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Research Article

General Decay and Blow-Up of Solutions for a System of Viscoelastic Equations of Kirchhoff Type with Strong Damping

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The general decay and blow-up of solutions for a system of viscoelastic equations of Kirchhoff type with strong damping is considered. We first establish two blow-up results: one is for certain solutions with nonpositive initial energy as well as positive initial energy by exploiting the convexity technique, the other is for certain solutions with arbitrarily positive initial energy based on the method of Li and Tsai. Then, we give a decay result of global solutions by the perturbed energy method under a weaker assumption on the relaxation functions.

1. Introduction

In this work, we investigate the following system of viscoelastic equations of Kirchhoff type:

$$\begin{gathered}
u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g_1(t-s)\,\Delta u(s)\,\mathrm{d}s - \Delta u_t \\
= f_1(u,v), \quad (x,t) \in \Omega \times (0,\infty), \\
v_{tt} - M\left(\|\nabla v\|_2^2\right)\Delta v + \int_0^t g_2(t-s)\,\Delta v(s)\,\mathrm{d}s - \Delta v_t \\
= f_2(u,v), \quad (x,t) \in \Omega \times (0,\infty), \\
u(x,t) = 0, \quad v(x,t) = 0, \\
(x,t) \in \partial\Omega \times [0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega,
\end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$, is a bounded domain with smooth boundary $\partial\Omega, M$ is a positive locally Lipschitz function, and $g_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1,2), (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ are given functions to be specified later.

To motivate our work, let us recall some previous results regarding viscoelastic equations of Kirchhoff type. The following problem:

$$\begin{gathered}
u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s + h(u_t) = f(u), \\
(x,t) \in \Omega \times (0,\infty), \\
u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega
\end{gathered} \tag{2}$$

is a model to describe the motion of deformable solids as hereditary effect is incorporated. It was first studied by Torrejón and Yong [1] who proved the existence of a weakly asymptotic stable solution for large analytical datum. Later, Muñoz Rivera [2] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Then, Wu and Tsai [3] treated problem (2) for $h(u_t) = -\Delta u_t$ and proved the global existence, decay, and blow-up with suitable conditions on initial data. They obtained the blow-up properties of local solution with small positive initial energy by the direct method of [4]. To obtain the decay result, they assumed that the nonnegative kernel $g'(t) \leq -rg(t)$, $\forall t \geq 0$ for some $r > 0$. This energy decay result was recently improved by Wu

in [5] under a weaker condition on g (i.e., $g'(t) \leq 0$ for $t \geq 0$). For a single wave equation of Kirchhoff type without the viscoelastic term, we refer the reader to Matsuyama and Ikehata [6] and Ono [7–10].

Many results concerning local existence, global existence, decay, and blow-up of solutions for a system of wave equations of Kirchhoff type without viscoelastic terms (i.e., $g_i = 0$, $i = 1, 2$) have also been extensively studied. For example, Park and Bae [11] considered the system of wave equations with nonlinear dampings for $f_1(u, v) = \mu|u|^{q-1}u$ and $f_2(u, v) = \mu|v|^{q-1}v$, $q \geq 1$, and showed the global existence and asymptotic behavior of solutions under some restrictions on the initial energy. Later, Benaissa and Messaoudi [12] discussed blow-up properties for negative initial energy. Recently, Wu and Tsai [13] studied the system (1) for $g_i = 0$ $(i = 1, 2)$. Under some suitable assumptions on f_i $(i = 1, 2)$, they proved local existence of solutions by applying the Banach fixed point theorem and the blow-up of solutions by using the method of Li and Tsai in [4], where three different cases on the sign of the initial energy $E(0)$ are considered.

In the case of $M \equiv 1$ and in the presence of viscoelastic term (i.e., $g \neq 0$), Cavalcanti et al. [14] studied the equation that was subject to a locally distributed dissipation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s + a(x)\,u_t + |u|^{\gamma}u = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{3}$$

with the same initial and boundary conditions as that of (2), and proved an exponential decay rate. This work extended the result of Zuazua [15], in which he considered (3) with $g = 0$ and the localized linear damping. By using the piecewise multipliers method, Cavalcanti and Oquendo [16] investigated the equation

$$\begin{aligned} u_{tt} - k_0\Delta u + \int_0^t \operatorname{div}\left[a(x)\,g(t-s)\,\nabla u(s)\right]\mathrm{d}s \\ + b(x)\,h(u_t) + f(u) = 0, \quad (x,t) \in \Omega \times (0,\infty), \end{aligned} \tag{4}$$

with the same initial and boundary conditions as that of (2). Under the similar conditions on the relaxation function g as above, and $a(x) + b(x) \geq \delta > 0$ for all $x \in \Omega$, they improved the results of [14] by establishing exponential stability for exponential decay function g and linear function h, and polynomial stability for polynomial decay function g and nonlinear function h, respectively.

Concerning blow-up results, Messaoudi [17] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s + au_t|u_t|^{m-2} = b|u|^{r-2}u, \quad (x,t) \in \Omega \times (0,\infty). \tag{5}$$

He proved that any weak solution with negative initial energy blows up in finite time if $r > m$ and

$$\int_0^\infty g(s)\,\mathrm{d}s \leq \frac{r-2}{r-2+1/r}, \tag{6}$$

while exists globally for any initial data in the appropriate space if $m \geq r$. This result was improved by the same author in [18] for positive initial energy under suitable conditions on g, m, and r. Recently, Liu [19] studied the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s - \omega\Delta u_t + \mu u_t = |u|^{r-2}u, \quad (x,t) \in \Omega \times (0,\infty), \tag{7}$$

with the same initial and boundary condition as that of (2). By virtue of convexity technique and supposing that

$$\int_0^\infty g(s)\,\mathrm{d}s \leq \frac{r-2}{r-2+1/\left[\left(1-\hat{\delta}\right)^2 r + 2\delta\left(1-\hat{\delta}\right)\right]}, \tag{8}$$

where $\hat{\delta} = \max\{0, \delta\}$, he proved that the solution with nonpositive initial energy as well as positive initial energy blows up in finite time.

We should mention that the following system:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\,\Delta u(\tau)\,\mathrm{d}\tau + |u_t|^{m-1}u_t = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\ &v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\,\Delta v(\tau)\,\mathrm{d}\tau + |v_t|^{\gamma-1}v_t = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\ &u(x,t) = 0, \quad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T), \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ &v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega \end{aligned} \tag{9}$$

was considered by Han and Wang in [20], where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n, n = 1, 2, 3$. Under suitable assumptions on the functions g_i, f_i $(i = 1, 2)$, the initial data and the parameters in the above problem established local existence, global existence, and blow-up property (the initial energy $E(0) < 0$). This latter blow-up result has been improved by Messaoudi and Said-Houari [21] into certain solutions with positive initial energy. Recently, Liang and Gao in [22] investigated the following problem:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau - \Delta u_t = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\ &v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,\mathrm{d}\tau - \Delta v_t = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \end{aligned} \tag{10}$$

with the same initial and boundary conditions as that of (9). Under suitable assumptions on the functions g_i, f_i $(i = 1, 2)$

and certain initial data in the stable set, they proved that the decay rate of the solution energy is exponential. Conversely, for certain initial data in the unstable set, they proved that there are solutions with positive initial energy that blow up in finite time. It is also worth mentioning the work [23] in which we studied system (1). Under suitable assumptions on the functions g_i, f_i $(i = 1,2)$ and certain initial conditions, we showed that the solutions are global in time and the energy decays exponentially. For other papers related to existence, uniform decay, and blow-up of solutions of nonlinear wave equations, we refer the reader to [14, 24–29] for existence and uniform decay, to [17, 30–34] for blow-up, and to [35–40] for the coupled system. To the best of our knowledge, the general decay and blow-up of solutions for systems of viscoelastic equations of Kirchhoff type with strong damping have not been well studied.

Motivated by the above mentioned research, we consider in the present work the coupled system (1) with nonzero g_i $(i = 1,2)$ and nonconstant $M(s)$. We note that in such a coupled system case we should overcome the additional difficulties brought by the treatment of the nonlinear coupled terms. We first establish two blow-up results: one is for certain solutions with nonpositive initial energy as well as positive initial energy, the other is for certain solutions with arbitrarily positive initial energy. Then, we give a decay result of global solutions under a weaker assumption on the relaxation functions $g_i(t)$ $(i = 1,2)$.

This paper is organized as follows. In the next section we present some assumptions, notations and known results and state the main results: Theorems 4, 5, 6, and 7. The two blow-up results, Theorems 5 and 6, are proved in Sections 3 and 4, respectively. Section 5 is devoted to the proof of the decay result—Theorem 7.

2. Preliminaries and Main Result

In this section we present some assumptions, notations, and known results and state the main results. First, we make the following assumptions.

(A1) $M(s)$ is a positive locally Lipschitz function for $s \geq 0$ with the Lipschitz constant L satisfying

$$M(s) \geq m_0 > 0. \tag{11}$$

(A2) $g_i(t) : [0,\infty) \to (0,\infty)$, $(i = 1,2)$ are strictly decreasing C^1 functions such that

$$m_0 - \int_0^\infty g_i(s)\,\mathrm{d}s = l_i > 0. \tag{12}$$

(A3) There exist two positive differentiable functions $\xi_1(t)$ and $\xi_2(t)$ such that

$$g_i'(t) \leq -\xi_i(t)\, g_i(t), \quad (i = 1,2), \text{ for } t \geq 0, \tag{13}$$

and $\xi_i(t)$ satisfies

$$\xi_i'(t) \leq 0, \quad \forall t > 0, \qquad \int_0^{+\infty} \xi_i(t)\,\mathrm{d}t = \infty, \quad (i = 1,2). \tag{14}$$

(A4) We make the following extra assumption on M:

$$2(p+2)\overline{M}(s) - 2M(s)s \geq 2(p+1)m_0 s, \quad \forall s \geq 0, \tag{15}$$

where $\overline{M}(s) = \int_0^s M(\tau)\mathrm{d}\tau$.

Remark 1. It is clear that $M(s) = m_0 + m_1 s^\gamma$ (which occurs physically in the study of vibrations of damped flexible space structures in a bounded domain in $\mathbb{R}^n$) satisfies (A4) for $s \geq 0$, $m_0 > 0$, $m_1 \geq 0$, $\gamma \geq 0$ as long as $p + 1 > \gamma$. Indeed, by straightforward calculations, we obtain

$$\begin{aligned} &2(p+2)\overline{M}(s) - 2M(s)s \\ &\quad = 2(p+2)\left(m_0 s + \frac{m_1}{\gamma+1}s^{\gamma+1}\right) - 2m_0 s - 2m_1 s^{\gamma+1} \\ &\quad = 2(p+1)m_0 s + \frac{2(p+1-\gamma)}{\gamma+1}m_1 s^{\gamma+1} \geq 2(p+1)m_0 s. \end{aligned} \tag{16}$$

Next, we introduce some notations. Consider

$$\begin{aligned} (g_i \diamond \nabla w)(t) &= \int_0^t g_i(t-s)\,\|\nabla w(t) - \nabla w(s)\|_2^2 \mathrm{d}s, \\ l &= \min\{l_1, l_2\}, \\ g_0(t) &= \max\{g_1(t), g_2(t)\}, \quad \forall t \geq 0, \\ \|\cdot\|_q &= \|\cdot\|_{L^q(\Omega)}, \quad 1 \leq q \leq \infty, \end{aligned} \tag{17}$$

the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u,v) = \int_\Omega u(x)\,v(x)\,\mathrm{d}x, \tag{18}$$

and the functions $f_1(u,v)$ and $f_2(u,v)$ (see also [21]):

$$\begin{aligned} f_1(u,v) &= \left[a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2}\right], \\ f_2(u,v) &= \left[a|u+v|^{2(p+1)}(u+v) + b|u|^{p+2}|v|^p v\right], \end{aligned} \tag{19}$$

where $a, b > 0$ are constants and p satisfies

$$\begin{aligned} &-1 < p, \quad \text{if } n = 1,2, \\ &-1 < p \leq \frac{(3-n)}{(n-2)}, \quad \text{if } n \geq 3. \end{aligned} \tag{20}$$

One can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(p+2)F(u,v), \quad \forall (u,v) \in \mathbb{R}^2, \tag{21}$$

where

$$F(u,v) = \frac{1}{2(p+2)}\left[a|u+v|^{2(p+2)} + 2b|uv|^{p+2}\right]. \tag{22}$$

Then, we give two lemmas which will be used throughout this work.

Lemma 2 (Sobolev-Poincaré inequality [41]). *If* $2 \le \rho \le 2n/(n-2)$, *then*

$$\|u\|_\rho \le C\|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega) \tag{23}$$

holds with some constant C.

Lemma 3 ([21, Lemma 3.2]). *Assume that* (20) *holds. Then there exists* $\eta_1 > 0$ *such that for any* $(u,v) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega))$, *one has*

$$\begin{aligned} &a\|u+v\|_{2(p+2)}^{2(p+2)} + 2b\|uv\|_{p+2}^{p+2} \\ &\quad \le \eta_1\left(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2\right)^{p+2}. \end{aligned} \tag{24}$$

We now state a local existence theorem for system (1), whose proof follows the arguments in [3, 13].

Theorem 4. *Suppose that* (20), $(A1)$, *and* $(A2)$ *hold, and that* $u_0, v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$. *Then problem* (1) *has a unique local solution*

$$\begin{aligned} &u, v \in C\left([0,T); H_0^1(\Omega) \cap H^2(\Omega)\right), \\ &u_t, v_t \in C\left([0,T); L^2(\Omega)\right) \cap L^2\left([0,T); H_0^1(\Omega)\right), \end{aligned} \tag{25}$$

for some $T > 0$. *Moreover, at least one of the following statements is valid:*

$$\begin{aligned} &(1)\ T = \infty, \\ &(2)\ \lim_{t \to T^-}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) = \infty. \end{aligned} \tag{26}$$

The energy associated with system (1) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{1}{2}\left(\int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\ & - \frac{1}{2}\left(\int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2 - \int_\Omega F(u,v)\,\mathrm{d}x \\ & + \frac{1}{2}\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\ & + \frac{1}{2}\overline{M}\left(\|\nabla u\|_2^2\right) + \frac{1}{2}\overline{M}\left(\|\nabla v\|_2^2\right), \quad \text{for } t \ge 0. \end{aligned} \tag{27}$$

As in [5], we can get

$$\begin{aligned} &E'(t) \\ &\quad = -\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) - \frac{1}{2}\left[g_1(t)\|\nabla u\|_2^2 + g_2(t)\|\nabla v\|_2^2\right] \\ &\qquad + \frac{1}{2}\left[\left(g_1' \diamond \nabla u\right)(t) + \left(g_2' \diamond \nabla v\right)(t)\right] \le 0, \quad \forall t \ge 0. \end{aligned} \tag{28}$$

Then we have

$$\begin{aligned} E(t) = {} & E(0) - \int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right)\mathrm{d}\tau \\ & - \frac{1}{2}\int_0^t \left[g_1(\tau)\|\nabla u(\tau)\|_2^2 + g_2(\tau)\|\nabla v(\tau)\|_2^2\right]\mathrm{d}\tau \\ & + \frac{1}{2}\int_0^t \left[\left(g_1' \diamond \nabla u\right)(\tau) + \left(g_2' \diamond \nabla v\right)(\tau)\right]\mathrm{d}\tau. \end{aligned} \tag{29}$$

We introduce

$$\begin{aligned} &B_1 = \eta_1^{1/2(p+2)}, \qquad \alpha_* = B_1^{-(p+2)/(p+1)}, \\ &E_1 = \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\alpha_*^2, \end{aligned} \tag{30}$$

where η_1 is the optimal constant in (24).

Our first result is concerned with the blow-up for certain local solutions with nonpositive initial energy as well as positive initial energy.

Theorem 5. *Assume that* $(A1)$-$(A2)$, $(A4)$, *and* (20) *hold. Let* (u,v) *be the unique local solution to system* (1). *For any fixed* $\delta < 1$, *assuming that we can choose* $(u_0, v_0), (u_1, v_1)$ *satisfy*

$$E(0) < \delta E_1, \qquad \left(l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2\right)^{1/2} > \alpha_*. \tag{31}$$

Suppose further that

$$\begin{aligned} &\int_0^\infty g_i(s)\,\mathrm{d}s \\ &\quad \le \frac{2(p+1)m_0}{2(p+1) + 1/\left\{2\left[\left(1-\widehat{\delta}\right)^2(p+2) + \widehat{\delta}\left(1-\widehat{\delta}\right)\right]\right\}}, \\ &\qquad (i = 1,2), \end{aligned} \tag{32}$$

where $\widehat{\delta} = \max\{0, \delta\}$, *then* $T < \infty$.

Our second result shows that certain local solutions with arbitrarily positive initial energy can also blow up.

Theorem 6. *Suppose that (A1)-(A2), (A4),* (20), *and*

$$\int_0^\infty g_i(s)\,\mathrm{d}s \le \frac{2(p+1)m_0}{2(p+1) + 1/2(p+2)}, \quad (i = 1,2), \tag{33}$$

hold. Assume further that $u_0, v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$ *and satisfy*

$$0 < E(0) < \min\left\{ \frac{\left(\int_\Omega u_0 u_1 \mathrm{d}x + \int_\Omega v_0 v_1 \mathrm{d}x\right)^2}{2\left[\|u_0\|_2^2 + \|v_0\|_2^2 + T_1\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]}, \frac{\sqrt{p+3}}{(p+2)\left(\sqrt{p+3}-\sqrt{p+1}\right)} \int_\Omega (u_0 u_1 + v_0 v_1)\,\mathrm{d}x - \frac{p+3}{2(p+2)}\left(\|u_0\|_2^2 + \|v_0\|_2^2 + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) \right\}, \tag{34}$$

then the solution of problem (1) *blows up at a finite time* T^* *in the sense of* (119) *below. Moreover, the upper bounds for* T^* *can be estimated by*

$$T^* \le 2^{(3p+5)/2(p+1)} \frac{(p+1)c}{2\sqrt{a_1}} \left\{1 - [1 + ch(0)]^{-1/(p+1)}\right\}, \tag{35}$$

where $h(0) = [\|u_0\|_2^2 + \|v_0\|_2^2 + T_1(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)]^{-(p+1)/2}$ *and* a_1 *is given in* (116) *below.*

Finally, we state the general decay result. For convenience, we choose especially $M(s) = m_0 + m_1 s^\gamma$, $m_0 > 0$, $m_1 \ge 0$, and $\gamma \ge 0$. Then the energy functional $E(t)$, defined by (27), becomes

$$\begin{aligned} E(t) &:= E(u(t), v(t)) \\ &= \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad + \frac{1}{2}\left[\left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2\right] \\ &\quad + \frac{1}{2}\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) + \frac{m_1}{\gamma+1}\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right)\right] \\ &\quad - \int_\Omega F(u, v)\,\mathrm{d}x. \end{aligned} \tag{36}$$

Theorem 7. *Suppose that* (20), (*A*1)-(*A*2), *and* (*A*3) *hold, and that* $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$ *and* $(v_0, v_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$ *and satisfy* $E(0) < E_1$ *and*

$$\left(l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2\right)^{1/2} < \alpha_*. \tag{37}$$

Then for each $t_0 > 0$, *there exist two positive constants* K *and* k *such that the energy of* (1) *satisfies*

$$E(t) \le K e^{-k\int_{t_0}^t \xi(s)\mathrm{d}s}, \quad t \ge t_0, \tag{38}$$

where $\xi(t) := \min\{\xi_1(t), \xi_2(t)\}$.

To achieve general decay result we will use a Lyapunov type technique for some perturbation energy following the method introduced in [42]. This result improves the one in Li et al. [23] in which only the exponential decay rates are considered.

3. Blow-Up of Solutions with Initial Data in the Unstable Set

In this section, we prove a finite time blow-up result for initial data in the unstable set. We need the following lemmas.

Lemma 8. *Suppose that* (20), (*A*1), (*A*2), *and* (*A*4) *hold. Let* (u, v) *be the solution of system* (1). *Assume further that* $E(0) < E_1$ *and*

$$\left(l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2\right)^{1/2} > \alpha_*. \tag{39}$$

Then there exists a constant $\alpha_3 > \alpha_*$ *such that*

$$\left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right)^{1/2} \ge \alpha_3, \quad \text{for } t \in [0, T). \tag{40}$$

Proof. We first note that, by (27), (24) and the definition of B_1, we have

$$\begin{aligned} E(t) &\ge \frac{1}{2}\left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2 \\ &\quad - \frac{1}{2(p+2)}\left(a\|u+v\|_{2(p+2)}^{2(p+2)} + 2b\|uv\|_{p+2}^{p+2}\right) \\ &\ge \frac{1}{2}\left[l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right] - \frac{1}{2(p+2)}\left(a\|u+v\|_{2(p+2)}^{2(p+2)} + 2b\|uv\|_{p+2}^{p+2}\right) \\ &\ge \frac{1}{2}\left[l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right] - \frac{B_1^{2(p+2)}}{2(p+2)}\left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right)^{p+2} \\ &= \frac{1}{2}\alpha^2 - \frac{B_1^{2(p+2)}}{2(p+2)}\alpha^{2(p+2)} := G(\alpha), \end{aligned} \tag{41}$$

where we have used $\overline{M}(s) = \int_0^s M(\tau)\mathrm{d}\tau \ge m_0 s \ge 0$ and

$$\alpha = \left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right)^{1/2}. \tag{42}$$

It is easy to verify that $G(\alpha)$ is increasing in $(0, \alpha_*)$, decreasing in (α_*, ∞), and that $G(\alpha) \to -\infty$, as $\alpha \to \infty$, and

$$G(\alpha)_{\max} = G(\alpha_*) = \frac{1}{2}\alpha_*^2 - \frac{B_1^{2(p+2)}}{2(p+2)}\alpha_*^{2(p+2)} = E_1, \tag{43}$$

where α_* is given in (30). Since $E(0) < E_1$, there exists $\alpha_3 > \alpha_*$ such that $G(\alpha_3) = E(0)$. Set $\alpha_0 = (l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2)^{1/2}$, then from (41) we can get $G(\alpha_0) \le E(0) = G(\alpha_3)$, which implies that $\alpha_0 \ge \alpha_3$.

Now we establish (40) by contradiction. First we assume that (40) is not true over $[0,T)$, then there exists $t_0 \in (0,T)$ such that

$$\left(l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla v(t_0)\|_2^2\right)^{1/2} < \alpha_3. \tag{44}$$

By the continuity of $l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2$ we can choose t_0 such that

$$\left(l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla v(t_0)\|_2^2\right)^{1/2} > \alpha_*. \tag{45}$$

Again, the use of (41) leads to

$$\begin{aligned} E(t_0) &\ge G\left[\left(l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla u(t_0)\|_2^2\right)^{1/2}\right] \\ &> G(\alpha_3) = E(0). \end{aligned} \tag{46}$$

This is impossible since $E(t) \le E(0)$ for all $t \in [0,T)$.

Hence (40) is established. □

Lemma 9 (see [43, 44]). *Let $L(t)$ be a positive, twice differentiable function, which satisfies, for $t > 0$, the inequality*

$$L(t)L''(t) - (1+\sigma)L'(t)^2 \ge 0 \tag{47}$$

for some $\sigma > 0$. If $L(0) > 0$ and $L'(0) > 0$, then there exists a time $T_* \le L(0)/[\sigma L'(0)]$ such that $\lim_{t\to T_*^-} L(t) = \infty$.

Proof of Theorem 5. Assume by contradiction that the solution (u,v) is global. Then we consider $L(t) : [0,T] \to \mathbb{R}^+$ defined by

$$\begin{aligned} L(t) &= \|u(t)\|_2^2 + \|v(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau + \int_0^t \|\nabla v(\tau)\|_2^2 d\tau \\ &\quad + (T-t)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) + \beta(t+T_0)^2, \end{aligned} \tag{48}$$

where T, β, and T_0 are positive constants to be chosen later. Then $L(t) > 0$ for all $t \in [0,T]$. Furthermore

$$\begin{aligned} L'(t) &= 2\int_\Omega u(t)u_t(t)\,dx + 2\int_\Omega v(t)v_t(t)\,dx \\ &\quad + \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 - \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) \\ &\quad + 2\beta(t+T_0) \\ &= 2\int_\Omega u(t)u_t(t)\,dx + 2\int_\Omega v(t)v_t(t)\,dx \\ &\quad + 2\int_0^t (\nabla u(\tau), \nabla u_t(\tau))\,d\tau \\ &\quad + 2\int_0^t (\nabla v(\tau), \nabla v_t(\tau))\,d\tau + 2\beta(t+T_0), \end{aligned} \tag{49}$$

and, consequently,

$$\begin{aligned} L''(t) &= 2\int_\Omega u(t)u_{tt}(t)\,dx + 2\int_\Omega v(t)v_{tt}(t)\,dx \\ &\quad + 2\|u_t(t)\|_2^2 + 2\|v_t(t)\|_2^2 + 2(\nabla u(t), \nabla u_t(t)) \\ &\quad + 2(\nabla v(t), \nabla v_t(t)) + 2\beta \end{aligned} \tag{50}$$

for almost every $t \in [0,T]$. Testing the first equation of system (1) with u and the second equation of system (1) with v, integrating the results over Ω, using integration by parts, and summing up, we have

$$\begin{aligned} &(u_{tt}(t), u(t)) + (v_{tt}(t), v(t)) + (\nabla u(t), \nabla u_t(t)) \\ &\quad + (\nabla v(t), \nabla v_t(t)) \\ &= -M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 - M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 \\ &\quad - \int_\Omega\int_0^t g_1(t-s)\Delta u(s)u(t)\,ds\,dx \\ &\quad - \int_\Omega\int_0^t g_2(t-s)\Delta v(s)v(t)\,ds\,dx \\ &\quad + 2(p+2)\int_\Omega F(u,v)\,dx, \end{aligned} \tag{51}$$

which implies

$$\begin{aligned} L''(t) &= 2\|u_t(t)\|_2^2 + 2\|v_t(t)\|_2^2 - 2M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 \\ &\quad - 2M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 + 2\beta \\ &\quad - 2\int_\Omega\int_0^t g_1(t-s)\Delta u(s)u(t)\,ds\,dx \\ &\quad - 2\int_\Omega\int_0^t g_2(t-s)\Delta v(s)v(t)\,ds\,dx \\ &\quad + 4(p+2)\int_\Omega F(u,v)\,dx. \end{aligned} \tag{52}$$

Therefore, we have

$$\begin{aligned} &L(t)L''(t) - \frac{p+3}{2}L'(t)^2 \\ &\quad = 2L(t)\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right. \\ &\qquad - M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 \\ &\qquad \left. -M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 + \beta\right) \end{aligned}$$

$$
\begin{aligned}
&-2L(t)\left(\int_\Omega\int_0^t g_1(t-s)\,\Delta u(s)\,u(t)\,\mathrm{d}s\,\mathrm{d}x\right.\\
&\qquad+\int_\Omega\int_0^t g_2(t-s)\,\Delta v(s)\,v(t)\,\mathrm{d}s\,\mathrm{d}x\\
&\qquad\left.-2(p+2)\int_\Omega F(u,v)\,\mathrm{d}x\right)-2(p+3)\\
&\times\left[\int_\Omega u(t)\,u_t(t)\,\mathrm{d}x+\int_\Omega v(t)\,v_t(t)\,\mathrm{d}x+\beta(t+T_0)\right.\\
&\qquad+\int_0^t(\nabla u(\tau),\nabla u_t(\tau))\,\mathrm{d}\tau\\
&\qquad\left.+\int_0^t(\nabla v(\tau),\nabla v_t(\tau))\,\mathrm{d}\tau\right]^2\\
&=2L(t)\left(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2-M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2\right.\\
&\qquad\left.-M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2+\beta\right)\\
&-2L(t)\left(\int_\Omega\int_0^t g_1(t-s)\,\Delta u(s)\,u(t)\,\mathrm{d}s\,\mathrm{d}x\right.\\
&\qquad+\int_\Omega\int_0^t g_2(t-s)\,\Delta v(s)\,v(t)\,\mathrm{d}s\,\mathrm{d}x\\
&\qquad\left.-2(p+2)\int_\Omega F(u,v)\,\mathrm{d}x\right)\\
&+2(p+3)\{H(t)\\
&\qquad-\left[L(t)-(T-t)\left(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2\right)\right]\\
&\qquad\times\Psi(t)\},
\end{aligned}
\tag{53}
$$

where $\Psi(t), H(t):[0,T]\to\mathbb{R}^+$ are the functions defined by

$$
\begin{aligned}
\Psi(t)&=\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\\
&\quad+\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau+\int_0^t\|\nabla v_t(\tau)\|_2^2\mathrm{d}\tau+\beta,\\
H(t)&=\left(\|u(t)\|_2^2+\|v(t)\|_2^2+\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau\right.\\
&\quad\left.+\int_0^t\|\nabla v(\tau)\|_2^2\mathrm{d}\tau+\beta(t+T_0)^2\right)\Psi(t)\\
&\quad-\left[\int_\Omega[u(t)\,u_t(t)+v(t)\,v_t(t)]\,\mathrm{d}x\right.\\
&\quad+\int_0^t[(\nabla u(\tau),\nabla u_t(\tau))+(\nabla v(\tau),\nabla v_t(\tau))]\,\mathrm{d}\tau\\
&\quad\left.+\beta(t+T_0)\right]^2.
\end{aligned}
\tag{54}
$$

Using the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\left(\int_\Omega u(t)\,u_t(t)\,\mathrm{d}x\right)^2\le\|u(t)\|_2^2\|u_t(t)\|_2^2,\\
&\text{(the similar inequality for } v(t) \text{ holds true)}\\
&\left(\int_0^t(\nabla u(\tau),\nabla u_t(\tau))\,\mathrm{d}\tau\right)^2\le\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau\\
&\qquad\text{(the similar inequality for } v(t)),\\
&\int_\Omega u(t)\,u_t(t)\,\mathrm{d}x\int_\Omega v(t)\,v_t(t)\,\mathrm{d}x\\
&\quad\le\|u(t)\|_2\|v_t(t)\|_2\|u_t(t)\|_2\|v(t)\|_2\\
&\quad\le\frac{1}{2}\|u(t)\|_2^2\|v_t(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2\|v(t)\|_2^2.
\end{aligned}
\tag{55}
$$

Similarly, we have

$$
\begin{aligned}
&\int_\Omega u(t)\,u_t(t)\,\mathrm{d}x\int_0^t(\nabla u(\tau),\nabla u_t(\tau))\,\mathrm{d}\tau\\
&\quad\le\frac{1}{2}\|u(t)\|_2^2\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau\\
&\qquad+\frac{1}{2}\|u_t(t)\|_2^2\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau,\\
&\text{(the similar inequality for } v(t) \text{ holds true)}\\
&\int_\Omega u(t)\,u_t(t)\,\mathrm{d}x\int_0^t(\nabla v(\tau),\nabla v_t(\tau))\,\mathrm{d}\tau\\
&\quad\le\frac{1}{2}\|u(t)\|_2^2\int_0^t\|\nabla v_t(\tau)\|_2^2\mathrm{d}\tau\\
&\qquad+\frac{1}{2}\|u_t(t)\|_2^2\int_0^t\|\nabla v(\tau)\|_2^2\mathrm{d}\tau,\\
&\int_\Omega v(t)\,v_t(t)\,\mathrm{d}x\int_0^t(\nabla u(\tau),\nabla u_t(\tau))\,\mathrm{d}\tau\\
&\quad\le\frac{1}{2}\|v(t)\|_2^2\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau\\
&\qquad+\frac{1}{2}\|v_t(t)\|_2^2\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau,\\
&\int_0^t(\nabla u(\tau),\nabla u_t(\tau))\,\mathrm{d}\tau\int_0^t(\nabla v(\tau),\nabla v_t(\tau))\,\mathrm{d}\tau\\
&\quad\le\frac{1}{2}\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau\int_0^t\|\nabla v_t(\tau)\|_2^2\mathrm{d}\tau\\
&\qquad+\frac{1}{2}\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau\int_0^t\|\nabla v(\tau)\|_2^2\mathrm{d}\tau.
\end{aligned}
\tag{56}
$$

By Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}
&\beta(t+T_0)\int_\Omega u(t)\,u_t(t)\,\mathrm{d}x \\
&\quad\le \beta(t+T_0)\,\|u(t)\|_2\|u_t(t)\|_2 \\
&\quad\le \frac{1}{2}\beta\|u(t)\|_2^2+\frac{1}{2}\beta(t+T_0)^2\|u_t(t)\|_2^2. \\
&\quad\text{(the similar inequality for } v(t) \text{ holds true)} \\
&\beta(t+T_0)\int_0^t(\nabla u(\tau),\nabla u_t(\tau))\,\mathrm{d}\tau \\
&\quad\le \beta(t+T_0)\int_0^t\|\nabla u(\tau)\|_2\|\nabla u_t(\tau)\|_2\mathrm{d}\tau \\
&\quad\le \frac{1}{2}\beta\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau+\frac{1}{2}\beta(t+T_0)^2\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau. \\
&\quad\text{(the similar inequality for } v(t) \text{ holds true)}.
\end{aligned} \tag{57}$$

The previous inequalities imply that $H(t)\ge 0$ for every $t\in[0,T]$. Using (53), we get

$$L(t)\,L''(t)-\frac{p+3}{2}L'(t)^2\ge L(t)\,\Phi(t) \tag{58}$$

for almost every $t\in[0,T]$, where $\Phi(t):[0,T]\to\mathbb{R}^+$ is the map defined by

$$\begin{aligned}
\Phi(t)=\;&-2(p+2)\left(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right) \\
&-2M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 \\
&-2M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 \\
&-2\left(\int_\Omega\int_0^t g_1(t-s)\,\Delta u(s)\,u(t)\,\mathrm{d}s\,\mathrm{d}x\right. \\
&\qquad\left.+\int_\Omega\int_0^t g_2(t-s)\,\Delta v(s)\,v(t)\,\mathrm{d}s\,\mathrm{d}x\right) \\
&-2(p+2)\,\beta \\
&-2(p+3)\left(\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau+\int_0^t\|\nabla v_t(\tau)\|_2^2\mathrm{d}\tau\right) \\
&+4(p+2)\int_\Omega F(u,v)\,\mathrm{d}x.
\end{aligned} \tag{59}$$

For the fourth term on the right hand side of (59), we have

$$\begin{aligned}
&-\int_\Omega\int_0^t g_1(t-s)\,\Delta u(s)\,u(t)\,\mathrm{d}s\,\mathrm{d}x \\
&\quad=\int_0^t g_1(t-s)\int_\Omega\nabla u(s)\cdot\nabla u(t)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad=\int_0^t g_1(t-s)\int_\Omega\nabla u(t)\cdot(\nabla u(s)-\nabla u(t))\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad+\left(\int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2.
\end{aligned} \tag{60}$$

Similarly,

$$\begin{aligned}
&-\int_\Omega\int_0^t g_2(t-s)\,\Delta v(s)\,v(t)\,\mathrm{d}s\,\mathrm{d}x \\
&\quad=\int_0^t g_2(t-s)\int_\Omega\nabla v(t)\cdot(\nabla v(s)-\nabla v(t))\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad+\left(\int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2.
\end{aligned} \tag{61}$$

Combining (59), (60) with (61), we get

$$\begin{aligned}
\Phi(t)=\;&-2(p+2)\left(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right) \\
&-2M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 \\
&-2M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 \\
&+2\int_0^t g_1(s)\,\mathrm{d}s\|\nabla u(t)\|_2^2+2\int_0^t g_2(s)\,\mathrm{d}s\|\nabla v(t)\|_2^2 \\
&+4(p+2)\int_\Omega F(u,v)\,\mathrm{d}x-2(p+2)\,\beta \\
&+2\int_0^t g_1(t-s)\int_\Omega\nabla u(t)\,(\nabla u(s)-\nabla u(t))\,\mathrm{d}x\,\mathrm{d}s \\
&-2(p+3)\int_0^t\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau \\
&+2\int_0^t g_2(t-s)\int_\Omega\nabla v(t)\,(\nabla v(s)-\nabla v(t))\,\mathrm{d}x\,\mathrm{d}s \\
&-2(p+3)\int_0^t\|\nabla v_t(\tau)\|_2^2\mathrm{d}\tau.
\end{aligned} \tag{62}$$

Since, we have

$$\begin{aligned}
&2\int_0^t g_i(t-s)\int_\Omega\nabla u(t)\cdot(\nabla u(s)-\nabla u(t))\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\ge-\varepsilon\int_0^t g_i(t-s)\,\|\nabla u(s)-\nabla u(t)\|_2^2\mathrm{d}s \\
&\qquad-\frac{1}{\varepsilon}\left(\int_0^t g_i(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\
&\quad=-\varepsilon(g_i\diamond\nabla u)(t)-\frac{1}{\varepsilon}\left(\int_0^t g_i(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2,\quad i=1,2,
\end{aligned} \tag{63}$$

for any $\varepsilon > 0$, inserting (63) into (62) and utilizing (27), we have

$$\begin{aligned}
\Phi(t) \geq & -2(p+2)\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) \\
& - \varepsilon\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\
& + 4(p+2)\int_\Omega F(u,v)\,\mathrm{d}x \\
& + \left(2 - \frac{1}{\varepsilon}\right) \\
& \times \left[\int_0^t g_1(s)\,\mathrm{d}s\|\nabla u(t)\|_2^2 + \int_0^t g_2(s)\,\mathrm{d}s\|\nabla v(t)\|_2^2\right] \\
& - 2(p+3)\left(\int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau + \int_0^t \|\nabla v_t(\tau)\|_2^2 \mathrm{d}\tau\right) \\
& - 2M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 \\
& - 2M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 \\
& - 2(p+2)\beta - 4(p+2)E(t) + 4(p+2)E(t) \\
= & -4(p+2)E(t) + 2(p+2)\overline{M}\left(\|\nabla u(t)\|_2^2\right) \\
& - 2M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2 \\
& + 2(p+2)\overline{M}\left(\|\nabla v(t)\|_2^2\right) \\
& - 2M\left(\|\nabla v(t)\|_2^2\right)\|\nabla v(t)\|_2^2 \\
& - 2\left(p+1+\frac{1}{2\varepsilon}\right)\int_0^t g_1(s)\,\mathrm{d}s\|\nabla u(t)\|_2^2 \\
& - 2(p+2)\beta \\
& - 2\left(p+1+\frac{1}{2\varepsilon}\right)\int_0^t g_2(s)\,\mathrm{d}s\|\nabla v(t)\|_2^2 \\
& - 2(p+3)\int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right)\mathrm{d}\tau \\
& + \left[2(p+2) - \varepsilon\right]\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right].
\end{aligned} \tag{64}$$

Using (29) and (A4), we have

$$\begin{aligned}
\Phi(t) \geq & -4(p+2)E(0) + 2(p+1) \\
& \times \left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\
& - \frac{1}{\varepsilon}\left(\int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\
& + 2(p+1)\left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2 \\
& - \frac{1}{\varepsilon}\left(\int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2 - 2(p+2)\beta \\
& + 2(p+1)\int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right)\mathrm{d}\tau \\
& + \left[2(p+2) - \varepsilon\right]\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\
= & -4(p+2)E(0) \\
& + \left[2(p+1)m_0 - \left(2(p+1) + \frac{1}{\varepsilon}\right)\int_0^t g_1(s)\,\mathrm{d}s\right] \\
& \times \|\nabla u(t)\|_2^2 - 2(p+2)\beta \\
& + \left[2(p+1)m_0 - \left(2(p+1) + \frac{1}{\varepsilon}\right)\int_0^t g_2(s)\,\mathrm{d}s\right] \\
& \times \|\nabla v(t)\|_2^2 \\
& + \left[2(p+2) - \varepsilon\right]\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\
& + 2(p+1)\int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right)\mathrm{d}\tau.
\end{aligned} \tag{65}$$

If $\delta < 0$, that is, $E(0) < 0$, we choose $\varepsilon = 2(p+2)$ in (65) and β small enough such that $\beta \leq -2E(0)$. Then by (32), we have

$$\begin{aligned}
\Phi(t) \geq & \left[2(p+1)m_0 - \left(2(p+1) + \frac{1}{2(p+2)}\right)\int_0^t g_1(s)\,\mathrm{d}s\right] \\
& \times \|\nabla u(t)\|_2^2 \\
& + \left[2(p+1)m_0 - \left(2(p+1) + \frac{1}{2(p+2)}\right)\int_0^t g_2(s)\,\mathrm{d}s\right] \\
& \times \|\nabla v(t)\|_2^2 \\
& + 2(p+1)\int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right)\mathrm{d}\tau \\
& + 2(p+2)\left(-2E(0) - \beta\right) \geq 0.
\end{aligned} \tag{66}$$

If $0 \leq \delta < 1$, that is, $0 \leq E(0) < \delta E_1 < E_1$, we choose $\varepsilon = 2(p+2)(1-\delta) + 2\delta$ and $\beta = 2(\delta E_1 - E(0))$ in (65). Then, we get

$$\begin{aligned}
& \Phi(t) \\
& \geq -4(p+2)\delta E_1 \\
& \quad + \left[2(p+1)m_0 - \left(2(p+1) + \frac{1}{2(p+2)(1-\delta) + 2\delta}\right) \times \int_0^t g_1(s)\,\mathrm{d}s\right] \\
& \quad \times \|\nabla u(t)\|_2^2
\end{aligned}$$

$$+\left[2(p+1)m_0-\left(2(p+1)+\frac{1}{2(p+2)(1-\delta)+2\delta}\right)\times\int_0^t g_2(s)\,ds\right]\times\|\nabla v(t)\|_2^2+2(p+1)\int_0^t\left(\|\nabla u_t(\tau)\|_2^2+\|\nabla v_t(\tau)\|_2^2\right)d\tau+2(p+1)\delta\left[(g_1\diamond\nabla u)(t)+(g_2\diamond\nabla v)(t)\right]. \tag{67}$$

By (32), we have (notice that $\widehat{\delta}=\delta$ due to $\delta\geq 0$)

$$\begin{aligned}\int_0^t g_i(s)\,ds &\leq \int_0^\infty g_i(s)\,ds\\ &\leq \frac{2(p+1)m_0}{2(p+1)+1/\left\{2\left[(1-\delta)^2(p+2)+\delta(1-\delta)\right]\right\}}\\ &=\frac{2(p+1)m_0-2\delta(p+1)m_0}{2(1-\delta)(p+1)+1/\{2(1-\delta)(p+2)+2\delta\}},\end{aligned}\quad (i=1,2); \tag{68}$$

that is,

$$2\delta(p+1)\left[m_0-\int_0^t g_i(s)\,ds\right]\leq 2(p+1)m_0-\left(2(p+1)+\frac{1}{2(p+2)(1-\delta)+2\delta}\right)\int_0^t g_i(s)\,ds, \tag{69}$$

for $i=1,2$. Therefore, from the above two inequalities and (A2), we can get

$$\Phi(t)\geq -4(p+2)\delta E_1+2\delta(p+1)\left(l_1\|\nabla u(t)\|_2^2+l_2\|\nabla v(t)\|_2^2\right). \tag{70}$$

Since

$$\left(l_1\|\nabla u_0\|_2^2+l_2\|\nabla v_0\|_2^2\right)^{1/2}>\alpha_*, \tag{71}$$

by Lemma 8, there exists a constant $\alpha_3>\alpha_*$ such that

$$\left(l_1\|\nabla u(t)\|_2^2+l_2\|\nabla v(t)\|_2^2\right)^{1/2}\geq\alpha_3, \tag{72}$$

which implies

$$\frac{p+1}{2(p+2)}\left(l_1\|\nabla u(t)\|_2^2+l_2\|\nabla v(t)\|_2^2\right)\geq\frac{p+1}{2(p+2)}\alpha_3^2>\frac{p+1}{2(p+2)}\alpha_*^2>E_1. \tag{73}$$

It follows from (67) and (73) that

$$\Phi(t)\geq 4(p+2)\left[\delta E_1-\delta E_1\right]\geq 0. \tag{74}$$

Therefore, by (58), (66), and (74), we obtain

$$L(t)L''(t)-\frac{p+3}{2}L'(t)^2\geq 0 \tag{75}$$

for almost every $t\in[0,T]$. By (49), we then choose T_0 sufficiently large such that

$$(p+1)\left(\int_\Omega u_0u_1\,dx+\int_\Omega v_0v_1\,dx+\beta T_0\right)>\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2>0, \tag{76}$$

consequently,

$$L'(0)=2\int_\Omega u_0u_1\,dx+2\int_\Omega v_0v_1\,dx+2\beta T_0>0. \tag{77}$$

Then by (76) and (77), we choose T large enough so that

$$T>\left(\|u_0\|_2^2+\|v_0\|_2^2+\beta T_0^2\right)\times\left((p+1)\left(\int_\Omega u_0u_1\,dx+\int_\Omega v_0v_1\,dx+\beta T_0\right)-\left(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2\right)\right)^{-1}>0, \tag{78}$$

which ensures that $T>(2/(p+1))(L(0)/L'(0))$. As $(p+3)/2>1$, letting $\sigma=(p+1)/2$, we can select T_* such that $T_*\leq L(0)/\sigma L'(0)\leq T$. By using Lemma 9, we get $\lim_{t\to T_*^-}L(t)=\infty$. This implies that

$$\lim_{t\to T_*^-}\left(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2\right)=\infty, \tag{79}$$

which is a contradiction. Thus, $T<\infty$. □

4. Blow-Up of Solutions with Arbitrarily Positive Initial Energy

In this section, we prove the second blow-up result (Theorem 6) for solutions with arbitrarily positive initial energy. In order to attain our aim, we need the following three lemmas.

Lemma 10 (see [4]). *Let $\delta^*>0$ and $B(t)\in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t)-4(\delta^*+1)B'(t)+4(\delta^*+1)B(t)\geq 0. \tag{80}$$

If

$$B'(0)>r_2B(0)+K_0, \tag{81}$$

then

$$B'(t)>K_0 \tag{82}$$

for $t>0$, where K_0 is a constant and $r_2=2(\delta^*+1)-2\sqrt{(\delta^*+1)\delta^*}$ is the smallest root of the equation

$$r^2-4(\delta^*+1)r+4(\delta^*+1)=0. \tag{83}$$

Lemma 11 (see [4]). *If $h(t)$ is a nonincreasing function on $[0,\infty)$ and satisfies the differential inequality*

$$h'(t)^2 \geq a_* + b_* h(t)^{2+1/\delta^*} \quad \text{for } t \geq 0, \tag{84}$$

where $a_* > 0$, $\delta^* > 0$, and $b_* > 0$, then there exists a finite time T^* such that $\lim_{t\to T^*-} h(t) = 0$ and the upper bound of T^* is estimated by

$$T^* \leq 2^{(3\delta^*+1)/2\delta^*} \frac{\delta^* c}{\sqrt{a_*}} \left\{1 - [1 + ch(0)]^{-1/2\delta^*}\right\}, \tag{85}$$

where $c = (a_*/b_*)^{2+1/\delta^*}$.

For the next lemma, we define

$$\begin{aligned} K(t) &:= K(u(t), v(t)) = \|u(t)\|_2^2 + \|v(t)\|_2^2 \\ &\quad + \int_0^t \|\nabla u(\tau)\|_2^2 \mathrm{d}\tau + \int_0^t \|\nabla v(\tau)\|_2^2 \mathrm{d}\tau, \quad t \geq 0. \end{aligned} \tag{86}$$

Lemma 12. *Assume that the conditions of Theorem 6 hold and let (u, v) be a solution of (1), then*

$$K'(t) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2, \quad t > 0. \tag{87}$$

Proof. By (86), we have

$$K'(t) = 2\int_\Omega u u_t \mathrm{d}x + 2\int_\Omega v v_t \mathrm{d}x + \|\nabla u\|_2^2 + \|\nabla v\|_2^2, \tag{88}$$

$$\begin{aligned} K''(t) &= 2\|u_t\|_2^2 + 2\|v_t\|_2^2 + 2\int_\Omega u u_{tt} \mathrm{d}x \\ &\quad + 2\int_\Omega v v_{tt} \mathrm{d}x + 2\int_\Omega \nabla u \cdot \nabla u_t \mathrm{d}x \\ &\quad + 2\int_\Omega \nabla v \cdot \nabla v_t \mathrm{d}x. \end{aligned} \tag{89}$$

Testing the first equation of system (1) with u and testing the second equation of system (1) with v and plugging the results into the expression of $K''(t)$ we obtain

$$\begin{aligned} K''(t) &= 2\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - 2M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 \\ &\quad - 2M\left(\|\nabla v\|_2^2\right)\|\nabla v\|_2^2 + 4(p+2)\int_\Omega F(u,v)\,\mathrm{d}x \\ &\quad + 2\int_0^t \int_\Omega g_1(t-s)\nabla u(s)\cdot\nabla u(t)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + 2\int_0^t \int_\Omega g_2(t-s)\nabla v(s)\cdot\nabla v(t)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{90}$$

By (27), (29), and (90), we get

$$\begin{aligned} &K''(t) - 2(p+3)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad \geq -4(p+2)E(0) \\ &\qquad + 4(p+2)\int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right)\mathrm{d}\tau \\ &\qquad + 2(p+2)\left[\overline{M}\left(\|\nabla u\|_2^2\right) + \overline{M}\left(\|\nabla v\|_2^2\right)\right] \\ &\qquad - \left[2M\left(\|\nabla u\|_2^2\right) + 2(p+2)\int_0^t g_1(s)\,\mathrm{d}s\right]\|\nabla u\|_2^2 \\ &\qquad - \left[2M\left(\|\nabla v\|_2^2\right) + 2(p+2)\int_0^t g_2(s)\,\mathrm{d}s\right]\|\nabla v\|_2^2 \\ &\qquad + 2(p+2)\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\ &\qquad - 2(p+2)\int_0^t \left[\left(g_1' \diamond \nabla u\right)(\tau) + \left(g_2' \diamond \nabla v\right)(\tau)\right]\mathrm{d}\tau \\ &\qquad + 2\int_0^t \int_\Omega g_1(t-s)\nabla u(s)\cdot\nabla u(t)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + 2\int_0^t \int_\Omega g_2(t-s)\nabla v(s)\cdot\nabla v(t)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{91}$$

By using Hölder's inequality and Young's inequality, we have

$$\begin{aligned} &2\int_0^t \int_\Omega g_1(t-s)\nabla u(s)\cdot\nabla u(t)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = 2\int_0^t g_1(t-s)\int_\Omega \nabla u(t)\left(\nabla u(s) - \nabla u(t)\right)\mathrm{d}x\,\mathrm{d}s \\ &\qquad + 2\left(\int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\ &\quad \geq -2(p+2)\left(g_1 \diamond \nabla u\right)(t) \\ &\qquad + \left(2 - \frac{1}{2(p+2)}\right)\left(\int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2. \end{aligned} \tag{92}$$

Similarly,

$$\begin{aligned} &2\int_0^t \int_\Omega g_2(t-s)\nabla v(s)\cdot\nabla v(t)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \geq -2(p+2)\left(g_2 \diamond \nabla v\right)(t) \\ &\qquad + \left(2 - \frac{1}{2(p+2)}\right)\left(\int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2. \end{aligned} \tag{93}$$

Then taking (92) and (93) into account, we obtain

$$\begin{aligned}
&K''(t) - 2(p+3)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&\quad \geq -4(p+2)E(0) + 4(p+2) \\
&\qquad \times \int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right) d\tau \\
&\qquad + \Bigg\{2(p+2)\overline{M}\left(\|\nabla u\|_2^2\right) \\
&\qquad - \Bigg[2M\left(\|\nabla u\|_2^2\right) \\
&\qquad + \left(2(p+1) + \frac{1}{2(p+2)}\right)\int_0^t g_1(s)\,ds\Bigg] \\
&\qquad \times \|\nabla u\|_2^2\Bigg\} \\
&\qquad + \Bigg\{2(p+2)\overline{M}\left(\|\nabla v\|_2^2\right) \\
&\qquad - \Bigg[2M\left(\|\nabla v\|_2^2\right) \\
&\qquad + \left(2(p+1) + \frac{1}{2(p+2)}\right)\int_0^t g_2(s)\,ds\Bigg] \\
&\qquad \times \|\nabla v\|_2^2\Bigg\}.
\end{aligned} \tag{94}$$

Thus, by (A4) and (33), we get

$$\begin{aligned}
&K''(t) - 2(p+3)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&\quad \geq -4(p+2)E(0) + 4(p+2) \\
&\qquad \times \int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right) d\tau.
\end{aligned} \tag{95}$$

We note that

$$\begin{aligned}
\|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2 &= 2\int_0^t \int_\Omega \nabla u \cdot \nabla u_t \,dx\,d\tau, \\
\|\nabla v(t)\|_2^2 - \|\nabla v_0\|_2^2 &= 2\int_0^t \int_\Omega \nabla v \cdot \nabla v_t \,dx\,d\tau.
\end{aligned} \tag{96}$$

By Hölder's inequality and Young's inequality, we obtain from (96)

$$\begin{aligned}
\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 &\leq \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 \\
&\quad + \int_0^t \left(\|\nabla u(\tau)\|_2^2 + \|\nabla v(\tau)\|_2^2\right) d\tau \\
&\quad + \int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right) d\tau.
\end{aligned} \tag{97}$$

By Hölder's inequality and Young's inequality again, it follows from (86), (88), and (97) that

$$\begin{aligned}
K'(t) &\leq K(t) + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2 \\
&\quad + \int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right) d\tau.
\end{aligned} \tag{98}$$

In view of (95) and (98), we have

$$K''(t) - 2(p+3)K'(t) + 2(p+3)K(t) + L_4 \geq 0, \tag{99}$$

where

$$L_4 := 4(p+2)E(0) + 2(p+3)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right). \tag{100}$$

Let

$$B(t) = K(t) + \frac{L_4}{2(p+3)}, \quad t > 0. \tag{101}$$

Then $B(t)$ satisfies (80) for $\delta^* = (p+1)/2$. By (81), we see that if

$$\begin{aligned}
K'(0) &> \left(p + 3 - \sqrt{(p+3)(p+1)}\right)\left[K(0) + \frac{L_4}{2(p+3)}\right] \\
&\quad + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2,
\end{aligned} \tag{102}$$

that is, if

$$\begin{aligned}
E(0) &< \frac{\sqrt{p+3}}{(p+2)\left(\sqrt{p+3} - \sqrt{p+1}\right)} \int_\Omega (u_0 u_1 + v_0 v_1)\,dx \\
&\quad - \frac{p+3}{2(p+2)}\left(\|u_0\|_2^2 + \|v_0\|_2^2 + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right),
\end{aligned} \tag{103}$$

which is satisfied by the second hypothesis to Theorem 6, then we get from Lemma 10 that $K'(t) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2, t > 0$. Thus, the proof of Lemma 12 is completed. □

In what follows, we find an estimate for the life span of $K(t)$ and prove Theorem 6.

Proof of Theorem 6. Let

$$h(t) = \left[K(t) + (T_1 - t)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]^{-\delta^*}, \quad \text{for } t \in [0, T_1], \tag{104}$$

where $\delta^* = (p+1)/2$ and $T_1 > 0$ is a certain constant which will be specified later. Then we have

$$h'(t) = -\delta^* h(t)^{1+1/\delta^*}\left(K'(t) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right), \tag{105}$$

$$h''(t) = -\delta^* h(t)^{1+2/\delta^*} V(t), \tag{106}$$

where

$$\begin{aligned}
V(t) &= K''(t)\left[K(t) + (T_1 - t)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right] \\
&\quad - (1 + \delta^*)\left(K'(t) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right)^2.
\end{aligned} \tag{107}$$

For simplicity, we denote

$$\begin{aligned} P &= \|u(t)\|_2^2 + \|v(t)\|_2^2, \\ Q &= \int_0^t \left(\|\nabla u(\tau)\|_2^2 + \|\nabla v(\tau)\|_2^2\right) d\tau, \\ R &= \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2, \\ S &= \int_0^t \left(\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2\right) d\tau. \end{aligned} \tag{108}$$

It follows from (88), (96), Hölder's inequality, and Young's inequality that

$$K'(t) \le 2\left(\sqrt{PR} + \sqrt{QS}\right) + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2. \tag{109}$$

By (95), we get

$$K''(t) \ge \left(-4 - 8\delta^*\right) E(0) + 4\left(1 + \delta^*\right)(R + S). \tag{110}$$

Applying (107)–(110), we obtain

$$\begin{aligned} V(t) \ge\ & \left[\left(-4 - 8\delta^*\right) E(0) + 4\left(1 + \delta^*\right)(R + S)\right] \\ & \times \left[K(t) + \left(T_1 - t\right)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right] \\ & - 4\left(1 + \delta^*\right)\left(\sqrt{PR} + \sqrt{QS}\right)^2. \end{aligned} \tag{111}$$

From (104) and (98) we deduce that

$$\begin{aligned} V(t) \ge\ & \left(-4 - 8\delta^*\right) E(0)\, h(t)^{-1/\delta^*} \\ & + 4\left(1 + \delta^*\right)(R + S)\left(T_1 - t\right)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) \\ & + 4\left(1 + \delta^*\right)\left[(R + S)(P + Q) - \left(\sqrt{PR} + \sqrt{QS}\right)^2\right]. \end{aligned} \tag{112}$$

By Cauchy-Schwarz inequality, the last term in the above inequality is nonnegative. Hence, we have

$$V(t) \ge \left(-4 - 8\delta^*\right) E(0)\, h(t)^{-1/\delta^*}, \quad t \ge 0. \tag{113}$$

Therefore by (106) and (113), we get

$$h''(t) \le \delta^*\left(4 + 8\delta^*\right) E(0)\, h(t)^{1+1/\delta^*}, \quad t \ge 0. \tag{114}$$

Note that by Lemma 12, $h'(t) < 0$ for $t > 0$. Multiplying (114) by $h'(t)$ and integrating from 0 to t, we obtain

$$h'(t)^2 \ge a_1 + b_1 h(t)^{2+1/\delta^*}, \quad \text{for } t \ge 0, \tag{115}$$

where

$$\begin{aligned} a_1 = {} & \delta^{*2} h(0)^{2+2/\delta^*} \\ & \times \left[\left(K'(0) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right)^2 \right. \\ & \left. -8E(0)\, h(0)^{-1/\delta^*}\right], \end{aligned} \tag{116}$$

$$b_1 = 8\delta^{*2} E(0). \tag{117}$$

We observe that $a_1 > 0$ if

$$\begin{aligned} E(0) &< \frac{\left(K'(0) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right)^2}{8\left[K(0) + T_1\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]} \\ &= \frac{\left(\int_\Omega u_0 u_1 dx + \int_\Omega v_0 v_1 dx\right)^2}{2\left[\|u_0\|_2^2 + \|v_0\|_2^2 + T_1\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]}. \end{aligned} \tag{118}$$

Then by Lemma 11, there exists a finite time T^* such that $\lim_{t \to T^{*-}} h(t) = 0$. Moreover, the upper bounds of T^* are estimated by

$$T^* \le 2^{(3\delta^*+1)/2\delta^*} \frac{\delta^* c}{\sqrt{a_1}} \left\{1 - \left[1 + ch(0)\right]^{-1/2\delta^*}\right\}. \tag{119}$$

Therefore

$$\begin{aligned} & \lim_{t \to T^{*-}} K(t) \\ & = \lim_{t \to T^{*-}} \left(\|u(t)\|_2^2 + \|v(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \right. \\ & \qquad \left. + \int_0^t \|\nabla v(\tau)\|_2^2 d\tau\right) = \infty. \end{aligned} \tag{120}$$

This completes the proof. □

Remark 13. The choice of T_1 in (104) is possible provided that $T_1 \ge T^*$.

5. General Decay of Solutions

In this section, we prove the general decay of solutions of system (1). The method of proof is similar to that of [42, Theorem 3.5]. We first state a lemma which is similar to the one first proved by Vitillaro in [33] to study a class of a scalar wave equation.

Lemma 14 ([23, Lemma 3.2]). *Suppose that* (20), *(A1), and* (*A*2) *hold. Let* (u, v) *be the solution of system* (1). *Assume further that* $E(0) < E_1$ *and*

$$\left(l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2\right)^{1/2} < \alpha_*. \tag{121}$$

Then

$$\begin{aligned} & \left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2 + \left(g_1 \diamond \nabla u\right)(t) + \left(g_2 \diamond \nabla v\right)(t)\right)^{1/2} \\ & \quad < \alpha_* \end{aligned} \tag{122}$$

for all $t \in [0, T)$.

The auxiliary functionals $I(t), J(t)$ of problem (1) are defined as

$$\begin{aligned} I(t) &:= I(u(t), v(t)) = \left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\ &\quad + \left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2 \\ &\quad + (g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \\ &\quad - 2(p+2)\int_\Omega F(u,v)\,\mathrm{d}x, \\ J(t) &:= J(u(t), v(t)) \\ &= \frac{1}{2}\left[\left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \right. \\ &\qquad \left. + \left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2\right] \\ &\quad + \frac{1}{2}\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \right. \\ &\qquad \left. + \frac{m_1}{\gamma+1}\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right)\right] \\ &\quad - \int_\Omega F(u,v)\,\mathrm{d}x. \end{aligned} \tag{123}$$

Lemma 15. *Suppose that* (20), *(A1), and* (*A*2) *hold. Let* (u, v) *be the solution of system* (1). *Assume further that* $E(0) < E_1$ *and*

$$\left(l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2\right)^{1/2} < \alpha_*. \tag{124}$$

Then

$$l\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \le \frac{2(p+2)}{p+1}E(t), \tag{125}$$

$$(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \le \frac{2(p+2)}{p+1}E(t) \tag{126}$$

for all $t \in [0, T)$.

Proof. Since $E(0) < E_1$ and

$$\left(l_1\|\nabla u_0\|_2^2 + l_2\|\nabla v_0\|_2^2\right)^{1/2} < \alpha_*, \tag{127}$$

it follows from Lemma 14 and (30) that

$$\begin{aligned} &l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2 \\ &\quad \le l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2 \\ &\qquad + (g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \\ &\quad < \alpha_*^2 = \eta_1^{-1/(p+1)}, \end{aligned} \tag{128}$$

which implies that

$$\begin{aligned} I(t) &\ge \left(m_0 - \int_0^\infty g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &\quad + \left(m_0 - \int_0^\infty g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2 \\ &\quad + (g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \\ &\quad - 2(p+2)\int_\Omega F(u,v)\,\mathrm{d}x \\ &\ge l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2 \\ &\quad - 2(p+2)\int_\Omega F(u,v)\,\mathrm{d}x \\ &= l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2 \\ &\quad - \left(a\|u+v\|_{2(p+2)}^{2(p+2)} + 2b\|uv\|_{p+2}^{p+2}\right) \\ &\ge l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2 \\ &\quad - \eta_1\left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right)^{p+2} \\ &= \left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right) \\ &\quad \times \left[1 - \eta_1\left(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla v(t)\|_2^2\right)^{p+1}\right] \ge 0 \end{aligned} \tag{129}$$

for $t \in [0, T)$, where we have used (24). Further, by (123), we have

$$\begin{aligned} J(t) &\ge \frac{1}{2}\left[\left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \right. \\ &\qquad + \left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2 + (g_1 \diamond \nabla u)(t) \\ &\qquad \left. + (g_2 \diamond \nabla v)(t)\right] \\ &\quad - \int_\Omega F(u,v)\,\mathrm{d}x - \frac{1}{2(p+2)}I(t) + \frac{1}{2(p+2)}I(t) \\ &\ge \left(\frac{1}{2} - \frac{1}{2(p+2)}\right) \\ &\quad \times \left[\left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \right. \\ &\qquad \left. + \left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2\right] \\ &\quad + \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\ &\quad + \frac{1}{2(p+2)}I(t) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{p+1}{2(p+2)} \\
&\quad \times \Big[l_1 \|\nabla u(t)\|_2^2 + l_2 \|\nabla v(t)\|_2^2 \\
&\qquad + (g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \Big] \\
&\quad + \frac{1}{2(p+2)} I(t) \\
&\geq 0.
\end{aligned} \tag{130}$$

From (130) and (36), we deduce that

$$\begin{aligned}
&l\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \\
&\quad \leq l_1 \|\nabla u(t)\|_2^2 + l_2 \|\nabla v(t)\|_2^2 \leq \frac{2(p+2)}{p+1} J(t) \\
&\quad \leq \frac{2(p+2)}{p+1} E(t), \\
&(g_1 \diamond \nabla u)(t) \\
&\quad + (g_2 \diamond \nabla v)(t) \leq \frac{2(p+2)}{p+1} J(t) \leq \frac{2(p+2)}{p+1} E(t).
\end{aligned} \tag{131}$$

□

We note that the following functional was introduced in [42]:

$$G_1(t) := E(t) + \varepsilon_1 \phi(t) + \varepsilon_2 \psi(t), \tag{132}$$

where ε_1 and ε_2 are some positive constants and

$$\phi(t) = \xi_1(t) \int_\Omega u(t) u_t(t)\, dx + \xi_2(t) \int_\Omega v(t) v_t(t)\, dx, \tag{133}$$

$$\begin{aligned}
\psi(t) = &-\xi_1(t) \int_\Omega u_t(t) \int_0^t g_1(t-s)(u(t)-u(s))\, ds\, dx \\
&- \xi_2(t) \int_\Omega v_t(t) \int_0^t g_2(t-s)(v(t)-v(s))\, ds\, dx.
\end{aligned} \tag{134}$$

Here, we use the same functional (132) but choose $\xi_i(t) \equiv 1$ $(i = 1, 2)$ in (133) and (134).

Lemma 16. *There exist two positive constants β_1 and β_2 such that*

$$\beta_1 G_1(t) \leq E(t) \leq \beta_2 G_1(t), \quad \forall t \geq 0. \tag{135}$$

Proof. From Hölder's inequality, Young's inequality, Lemma 2, (36), and (125), we deduce

$$\begin{aligned}
|\phi(t)| &\leq \|u\|_2 \|u_t\|_2 + \|v\|_2 \|v_t\|_2 \\
&\leq \frac{1}{2}\left(\|u\|_2^2 + \|u_t\|_2^2 + \|v\|_2^2 + \|v_t\|_2^2\right) \\
&\leq \frac{1}{2} C^2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&\leq C^2 \frac{p+2}{l(p+1)} E(t) + E(t) = \left(1 + C^2 \frac{p+2}{l(p+1)}\right) E(t).
\end{aligned} \tag{136}$$

Applying Hölder's inequality, Young's inequality, and Lemma 2 again, it follows that

$$\begin{aligned}
&\int_\Omega u_t(t) \int_0^t g_1(t-s)(u(t)-u(s))\, ds\, dx \\
&\quad \leq \|u_t\|_2 \\
&\qquad \times \left[\int_\Omega \left(\int_0^t g_1(t-s)(u(t)-u(s))\, ds \right)^2 dx \right]^{1/2} \\
&\quad \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2} \int_\Omega \left(\int_0^t g_1(t-s)(u(t)-u(s))\, ds \right)^2 dx \\
&\quad \leq \frac{1}{2}\|u_t\|_2^2 \\
&\qquad + \frac{1}{2}(m_0 - l) \int_\Omega \int_0^t g_1(t-s)(u(t)-u(s))^2 ds\, dx \\
&\quad \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2} C^2 (m_0 - l)(g_1 \diamond \nabla u)(t).
\end{aligned} \tag{137}$$

Similarly, applying Hölder's inequality, Young's inequality, and Lemma 2 again, we have

$$\begin{aligned}
&\int_\Omega v_t(t) \int_0^t g_2(t-s)(v(t)-v(s))\, ds\, dx \\
&\quad \leq \frac{1}{2}\|v_t(t)\|_2^2 + \frac{1}{2} C^2 (m_0 - l)(g_2 \diamond \nabla v)(t).
\end{aligned} \tag{138}$$

It follows from (137), (138), (36), and (126) that

$$\begin{aligned}
|\psi(t)| &\leq \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&\qquad + \frac{1}{2} C^2 (m_0 - l)\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\
&\leq E(t) + \frac{p+2}{p+1} C^2 (m_0 - l) E(t) \\
&= \left(1 + \frac{p+2}{p+1} C^2 (m_0 - l)\right) E(t).
\end{aligned} \tag{139}$$

If we take ϵ_1 and ϵ_2 to be sufficiently small, then (135) follows from (132), (136), and (139). □

Lemma 17. *Under the conditions of Theorem 7, the functional $\phi(t)$, defined by* (133) *(with $\xi_i(t) \equiv 1$ $(i = 1, 2)$), satisfies*

$$\begin{aligned}\phi'(t) \le{}& \|u_t\|_2^2 + \|v_t\|_2^2 - \left[l - \epsilon\left(m_0 - l + \frac{1}{2}\right)\right]\|\nabla u\|_2^2 \\ &- \left[l - \epsilon\left(m_0 - l + \frac{1}{2}\right)\right]\|\nabla v\|_2^2 \\ &+ \frac{1}{4\epsilon}\left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right] \\ &+ \frac{1}{2\epsilon}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \\ &+ 2(p+2)\int_\Omega F(u, v)\, \mathrm{d}x \\ &- m_1\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right), \quad \forall \epsilon > 0.\end{aligned} \tag{140}$$

Proof. By using the equations in (1) and setting $\xi_i(t) \equiv 1$ $(i = 1, 2)$ in (133), we easily see that

$$\begin{aligned}\phi'(t) ={}& \|u_t\|_2^2 + \|v_t\|_2^2 + \int_\Omega u u_{tt}\mathrm{d}x + \int_\Omega v v_{tt}\mathrm{d}x \\ ={}& \|u_t\|_2^2 + \|v_t\|_2^2 \\ &- M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 - M\left(\|\nabla v\|_2^2\right)\|\nabla v\|_2^2 \\ &+ \int_\Omega \nabla u(t) \cdot \int_0^t g_1(t-s)\nabla u(s)\,\mathrm{d}s\,\mathrm{d}x \\ &+ \int_\Omega \nabla v(t) \cdot \int_0^t g_2(t-s)\nabla v(s)\,\mathrm{d}s\,\mathrm{d}x \\ &- \int_\Omega \nabla u_t \cdot \nabla u\mathrm{d}x - \int_\Omega \nabla v_t \cdot \nabla v\mathrm{d}x + 2(p+2) \\ &\times \int_\Omega F(u, v)\,\mathrm{d}x.\end{aligned} \tag{141}$$

By Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}&\int_\Omega \nabla u(t) \cdot \int_0^t g_1(t-s)\nabla u(s)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad\le \left(\int_0^t g_1(t-s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\ &\qquad + \int_\Omega \int_0^t g_1(t-s)\,|\nabla u(t)| \\ &\qquad \times |\nabla u(s) - \nabla u(t)|\,\mathrm{d}s\,\mathrm{d}x \le (1+\epsilon)\left(\int_0^t g_1(s)\,\mathrm{d}s\right) \\ &\qquad \times \|\nabla u(t)\|_2^2 + \frac{1}{4\epsilon}(g_1 \diamond \nabla u)(t) \le (1+\epsilon)(m_0 - l) \\ &\qquad \times \|\nabla u(t)\|_2^2 + \frac{1}{4\epsilon}(g_1 \diamond \nabla u)(t), \quad \forall \epsilon > 0.\end{aligned} \tag{142}$$

In the same way, we can get

$$\begin{aligned}&\int_\Omega \nabla v(t) \cdot \int_0^t g_2(t-s)\nabla v(s)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad\le (1+\epsilon)(m_0 - l)\|\nabla v(t)\|_2^2 + \frac{1}{4\epsilon}(g_2 \diamond \nabla v)(t),\end{aligned} \tag{143}$$

$$\int_\Omega \nabla u_t \cdot \nabla u\mathrm{d}x \le \frac{\epsilon}{2}\|\nabla u\|_2^2 + \frac{1}{2\epsilon}\|\nabla u_t\|_2^2, \tag{144}$$

$$\int_\Omega \nabla v_t \cdot \nabla v\mathrm{d}x \le \frac{\epsilon}{2}\|\nabla v\|_2^2 + \frac{1}{2\epsilon}\|\nabla v_t\|_2^2, \quad \forall \epsilon > 0. \tag{145}$$

Using (141)–(145), we obtain

$$\begin{aligned}\phi'(t) \le{}& \|u_t\|_2^2 + \|v_t\|_2^2 - m_0\|\nabla u\|_2^2 \\ &- m_0\|\nabla v\|_2^2 - m_1\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right) \\ &+ (1+\epsilon)(m_0 - l)\left(\|\nabla u\|_2^2 + \|\nabla v(t)\|_2^2\right) \\ &+ \frac{1}{4\epsilon}\left((g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right) \\ &+ \frac{\epsilon}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{2\epsilon}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \\ &+ 2(p+2)\int_\Omega F(u, v)\,\mathrm{d}x \\ ={}& \|u_t\|_2^2 + \|v_t\|_2^2 \\ &- \left[l - \epsilon\left(m_0 - l + \frac{1}{2}\right)\right]\|\nabla u\|_2^2 - \left[l - \epsilon\left(m_0 - l + \frac{1}{2}\right)\right] \\ &\times \|\nabla v\|_2^2 + \frac{1}{4\epsilon}(g_1 \diamond \nabla u)(t) + \frac{1}{4\epsilon}(g_2 \diamond \nabla v)(t) \\ &+ \frac{1}{2\epsilon}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \\ &- m_1\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right) \\ &+ 2(p+2)\int_\Omega F(u, v)\,\mathrm{d}x, \quad \forall \epsilon > 0.\end{aligned} \tag{146}$$

So, Lemma 17 is established. □

Lemma 18. *Under the conditions of Theorem 7, the functional $\psi(t)$, defined by* (134) *(with $\xi_i(t) \equiv 1$ $(i = 1, 2)$), satisfies*

$$\begin{aligned}\psi'(t) \le{}& \left[\eta(m_0 - l)(3m_0 - 2l) + 3C\eta C_7\right] \\ &\times \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \eta\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \\ &+ \eta m_1(m_0 - l)\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right) \\ &+ \left[\left(2\eta + \frac{2 + C^2}{4\eta}\right)(m_0 - l) + \frac{C_6}{4\eta}\right] \\ &\times \left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t)\right]\end{aligned}$$

$$
\begin{aligned}
&-\frac{g_0(0)}{4\eta}C^2\left[\left(g_1'\diamond\nabla u\right)(t)+\left(g_2'\diamond\nabla v\right)(t)\right]\\
&+\left(\eta-\int_0^t g_1(s)\,ds\right)\|u_t\|_2^2\\
&+\left(\eta-\int_0^t g_2(s)\,ds\right)\|v_t\|_2^2,\quad \forall\eta>0,
\end{aligned}
\tag{147}
$$

where

$$
\begin{aligned}
&g_0(0)=\max\{g_1(0),g_2(0)\},\\
&C_6:=m_0+m_1\left(\frac{2(p+2)}{l(p+1)}E(0)\right)^{\gamma},\\
&C_7:=C^{2(2p+3)}\left[\frac{2(p+2)}{l(p+1)}E(0)\right]^{2(p+1)}.
\end{aligned}
\tag{148}
$$

Proof. By using the equations in (1) and set $\xi_i(t)\equiv 1$ $(i=1,2)$ in (134), we observe that

$$
\begin{aligned}
\psi'(t)=&-\int_\Omega u_{tt}(t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega v_{tt}(t)\int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx\\
&-\int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega v_t(t)\int_0^t g_2'(t-s)(v(t)-v(s))\,ds\,dx\\
&-\int_\Omega u_t(t)\int_0^t g_1(t-s)u_t(t)\,ds\,dx\\
&-\int_\Omega v_t(t)\int_0^t g_2(t-s)v_t(t)\,ds\,dx\\
=&\int_\Omega M\left(\|\nabla u\|_2^2\right)\nabla u(t)\\
&\cdot\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&+\int_\Omega M\left(\|\nabla v\|_2^2\right)\nabla v(t)\\
&\cdot\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx\\
&-\int_\Omega\left(\int_0^t g_1(t-s)\nabla u(s)\,ds\right)\\
&\times\left[\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\right]dx\\
&-\int_\Omega\left(\int_0^t g_2(t-s)\nabla v(s)\,ds\right)\\
&\times\left[\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\right]dx\\
&+\left[\int_\Omega\nabla u_t(t)\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\right.\\
&\qquad\left.+\int_\Omega\nabla v_t(t)\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx\right]\\
&-\left[\int_\Omega f_1(u,v)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\right.\\
&\qquad\left.+\int_\Omega f_2(u,v)\int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx\right]\\
&-\left[\int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx\right.\\
&\qquad\left.+\int_\Omega v_t(t)\int_0^t g_2'(t-s)(v(t)-v(s))\,ds\,dx\right]\\
&-\left(\int_0^t g_1(s)\,ds\right)\|u_t(t)\|_2^2\\
&-\left(\int_0^t g_2(s)\,ds\right)\|v_t(t)\|_2^2\\
:=&\,I_1+\cdots+I_7-\left(\int_0^t g_1(s)\,ds\right)\|u_t(t)\|_2^2\\
&-\left(\int_0^t g_2(s)\,ds\right)\|v_t(t)\|_2^2.
\end{aligned}
\tag{149}
$$

In what follows we will estimate $I_1,\ldots,I_7$ in (149). By Young's inequality, (A2), and (125), we get

$$
\begin{aligned}
|I_1|\le&\,\eta M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2\int_0^t g_1(s)\,ds\\
&+\frac{1}{4\eta}M\left(\|\nabla u\|_2^2\right)(g_1\diamond\nabla u)(t)\\
=&\,\eta\left(m_0\|\nabla u\|_2^2+m_1\|\nabla u\|_2^{2(\gamma+1)}\right)\int_0^t g_1(s)\,ds\\
&+\frac{1}{4\eta}\left(m_0+m_1\|\nabla u\|_2^{2\gamma}\right)(g_1\diamond\nabla u)(t)\\
\le&\,\eta(m_0-l)\left(m_0\|\nabla u\|_2^2+m_1\|\nabla u\|_2^{2(\gamma+1)}\right)\\
&+\frac{1}{4\eta}\left[m_0+m_1\left(\frac{2(p+2)}{l(p+1)}E(0)\right)^{\gamma}\right](g_1\diamond\nabla u)(t)\\
\le&\,\eta m_0(m_0-l)\|\nabla u\|_2^2+\eta m_1(m_0-l)\|\nabla u\|_2^{2(\gamma+1)}\\
&+\frac{C_6}{4\eta}(g_1\diamond\nabla u)(t),\quad\forall\eta>0.
\end{aligned}
\tag{150}
$$

Similarly, we have

$$\begin{aligned}|I_2| \le{}& \eta m_0 (m_0 - l) \|\nabla v\|_2^2 + \eta m_1 (m_0 - l) \|\nabla v\|_2^{2(\gamma+1)} \\ &+ \frac{C_6}{4\eta} (g_2 \diamond \nabla v)(t), \quad \forall \eta > 0.\end{aligned} \tag{151}$$

For I_3 in (149), applying (A2), Hölder's inequality, and Young's inequality, we deduce

$$\begin{aligned}|I_3| \le{}& \eta \int_\Omega \left(\int_0^t g_1(t-s) \nabla u(s)\, \mathrm{d}s \right)^2 \mathrm{d}x \\ &+ \frac{1}{4\eta} \int_\Omega \left(\int_0^t g_1(t-s) (\nabla u(t) - \nabla u(s))\, \mathrm{d}s \right)^2 \mathrm{d}x \\ \le{}& \eta \int_\Omega \left[\int_0^t g_1(t-s) (|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|)\, \mathrm{d}s \right]^2 \mathrm{d}x \\ &+ \frac{1}{4\eta} (m_0 - l) (g_1 \diamond \nabla u)(t) \\ \le{}& 2\eta \left(\int_0^t g_1(s)\, \mathrm{d}s \right)^2 \|\nabla u\|_2^2 \\ &+ \left(2\eta + \frac{1}{4\eta} \right) (m_0 - l) (g_1 \diamond \nabla u)(t) \\ \le{}& 2\eta (m_0 - l)^2 \|\nabla u\|_2^2 \\ &+ \left(2\eta + \frac{1}{4\eta} \right) (m_0 - l) (g_1 \diamond \nabla u)(t), \quad \forall \eta > 0.\end{aligned} \tag{152}$$

Similarly,

$$\begin{aligned}|I_4| \le{}& 2\eta (m_0 - l)^2 \|\nabla v\|_2^2 \\ &+ \left(2\eta + \frac{1}{4\eta} \right) (m_0 - l) (g_2 \diamond \nabla v)(t), \quad \forall \eta > 0.\end{aligned} \tag{153}$$

In the same way, we have

$$\begin{aligned}|I_5| \le{}& \eta \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \right) \\ &+ \frac{1}{4\eta} (m_0 - l) \left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \right], \\ &\forall \eta > 0.\end{aligned} \tag{154}$$

By Young's inequality, Lemma 2, and (125) we see that

$$\begin{aligned}&\left| \int_\Omega f_1(u,v) \int_0^t g_1(t-s) (u(t) - u(s))\, \mathrm{d}s\, \mathrm{d}x \right| \\ &\quad \le \eta \int_\Omega \left(a|u+v|^{2p+3} + b|u|^{p+1}|v|^{p+2} \right)^2 \mathrm{d}x \\ &\qquad + \frac{C^2 (m_0 - l)}{4\eta} (g_1 \diamond \nabla u)(t) \\ &\quad \le C\eta \int_\Omega \left(|u|^{2(2p+3)} + |v|^{2(2p+3)} + |u|^{2(p+1)} |v|^{2(p+2)} \right) \mathrm{d}x \\ &\qquad + \frac{C^2 (m_0 - l)}{4\eta} (g_1 \diamond \nabla u)(t) \\ &\quad \le C\eta \left[\|u\|_{2(2p+3)}^{2(2p+3)} + \|v\|_{2(2p+3)}^{2(2p+3)} + \|u\|_{2(2p+3)}^{2(p+1)} \|v\|_{2(2p+3)}^{2(p+2)} \right] \\ &\qquad + \frac{C^2 (m_0 - l)}{4\eta} (g_1 \diamond \nabla u)(t) \\ &\quad \le C\eta C^{2(2p+3)} \\ &\qquad \times \left[\|\nabla u\|_2^{2(2p+3)} + \|\nabla v\|_2^{2(2p+3)} + \|\nabla u\|_2^{2(p+1)} \|\nabla v\|_2^{2(p+2)} \right] \\ &\qquad + \frac{C^2 (m_0 - l)}{4\eta} (g_1 \diamond \nabla u)(t) \\ &\quad \le C\eta C^{2(2p+3)} \left(\frac{2(p+2)}{l(p+1)} E(0) \right)^{2p+2} \\ &\qquad \times \left[\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\nabla u\|_2 \|\nabla v\|_2 \right] \\ &\qquad + \frac{C^2 (m_0 - l)}{4\eta} (g_1 \diamond \nabla u)(t) \\ &\quad \le \frac{3}{2} C\eta C_7 \left[\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right] \\ &\qquad + \frac{C^2 (m_0 - l)}{4\eta} (g_1 \diamond \nabla u)(t), \quad \forall \eta > 0.\end{aligned} \tag{155}$$

Hence, we infer that

$$\begin{aligned}|I_6| \le{}& 3C\eta C_7 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) \\ &+ \frac{C^2 (m_0 - l)}{4\eta} \left[(g_1 \diamond \nabla u)(t) + (g_2 \diamond \nabla v)(t) \right], \\ &\forall \eta > 0.\end{aligned} \tag{156}$$

By Young's inequality and Lemma 2 again, we obtain

$$\begin{aligned}|I_7| \le{}& \eta \|u_t\|_2^2 + \frac{1}{4\eta} \int_\Omega \left(\int_0^t g_1'(t-s) (u(t) - u(s))\, \mathrm{d}s \right)^2 \mathrm{d}x \\ &+ \eta \|v_t\|_2^2 + \frac{1}{4\eta} \int_\Omega \left(\int_0^t g_2'(t-s) (v(t) - v(s))\, \mathrm{d}s \right)^2 \mathrm{d}x \\ \le{}& \eta \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) + \frac{C^2}{4\eta} \left(\int_0^t g_1'(s)\, \mathrm{d}s \right) (g_1' \diamond \nabla u)(t) \\ &+ \frac{C^2}{4\eta} \left(\int_0^t g_2'(s)\, \mathrm{d}s \right) (g_2' \diamond \nabla v)(t)\end{aligned}$$

$$\leq \eta\left(\|u_t\|_2^2+\|v_t\|_2^2\right)-\frac{g_1(0)C^2}{4\eta}\left(g_1'\diamond\nabla u\right)(t)$$
$$-\frac{g_2(0)C^2}{4\eta}\left(g_2'\diamond\nabla v\right)(t),\quad \forall\eta>0. \tag{157}$$

Combining (149)–(157), we complete the proof of Lemma 18. □

Proof of Theorem 7. Since g_i are positive, we have, for any $t_0>0$,

$$\int_0^t g_i(s)\,\mathrm{d}s\geq\int_0^{t_0} g_i(s)\,\mathrm{d}s\quad (i=1,2),\ t\geq t_0, \tag{158}$$

denote

$$g_3=\min\left\{\int_0^{t_0} g_1(s)\,\mathrm{d}s,\int_0^{t_0} g_2(s)\,\mathrm{d}s\right\}. \tag{159}$$

By using (28), (132), (140), and (147), a series of computations, for $t\geq t_0$, we have

$$\begin{aligned}
G_1'(t)&=E'(t)+\varepsilon_1\phi'(t)+\varepsilon_2\psi'(t)\\
&\leq-\left\{\varepsilon_1\left[l-\epsilon\left(m_0-l+\frac{1}{2}\right)\right]\right.\\
&\qquad\left.-\varepsilon_2\eta\left[(m_0-l)(3m_0-2l)+3CC_7\right]\right\}\\
&\quad\times\left[\|\nabla u\|_2^2+\|\nabla v\|_2^2\right]\\
&\quad-\left[m_1\varepsilon_1-m_1\varepsilon_2\eta(m_0-l)\right]\\
&\quad\times\left(\|\nabla u\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\right)\\
&\quad-\left(1-\frac{\varepsilon_1}{2\epsilon}-\varepsilon_2\eta\right)\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)\\
&\quad+\left\{\frac{\varepsilon_1}{4\epsilon}+\varepsilon_2\left[\left(2\eta+\frac{2+C^2}{4\eta}\right)(m_0-l)+\frac{C_6}{4\eta}\right]\right\}\\
&\quad\times\left[(g_1\diamond\nabla u)(t)+(g_2\diamond\nabla v)(t)\right]\\
&\quad-\left(\varepsilon_2g_3-\varepsilon_2\eta-\varepsilon_1\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\
&\quad+2\varepsilon_1(p+2)\int_\Omega F(u,v)\,\mathrm{d}x\\
&\quad+\left(\frac{1}{2}-\frac{g_0(0)}{4\eta}\varepsilon_2C^2\right)\\
&\quad\times\left[\left(g_1'\diamond\nabla u\right)(t)+\left(g_2'\diamond\nabla v\right)(t)\right].
\end{aligned} \tag{160}$$

We choose ε_1, ε_2, and η small enough, such that

$$\varepsilon_2\eta(m_0-l)<\varepsilon_1<\varepsilon_2(g_3-\eta) \tag{161}$$

and $\epsilon=\varepsilon_1$; then we can check that

$$\begin{aligned}
C_8&=\varepsilon_2g_3-\varepsilon_2\eta-\varepsilon_1>0,\\
C_{10}&=m_1\varepsilon_1-m_1\varepsilon_2\eta(m_0-l)>0,\\
C_{11}&=\frac{1}{2}-\frac{g_0(0)}{4\eta}\varepsilon_2C^2>0,\\
C_{12}&=1-\frac{\varepsilon_1}{2\epsilon}-\varepsilon_2\eta>0.
\end{aligned} \tag{162}$$

We choose η, ε_1, and ε_2 so small that (162) remains valid and, further

$$\begin{aligned}
C_9&=\varepsilon_1\left[l-\epsilon\left(m_0-l+\frac{1}{2}\right)\right]\\
&\quad-\varepsilon_2\eta\left[(m_0-l)(3m_0-2l)+3CC_7\right]>0.
\end{aligned} \tag{163}$$

So, we arrive at

$$\begin{aligned}
G_1'(t)\leq&-C_8\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\
&-C_9\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+2\varepsilon_1(p+2)\int_\Omega F(u,v)\,\mathrm{d}x\\
&-C_{10}\left(\|\nabla u\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\right)\\
&+C_{11}\left[\left(g_1'\diamond\nabla u\right)(t)+\left(g_2'\diamond\nabla v\right)(t)\right]\\
&+\left\{\frac{\varepsilon_1}{4\epsilon}+\varepsilon_2\left[\left(2\eta+\frac{2+C^2}{4\eta}\right)(m_0-l)+\frac{C_6}{4\eta}\right]\right\}\\
&\times\left[(g_1\diamond\nabla u)(t)+(g_2\diamond\nabla v)(t)\right],
\end{aligned} \tag{164}$$

which yields (if needed, one can choose ε_2 sufficiently small)

$$G_1'(t)\leq-c_*E(t)+C\left[(g_1\diamond\nabla u)(t)+(g_2\diamond\nabla v)(t)\right],\quad\forall t\geq t_0, \tag{165}$$

where c_* is some positive constant. It follows from (165), (A3), and (28) that

$$\begin{aligned}
\xi(t)G_1'(t)&\leq-c_*\xi(t)E(t)\\
&\quad+C\xi(t)\left[(g_1\diamond\nabla u)(t)+(g_2\diamond\nabla v)(t)\right]\\
&\leq-c_*\xi(t)E(t)+C\xi_1(t)(g_1\diamond\nabla u)(t)\\
&\quad+C\xi_2(t)(g_2\diamond\nabla v)(t)\\
&\leq-c_*\xi(t)E(t)\\
&\quad-C\left[\left(g_1'\diamond\nabla u\right)(t)+\left(g_2'\diamond\nabla v\right)(t)\right]\\
&\leq-c_*\xi(t)E(t)-CE'(t),\quad\forall t\geq t_0.
\end{aligned} \tag{166}$$

That is

$$L_*'(t)\leq-c_*\xi(t)E(t)\leq-k\xi(t)L_*(t),\quad\forall t\geq t_0, \tag{167}$$

where $L_*(t) = \xi(t)G_1(t) + CE(t)$ is equivalent to $E(t)$ due to (135) and k is a positive constant. A simple integration of (167) leads to

$$L_*(t) \le L_*(t_0)\, e^{-k\int_{t_0}^{t}\xi(s)\mathrm{d}s}, \quad \forall t \ge t_0. \tag{168}$$

This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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