_{w_2}^q}^p \\ & \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f\chi_k\|_{L_{w_2}^q}^p \leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p. \end{aligned} \quad (57)$$

For any $x \in C_k$, it follows from inequalities (33) and (34) that

$$\begin{aligned} |\mathcal{S}_\beta(f_2)(x)| & \leq \sum_{\ell=-\infty}^{k-2} |\mathcal{S}_\beta(f\chi_\ell)(x)| \\ & \leq C \sum_{\ell=-\infty}^{k-2} \frac{1}{|x|^n} \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \\ & \leq C \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q}. \end{aligned} \quad (58)$$

By using Lemma 9, inequality (38), and the fact that $\alpha q_1 = n(1 - q_2/q)$, we deduce that

$$\begin{aligned}
 |\mathcal{S}_\beta(f_2)(x)| &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \sum_{\ell=-\infty}^{k-2} [w_1(B_\ell)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q} \cdot \frac{|B_\ell|}{|B_k|} \\
 &\quad \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_{\ell+2})]^{1/q}} \cdot \frac{[w_1(B_k)]^{\alpha/n}}{[w_1(B_{\ell+2})]^{1/q}} \\
 &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \sum_{\ell=-\infty}^{k-2} [w_1(B_\ell)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q} \\
 &\quad \cdot \left(\frac{|B_{\ell+2}|}{|B_k|} \right)^{1-\alpha q_1/n-q_2/q} \\
 &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \sum_{\ell=-\infty}^{k-2} [w_1(B_\ell)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{59}
 \end{aligned}$$

Moreover, since $0 < p \leq 1$, then we have that, for any $x \in C_k$,

$$\begin{aligned}
 |\mathcal{S}_\beta(f_2)(x)| &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \left(\sum_{\ell=-\infty}^{k-2} [w_1(B_\ell)]^{\alpha p/n} \|f\chi_\ell\|_{L_{w_2}^q}^p \right)^{1/p} \\
 &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}. \tag{60}
 \end{aligned}$$

Set $A_k = [w_1(B_k)]^{-\alpha/n} [w_2(B_k)]^{-1/q}$. If $\{x \in C_k : |\mathcal{S}_\beta(f_2)(x)| > \lambda/3\} = \emptyset$, then the inequality

$$I'_2 \leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p \tag{61}$$

holds trivially. Now we suppose that $\{x \in C_k : |\mathcal{S}_\beta(f_2)(x)| > \lambda/3\} \neq \emptyset$. First it is easy to verify that $\lim_{k \rightarrow \infty} A_k = 0$. Then, for any fixed $\lambda > 0$, we are able to find a maximal positive integer k_λ such that

$$\frac{\lambda}{3} \leq C \cdot A_{k_\lambda} \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}. \tag{62}$$

Hence

$$\begin{aligned}
 I'_2 &\leq \lambda^p \sum_{k=-\infty}^{k_\lambda} [w_1(B_k)]^{\alpha p/n} [w_2(B_k)]^{p/q} \\
 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p \sum_{k=-\infty}^{k_\lambda} \frac{[w_1(B_k)]^{\alpha p/n}}{[w_1(B_{k_\lambda})]^{1/p}} \cdot \frac{[w_2(B_k)]^{p/q}}{[w_2(B_{k_\lambda})]^{1/p}}. \tag{63}
 \end{aligned}$$

Because $B_k \subseteq B_{k_\lambda}$, then by Lemma 10, with the same notations δ_i as in (49), we can get

$$\frac{w_i(B_k)}{w_i(B_{k_\lambda})} \leq C \left(\frac{|B_k|}{|B_{k_\lambda}|} \right)^{\delta_i}, \quad \text{for } i = 1 \text{ and } 2. \tag{64}$$

Therefore

$$\begin{aligned}
 I'_2 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p \sum_{k=-\infty}^{k_\lambda} \left(\frac{|B_k|}{|B_{k_\lambda}|} \right)^{\alpha \delta_1 p/n + \delta_2 p/q} \\
 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p. \tag{65}
 \end{aligned}$$

On the other hand, it follows from inequalities (34) and (45) that

$$\begin{aligned}
 |\mathcal{S}_\beta(f_3)(x)| &\leq \sum_{\ell=k+2}^{\infty} |\mathcal{S}_\beta(f\chi_\ell)(x)| \\
 &\leq C \sum_{\ell=k+2}^{\infty} \int_{2^{\ell-1} < |z| \leq 2^\ell} \frac{|f(z)|}{|z|^n} dz \\
 &\leq C \sum_{\ell=k+2}^{\infty} [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{66}
 \end{aligned}$$

In the present situation, since $B_k \subseteq B_{\ell-2}$ with $\ell \geq k+2$, then it follows from inequality (49) that

$$\begin{aligned}
 |\mathcal{S}_\beta(f_3)(x)| &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q} \\
 &\quad \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_{\ell-2})]^{1/q}} \cdot \frac{[w_1(B_k)]^{\alpha/n}}{[w_1(B_{\ell-2})]^{1/q}} \\
 &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q} \\
 &\quad \cdot \left(\frac{|B_k|}{|B_{\ell-2}|} \right)^{\alpha \delta_1/n + \delta_2/q} \\
 &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\
 &\quad \times \sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha/n} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{67}
 \end{aligned}$$

Furthermore, recalling that $0 < p \leq 1$, then, for any $x \in C_k$, we have

$$\begin{aligned} |\mathcal{S}_\beta(f_3)(x)| &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \\ &\quad \times \left(\sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha p/n} \|f \chi_\ell\|_{L_{w_2}^p}^p \right)^{1/p} \\ &\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n} [w_2(B_k)]^{1/q}} \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}. \end{aligned} \quad (68)$$

Repeating the arguments used for the term I'_2 , we can also obtain

$$I'_3 \leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}^p. \quad (69)$$

Combining the above estimates for I'_1 , I'_2 , and I'_3 and then taking the supremum over all $\lambda > 0$, we finish the proof of Theorem 4. $\square$

4. Proofs of Theorems 5 and 6

In order to prove the main theorems of this section, let us first establish the following results.

Proposition 11. *Let $0 < \beta \leq 1$, $q = 2$, and $w \in A_{q_2}$ with $1 \leq q_2 \leq q$. Then, for any $j \in \mathbb{Z}_+$, one has*

$$\|\mathcal{S}_{\beta, 2^j}(f)\|_{L_w^2} \leq C \cdot 2^{jnq_2/2} \|\mathcal{S}_\beta(f)\|_{L_w^2}. \quad (70)$$

Proof. Since $w \in A_{q_2}$, then, by Lemma 9, we know that for any $(y, t) \in \mathbb{R}_+^{n+1}$,

$$\begin{aligned} w(B(y, 2^j t)) &= w(2^j B(y, t)) \leq C \cdot 2^{jnq_2} w(B(y, t)) \\ j &= 1, 2, \dots \end{aligned} \quad (71)$$

Therefore

$$\begin{aligned} \|\mathcal{S}_{\beta, 2^j}(f)\|_{L_w^2}^2 &= \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}_+^{n+1}} (A_\beta(f)(y, t))^2 \chi_{|x-y| < 2^j t} \frac{dy dt}{t^{n+1}} \right) w(x) dx \\ &= \iint_{\mathbb{R}_+^{n+1}} \left(\int_{|x-y| < 2^j t} w(x) dx \right) (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{jnq_2} \iint_{\mathbb{R}_+^{n+1}} \left(\int_{|x-y| < t} w(x) dx \right) (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &= C \cdot 2^{jnq_2} \|\mathcal{S}_\beta(f)\|_{L_w^2}^2. \end{aligned} \quad (72)$$

Taking square-roots on both sides of the above inequality, we are done. $\square$

Proposition 12. *Let $0 < \beta \leq 1$, $2 < q < \infty$, and $w \in A_{q_2}$ with $1 \leq q_2 \leq q$. Then, for any $j \in \mathbb{Z}_+$, one has*

$$\|\mathcal{S}_{\beta, 2^j}(f)\|_{L_w^q} \leq C \cdot 2^{jnq_2/2} \|\mathcal{S}_\beta(f)\|_{L_w^q}. \quad (73)$$

Proof. For any $j \in \mathbb{Z}_+$ and $0 < \beta \leq 1$, it is easy to see that

$$\|\mathcal{S}_{\beta, 2^j}(f)\|_{L_w^q}^2 = \|\mathcal{S}_{\beta, 2^j}(f)\|_{L_w^{q/2}}^2. \quad (74)$$

Since $q/2 > 1$, then, by duality, we have

$$\begin{aligned} \|\mathcal{S}_{\beta, 2^j}(f)\|_{L_w^{q/2}}^2 &= \sup_{\|b\|_{L_w^{(q/2)'}} \leq 1} \left| \int_{\mathbb{R}^n} \mathcal{S}_{\beta, 2^j}(f)(x)^2 b(x) w(x) dx \right| \\ &= \sup_{\|b\|_{L_w^{(q/2)'}} \leq 1} \left| \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}_+^{n+1}} (A_\beta(f)(y, t))^2 \right. \right. \\ &\quad \times \left. \left. \chi_{|x-y| < 2^j t} \frac{dy dt}{t^{n+1}} \right) b(x) w(x) dx \right| \\ &= \sup_{\|b\|_{L_w^{(q/2)'}} \leq 1} \left| \iint_{\mathbb{R}_+^{n+1}} \left(\int_{|x-y| < 2^j t} b(x) w(x) dx \right) \right. \\ &\quad \times \left. (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right|. \end{aligned} \quad (75)$$

For $w \in A_{q_2}$, we denote the weighted maximal operator by M_w ; that is,

$$M_w(f)(x) = \sup_{x \in B} \frac{1}{w(B)} \int_B |f(y)| w(y) dy, \quad (76)$$

where the supremum is taken over all balls B which contain x . Then, by Lemma 9, we can get

$$\begin{aligned} &\int_{|x-y| < 2^j t} b(x) w(x) dx \\ &\leq C \cdot 2^{jnq_2} w(B(y, t)) \cdot \frac{1}{w(B(y, 2^j t))} \\ &\quad \times \int_{B(y, 2^j t)} b(x) w(x) dx \\ &\leq C \cdot 2^{jnq_2} w(B(y, t)) \inf_{x \in B(y, 2^j t)} M_w(b)(x) \\ &\leq C \cdot 2^{jnq_2} \int_{|x-y| < t} M_w(b)(x) w(x) dx. \end{aligned} \quad (77)$$

Substituting inequality (77) into (75) and then using Hölder's inequality together with the $L_w^{(q/2)'}$ boundedness of M_w , we thus obtain

$$\begin{aligned}
& \|\mathcal{S}_{\beta,2^j}(f)\|_{L_w^{(q/2)}}^2 \\
& \leq C \cdot 2^{jnq_2} \sup_{\|b\|_{L_w^{(q/2)'}} \leq 1} \left| \int_{\mathbb{R}^n} \mathcal{S}_{\beta}(f)(x)^2 M_w(b)(x) w(x) dx \right| \\
& \leq C \cdot 2^{jnq_2} \|\mathcal{S}_{\beta}(f)\|_{L_w^{q/2}}^2 \sup_{\|b\|_{L_w^{(q/2)'}} \leq 1} \|M_w(b)\|_{L_w^{(q/2)'}} \\
& \leq C \cdot 2^{jnq_2} \|\mathcal{S}_{\beta}(f)\|_{L_w^{q/2}}^2 \\
& = C \cdot 2^{jnq_2} \|\mathcal{S}_{\beta}(f)\|_{L_w^q}^2.
\end{aligned} \tag{78}$$

This estimate together with (74) implies the desired result. $\square$

Proposition 13. *Let $0 < \beta \leq 1$, $1 < q < 2$, and $w \in A_{q_2}$ with $1 \leq q_2 \leq q$. Then, for any $j \in \mathbb{Z}_+$, one has*

$$\|\mathcal{S}_{\beta,2^j}(f)\|_{L_w^q} \leq C \cdot 2^{jnq_2/q} \|\mathcal{S}_{\beta}(f)\|_{L_w^q}. \tag{79}$$

Proof. We will adopt the same method given in [26]. For any $j \in \mathbb{Z}_+$, set $\Omega_{\lambda} = \{x \in \mathbb{R}^n : \mathcal{S}_{\beta}(f)(x) > \lambda\}$ and $\Omega_{\lambda,j} = \{x \in \mathbb{R}^n : \mathcal{S}_{\beta,2^j}(f)(x) > \lambda\}$. We also set

$$\Omega_{\lambda}^* = \left\{ x \in \mathbb{R}^n : M_w(\chi_{\Omega_{\lambda}})(x) > \frac{1}{2^{(jnq_2+1)} \cdot [w]_{A_{q_2}}} \right\}. \tag{80}$$

Observe that $w(\Omega_{\lambda,j}) \leq w(\Omega_{\lambda}^*) + w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*))$. Thus, for any $j \in \mathbb{Z}_+$,

$$\begin{aligned}
\|\mathcal{S}_{\beta,2^j}(f)\|_{L_w^q}^q &= \int_0^{\infty} q\lambda^{q-1} w(\Omega_{\lambda,j}) d\lambda \\
&\leq \int_0^{\infty} q\lambda^{q-1} w(\Omega_{\lambda}^*) d\lambda \\
&\quad + \int_0^{\infty} q\lambda^{q-1} w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*)) d\lambda \\
&= \text{I} + \text{II}.
\end{aligned} \tag{81}$$

The weighted weak type estimate of M_w yields

$$\text{I} \leq C \cdot 2^{jnq_2} \int_0^{\infty} q\lambda^{q-1} w(\Omega_{\lambda}) d\lambda \leq C \cdot 2^{jnq_2} \|\mathcal{S}_{\beta}(f)\|_{L_w^q}^q. \tag{82}$$

To estimate II, we now claim that the following inequality holds:

$$\begin{aligned}
& \int_{\mathbb{R}^n \setminus \Omega_{\lambda}^*} \mathcal{S}_{\beta,2^j}(f)(x)^2 w(x) dx \\
& \leq C \cdot 2^{jnq_2} \int_{\mathbb{R}^n \setminus \Omega_{\lambda}} \mathcal{S}_{\beta}(f)(x)^2 w(x) dx.
\end{aligned} \tag{83}$$

Assuming the claim for the moment, then, it follows from Chebyshev's inequality and inequality (83) that

$$\begin{aligned}
& w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*)) \\
& \leq \lambda^{-2} \int_{\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*)} \mathcal{S}_{\beta,2^j}(f)(x)^2 w(x) dx \\
& \leq \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_{\lambda}^*} \mathcal{S}_{\beta,2^j}(f)(x)^2 w(x) dx \\
& \leq C \cdot 2^{jnq_2} \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_{\lambda}} \mathcal{S}_{\beta}(f)(x)^2 w(x) dx.
\end{aligned} \tag{84}$$

Hence

$$\begin{aligned}
\text{II} &\leq C \cdot 2^{jnq_2} \int_0^{\infty} q\lambda^{q-1} \\
&\quad \times \left(\lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_{\lambda}} \mathcal{S}_{\beta}(f)(x)^2 w(x) dx \right) d\lambda.
\end{aligned} \tag{85}$$

Changing the order of integration yields

$$\begin{aligned}
\text{II} &\leq C \cdot 2^{jnq_2} \int_{\mathbb{R}^n} \mathcal{S}_{\beta}(f)(x)^2 \\
&\quad \times \left(\int_{|\mathcal{S}_{\beta}(f)(x)|}^{\infty} q\lambda^{q-3} d\lambda \right) w(x) dx \\
&\leq C \cdot 2^{jnq_2} \frac{q}{2-q} \cdot \|\mathcal{S}_{\beta}(f)\|_{L_w^q}^q.
\end{aligned} \tag{86}$$

Combining the above estimate (86) with (82) and taking q th root on both sides, we are done. So it remains to prove inequality (83). Set $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_{\lambda}^*) = \cup_{x \in \mathbb{R}^n \setminus \Omega_{\lambda}^*} \Gamma_{2^j}(x)$ and $\Gamma(\mathbb{R}^n \setminus \Omega_{\lambda}) = \cup_{x \in \mathbb{R}^n \setminus \Omega_{\lambda}} \Gamma(x)$. For each given $(y, t) \in \Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_{\lambda}^*)$, by Lemma 9, we thus have

$$w(B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*)) \leq C \cdot 2^{jnq_2} w(B(y, t)). \tag{87}$$

It is not difficult to check that $w(B(y, t) \cap \Omega_{\lambda}) \leq w(B(y, t))/2$ and $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_{\lambda}^*) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_{\lambda})$. In fact, for any $(y, t) \in \Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_{\lambda}^*)$, there exists a point $x \in \mathbb{R}^n \setminus \Omega_{\lambda}^*$ such that $(y, t) \in \Gamma_{2^j}(x)$. Then we can deduce that

$$\begin{aligned}
w(B(y, t) \cap \Omega_{\lambda}) &\leq w(B(y, 2^j t) \cap \Omega_{\lambda}) \\
&= \int_{B(y, 2^j t)} \chi_{\Omega_{\lambda}}(z) w(z) dz \\
&\leq [w]_{A_{q_2}} \cdot 2^{jnq_2} w(B(y, t)) \\
&\quad \cdot \frac{1}{w(B(y, 2^j t))} \int_{B(y, 2^j t)} \chi_{\Omega_{\lambda}}(z) w(z) dz.
\end{aligned} \tag{88}$$

Note that $x \in B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*)$. So we have

$$\begin{aligned}
w(B(y, t) \cap \Omega_{\lambda}) &\leq [w]_{A_{q_2}} \cdot 2^{jnq_2} w(B(y, t)) M_w(\chi_{\Omega_{\lambda}^*})(x) \\
&\leq \frac{w(B(y, t))}{2}.
\end{aligned} \tag{89}$$

Hence

$$\begin{aligned} w(B(y, t)) &= w(B(y, t) \cap \Omega_\lambda) + w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)) \\ &\leq \frac{w(B(y, t))}{2} + w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)), \end{aligned} \quad (90)$$

which is equivalent to

$$w(B(y, t)) \leq 2 \cdot w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)). \quad (91)$$

The above inequality implies in particular that there is a point $z \in B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda) \neq \emptyset$. In this case, we have $(y, t) \in \Gamma(z)$ with $z \in \mathbb{R}^n \setminus \Omega_\lambda$, which implies that $\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_\lambda)$. Thus we obtain

$$\begin{aligned} w(B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) \\ \leq C \cdot 2^{jnq_2} w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)). \end{aligned} \quad (92)$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \Omega_\lambda^*} \mathcal{S}_{\beta, 2^j}(f)(x)^2 w(x) dx \\ &= \int_{\mathbb{R}^n \setminus \Omega_\lambda^*} \left(\iint_{\Gamma_{2^j}(x)} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right) w(x) dx \\ &\leq \iint_{\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*)} \left(\int_{B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)} w(x) dx \right) \\ &\quad \times (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{jnq_2} \iint_{\Gamma(\mathbb{R}^n \setminus \Omega_\lambda)} \left(\int_{B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)} w(x) dx \right) \\ &\quad \times (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \cdot 2^{jnq_2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} \mathcal{S}_\beta(f)(x)^2 w(x) dx, \end{aligned} \quad (93)$$

which is exactly what we want. This completes the proof of Proposition 13. $\square$

We are now in a position to give the proofs of the main theorems.

Proof of Theorem 5. From the definition of $\mathcal{G}_{\lambda, \beta}^*$, we readily see that

$$\begin{aligned} &|\mathcal{G}_{\lambda, \beta}^*(f)(x)|^2 \\ &= \iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\quad + \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t \leq |x-y| < 2^j t} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \\ &\quad \times (A_\beta(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \left[\mathcal{S}_\beta(f)(x)^2 + \sum_{j=1}^\infty 2^{-j\lambda n} \mathcal{S}_{\beta, 2^j}(f)(x)^2 \right]. \end{aligned} \quad (94)$$

Let $f \in \dot{K}_q^{\alpha, p}(w_1, w_2)$. We decompose $f(x) = f_1(x) + f_2(x) + f_3(x)$ as in the proof of Theorem 3; then we have

$$\begin{aligned} &\|\mathcal{G}_{\lambda, \beta}^*(f)\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)}^p \\ &\leq C \sum_{i=1}^3 \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|\mathcal{G}_{\lambda, \beta}^*(f_i) \chi_k\|_{L_{w_2}^q}^p \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (95)$$

Note that $\lambda > \max\{q_2, 3\} \geq \max\{q_2, 2q_2/q\}$ when $q_2 \leq q$. Since $w_2 \in A_{q_2}$ and $1 \leq q_2 \leq q$, then $w_2 \in A_q$. Applying Propositions 11–13, Theorem A, and inequality (94), we obtain

$$\begin{aligned} &\|\mathcal{G}_{\lambda, \beta}^*(f_1)\|_{L_{w_2}^q} \\ &\leq C \left(\|\mathcal{S}_\beta(f_1)\|_{L_{w_2}^q} + \sum_{j=1}^\infty 2^{-j\lambda n/2} \|\mathcal{S}_{\beta, 2^j}(f_1)\|_{L_{w_2}^q} \right) \\ &\leq C \|f_1\|_{L_{w_2}^q} \left(1 + \sum_{j=1}^\infty 2^{-j\lambda n/2} [2^{jnq_2/2} + 2^{jnq_2/q}] \right) \\ &\leq C \|f_1\|_{L_{w_2}^q}. \end{aligned} \quad (96)$$

From the above estimate (96) and Lemma 9, it follows that

$$\begin{aligned} I_1 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|\mathcal{G}_{\lambda, \beta}^*(f_1)\|_{L_{w_2}^q}^p \\ &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f_1\|_{L_{w_2}^q}^p \\ &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f \chi_k\|_{L_{w_2}^q}^p \\ &\leq C \|f\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)}^p. \end{aligned} \quad (97)$$

For any $j \in \mathbb{Z}_+$, $x \in C_k$, $(y, t) \in \Gamma_{2^j}(x)$, and $z \in \{2^{\ell-1} < |z| \leq 2^\ell\} \cap B(y, t)$ with $\ell \leq k-2$, then, by a simple calculation, we can easily deduce that

$$t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |x| - |z| \geq \frac{|x|}{2}. \quad (98)$$

Thus, by inequality (31) and Minkowski's inequality, we get

$$\begin{aligned}
& |\mathcal{S}_{\beta, 2^j}(f\chi_\ell)(x)| \\
&= \left(\iint_{\Gamma_{2^j}(x)} \left(\sup_{\varphi \in \mathcal{C}_\beta} |(f\chi_\ell) * \varphi_t(y)| \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
&\leq C \left(\int_{|x|/2^{j+2}}^\infty \int_{|x-y| < 2^j t} \left| t^{-n} \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
&\leq C \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \left(\int_{|x|/2^{j+2}}^\infty 2^{jn} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
&\leq C \cdot 2^{3jn/2} \frac{1}{|x|^n} \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right). \tag{99}
\end{aligned}$$

Moreover, by using Minkowski's inequality, (34), and (99), we obtain

$$\begin{aligned}
& \|\mathcal{S}_{\beta, 2^j}(f_2)\chi_k\|_{L_{w_2}^q} \\
&\leq \sum_{\ell=-\infty}^{k-2} \|\mathcal{S}_{\beta, 2^j}(f\chi_\ell)\chi_k\|_{L_{w_2}^q} \\
&\leq C \cdot 2^{3jn/2} \sum_{\ell=-\infty}^{k-2} \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \\
&\quad \times \left(\int_{2^{k-1} < |x| \leq 2^k} \frac{w_2(x)}{|x|^{nq}} dx \right)^{1/q} \\
&\leq C \cdot 2^{3jn/2} \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{100}
\end{aligned}$$

Consequently

$$\begin{aligned}
J_2 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\
&\quad \times \left(\|\mathcal{S}_\beta(f_2)\chi_k\|_{L_{w_2}^q} \right. \\
&\quad \left. + \sum_{j=1}^\infty 2^{-j\lambda n/2} \|\mathcal{S}_{\beta, 2^j}(f_2)\chi_k\|_{L_{w_2}^q} \right)^p
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\
&\quad \times \left(\sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p \\
&\quad \times \left(1 + \sum_{j=1}^\infty 2^{-j\lambda n/2} \cdot 2^{3jn/2} \right)^p \\
&\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\
&\quad \times \left(\sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p, \tag{101}
\end{aligned}$$

where the last inequality holds under our assumption $\lambda > 3$. On the other hand, for any $j \in \mathbb{Z}_+$, $x \in C_k$, $(y, t) \in \Gamma_{2^j}(x)$, and $z \in \{2^{\ell-1} < |z| \leq 2^\ell\} \cap B(y, t)$ with $\ell \geq k+2$, it is easy to verify that

$$t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |z| - |x| \geq \frac{|z|}{2}. \tag{102}$$

Then it follows from inequality (31) and Minkowski's inequality that

$$\begin{aligned}
& |\mathcal{S}_{\beta, 2^j}(f\chi_\ell)(x)| \\
&\leq C \left(\int_{|x|/2^{j+2}}^\infty \int_{|x-y| < 2^j t} \left| t^{-n} \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
&\leq C \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \left(\int_{|x|/2^{j+2}}^\infty 2^{jn} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
&\leq C \cdot 2^{3jn/2} \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} \frac{|f(z)|}{|z|^n} dz \right). \tag{103}
\end{aligned}$$

Furthermore, by Minkowski's inequality, (34), and (103), we have

$$\begin{aligned}
\|\mathcal{S}_{\beta, 2^j}(f_3)\chi_k\|_{L_{w_2}^q} &\leq \sum_{\ell=k+2}^\infty \|\mathcal{S}_{\beta, 2^j}(f\chi_\ell)\chi_k\|_{L_{w_2}^q} \\
&\leq C \cdot 2^{3jn/2} \sum_{\ell=k+2}^\infty \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{104}
\end{aligned}$$

Therefore

$$\begin{aligned}
 J_3 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\
 &\quad \times \left(\left\| \mathcal{S}_\beta(f_3) \chi_k \right\|_{L_{w_2}^q} \right. \\
 &\quad \left. + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \left\| \mathcal{S}_{\beta,2^j}(f_3) \chi_k \right\|_{L_{w_2}^q} \right)^p \\
 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\sum_{\ell=k+2}^{\infty} \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p \\
 &\quad \times \left(1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3jn/2} \right)^p \\
 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\sum_{\ell=k+2}^{\infty} \frac{[w_2(B_k)]^{1/q}}{[w_2(B_\ell)]^{1/q}} \|f\chi_\ell\|_{L_{w_2}^q} \right)^p, \tag{105}
 \end{aligned}$$

where the last inequality also holds since $\lambda > 3$. Following along the same lines as in Theorem 3, we can also show that

$$\begin{aligned}
 J_2 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p, \\
 J_3 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p. \tag{106}
 \end{aligned}$$

Summing up the above estimates for J_1 , J_2 , and J_3 , we complete the proof of Theorem 5. $\square$

Proof of Theorem 6. Let $f \in \dot{K}_q^{\alpha,p}(w_1,w_2)$. We set $f(x) = f_1(x) + f_2(x) + f_3(x)$ as in the proof of Theorem 4; then for any given $\sigma > 0$, we can write

$$\begin{aligned}
 &\sigma^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} w_2(\{x \in C_k : |\mathcal{G}_{\lambda,\beta}^*(f)(x)| > \sigma\})^{p/q} \\
 &\leq \sum_{i=1}^3 \sigma^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \\
 &\quad \times w_2\left(\left\{x \in C_k : |\mathcal{G}_{\lambda,\beta}^*(f_i)(x)| > \frac{\sigma}{3}\right\}\right)^{p/q} \\
 &= J'_1 + J'_2 + J'_3. \tag{107}
 \end{aligned}$$

Since $\lambda > \max\{q_2, 3\} \geq \max\{q_2, 2q_2/q\}$ when $q_2 \leq q$, applying Chebyshev's inequality, Lemma 9, and (96), we obtain

$$\begin{aligned}
 J'_1 &\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left(\frac{3^q}{\sigma^q} \left\| \mathcal{G}_{\lambda,\beta}^*(f_1) \right\|_{L_{w_2}^q}^q \right)^{p/q} \\
 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f_1\|_{L_{w_2}^q}^p \\
 &\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f\chi_k\|_{L_{w_2}^q}^p \leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p. \tag{108}
 \end{aligned}$$

For the term J'_2 , when $x \in C_k$, then it follows from (94), (99), (34), and the fact $\lambda > 3$ that

$$\begin{aligned}
 &|\mathcal{G}_{\lambda,\beta}^*(f_2)(x)| \\
 &\leq \sum_{\ell=-\infty}^{k-2} |\mathcal{G}_{\lambda,\beta}^*(f\chi_\ell)(x)| \\
 &\leq C \sum_{\ell=-\infty}^{k-2} \left(|\mathcal{S}_\beta(f\chi_\ell)(x)| + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} |\mathcal{S}_{\beta,2^j}(f\chi_\ell)(x)| \right) \\
 &\leq C \left(\sum_{\ell=-\infty}^{k-2} \frac{1}{|x|^n} \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \\
 &\quad \times \left(1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3jn/2} \right) \\
 &\leq C \sum_{\ell=-\infty}^{k-2} \frac{1}{|x|^n} \left(\int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| dz \right) \\
 &\leq C \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{109}
 \end{aligned}$$

For the last term J'_3 , when $x \in C_k$, by using (94), (103), (34), and the fact that $\lambda > 3$, we get

$$\begin{aligned}
 &|\mathcal{G}_{\lambda,\beta}^*(f_3)(x)| \\
 &\leq \sum_{\ell=k+2}^{\infty} |\mathcal{G}_{\lambda,\beta}^*(f\chi_\ell)(x)| \\
 &\leq C \sum_{\ell=k+2}^{\infty} \left(|\mathcal{S}_\beta(f\chi_\ell)(x)| + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} |\mathcal{S}_{\beta,2^j}(f\chi_\ell)(x)| \right) \\
 &\leq C \left(\sum_{\ell=k+2}^{\infty} \int_{2^{\ell-1} < |z| \leq 2^\ell} \frac{|f(z)|}{|z|^n} dz \right) \left(1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3jn/2} \right) \\
 &\leq C \sum_{\ell=k+2}^{\infty} \int_{2^{\ell-1} < |z| \leq 2^\ell} \frac{|f(z)|}{|z|^n} dz \\
 &\leq C \sum_{\ell=k+2}^{\infty} [w_2(B_\ell)]^{-1/q} \|f\chi_\ell\|_{L_{w_2}^q}. \tag{110}
 \end{aligned}$$

The rest of the proof is exactly the same as that of Theorem 4, and we finally obtain

$$\begin{aligned}
 J'_2 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p, \\
 J'_3 &\leq C \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p. \tag{111}
 \end{aligned}$$

Combining the above estimates for J'_1 , J'_2 , and J'_3 and then taking the supremum over all $\sigma > 0$, we conclude the proof of Theorem 6. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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