Research Article

Multilinear Singular Integral Operators on Generalized Weighted Morrey Spaces

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The purpose of this paper is to discuss the boundedness properties of multilinear Calderón-Zygmund operator and its commutator on the generalized weighted Morrey spaces.

1. Introduction and Results

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [1] and was later systematically studied by Grafakos and Torres in [2–4]. In 2009, the weighted estimates of multilinear Calderón-Zygmund singular integral operator and its commutator were established in [5] by Lerner et al. In 2013, the results of [5] were extended to the weighted Morrey space (see [6, 7]). In this paper, we will discuss the boundedness properties of multilinear Calderón-Zygmund operator and its commutator on the generalized weighted Morrey spaces.

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the m -fold product space ($m \in \mathbb{N}$). We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and denote by $\mathcal{S}'(\mathbb{R}^n)$ its dual space of the set of all tempered distributions on $\mathbb{R}^n$.

We say that a locally integrable function $K(y_0, y_1, \dots, y_m)$ defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ is a kernel in the class m -CZK(A, ϵ) if it satisfies the size estimate

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_l - y_k|\right)^{mn}} \quad (1)$$

for some $A > 0$ and all $(y_0, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $y_l \neq y_k$ for some l, k . Moreover, assume that for some $\epsilon > 0$ and all $0 \leq j \leq m$ it satisfies the smoothness estimates

$$\begin{aligned} & |K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m)| \\ & \leq \frac{A|y_j - y'_j|^\epsilon}{\left(\sum_{k,l=0}^m |y_l - y_k|\right)^{mn+\epsilon}} \end{aligned} \quad (2)$$

whenever

$$|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|. \quad (3)$$

Let T be a multilinear operator initially defined on the m -fold product of Schwartz spaces, and, by taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (4)$$

Following [2], we say that T is an m -linear Calderón-Zygmund operator if, for some $q_1, \dots, q_m \in [1, \infty)$ and $q \in (0, \infty)$ with $1/q = \sum_{i=1}^m 1/q_i$, it extends to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ and if there exists a kernel function $K(x, y_1, \dots, y_m)$ in

the class m -CZK(A, ϵ), defined away from the diagonal $x = y_1 = \dots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$\begin{aligned} T(\vec{f})(x) &= T(f_1, \dots, f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m \end{aligned} \quad (5)$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_j$ and $f_1, \dots, f_m$ are C^∞ functions with compact support.

Theorem 1 (see [5]). *Let $m \geq 2$ and let T be an m -linear Calderón-Zygmund operator. Suppose that $1/p = \sum_{i=1}^m 1/p_i$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfies the $A_{\vec{p}}$ condition, and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$.*

(1) *If $1 < p_i < \infty$, $i = 1, \dots, m$, then*

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \quad (6)$$

(2) *If $1 \leq p_i < \infty$, $i = 1, \dots, m$, and at least one of the $p_i = 1$, then*

$$\|T(\vec{f})\|_{WLP(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \quad (7)$$

The purpose of this paper is to discuss the boundedness properties of m -linear Calderón-Zygmund operator and its commutator on the generalized weighted Morrey spaces $M_{\varphi}^p(\omega)$.

The classical Morrey spaces $L^{p,\lambda}$ were originally introduced by Morrey in [8] to study the local behavior of solutions to the second-order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator, and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [9–11]. Moreover, various Morrey spaces are defined in the process of study. Mizuhara [12] introduced the generalized Morrey space $L^{p,\varphi}$; Komori and Shirai [13] defined the weighted Morrey spaces $L^{p,\kappa}(\omega)$; Guliyev [14] gave a concept of generalized weighted Morrey space $M_{\varphi}^p(\omega)$ which could be viewed as extension of both $L^{p,\varphi}$ and $L^{p,\kappa}(\omega)$. The boundedness of some operators on these Morrey spaces can be seen in [12–16].

Our first result can be formulated as follows.

Theorem 2. *Let $m \geq 2$, let T be an m -linear Calderón-Zygmund operator, and let $\vec{\varphi}_k = (\varphi_{k1}, \dots, \varphi_{km})$, $k = 1, 2$. Suppose that $1/p = \sum_{i=1}^m 1/p_i$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfies the $A_{\vec{p}}$ condition with $\omega_1, \dots, \omega_m \in A_{\infty}$, and $(\vec{\varphi}_1, \vec{\varphi}_2)$ satisfies the condition*

$$\begin{aligned} &\int_s^\infty \frac{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^m \omega_i(B(x, r))^{1/p_i}} \frac{dr}{r} \\ &\leq C \varphi_2(x, s). \end{aligned} \quad (8)$$

(1) *If $1 < p_i < \infty$, $i = 1, \dots, m$, then*

$$\|T(\vec{f})\|_{M_{\varphi_2}^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}. \quad (9)$$

(2) *If $1 \leq p_i < \infty$, $i = 1, \dots, m$, and at least one of the $p_i = 1$, then*

$$\|T(\vec{f})\|_{WM_{\varphi_2}^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}, \quad (10)$$

where $\varphi_2 = \prod_{i=1}^m \varphi_{2i}$ and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$.

Given a locally integrable vector function $\vec{b} = (b_1, \dots, b_m) \in (BMO)^m$, the commutator of $\vec{b}$ and the m -linear Calderón-Zygmund operator T , denoted here by $T_{\Sigma b}$, was introduced in [5] and is defined via

$$T_{\Sigma b}(\vec{f}) = \sum_{i=1}^m T_{b_i}^i(\vec{f}), \quad (11)$$

where

$$T_{b_i}^i(\vec{f}) = b_i T(\vec{f}) - T(f_1, \dots, b_i f_i, \dots, f_m). \quad (12)$$

And the iterated commutator $T_{\Pi b}$ was introduced in [17] and is defined by

$$T_{\Pi b}(\vec{f}) = [b_1, \dots, [b_{m-1}, [b_m, T]_{m-1} \cdots]_1](\vec{f}). \quad (13)$$

To clarify the notation, if T is associated in the usual way with a Calderón-Zygmund kernel K , then, at a formal level,

$$\begin{aligned} T_{\Sigma b}(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} \sum_{i=1}^m (b_i(x) - b_i(y_i)) K(x, y_1, \dots, y_m) \\ &\quad \times f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \\ T_{\Pi b}(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} \prod_{i=1}^m (b_i(x) - b_i(y_i)) K(x, y_1, \dots, y_m) \\ &\quad \times f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \end{aligned} \quad (14)$$

For T is an m -linear Calderón-Zygmund operator, $\vec{\omega} \in A_{\vec{p}}$, and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ with $1/p = 1/p_1 + \dots + 1/p_m$ and $\vec{b} \in (BMO)^m$, Lerner et al. [5] proved that $T_{\Sigma b}$ bounded from $L^{p_1}(\omega_1) \times \dots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ and Pérez et al. [17] extended the result to $T_{\Pi b}$. Our second result is to extend their results to the generalized weighted Morrey space.

Theorem 3. *Let $m \geq 2$ and let T be an m -linear Calderón-Zygmund operator. Suppose that $p_1, \dots, p_m \in (1, \infty)$, $p \in (0, \infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfies the $A_{\vec{p}}$ condition with $\omega_1, \dots, \omega_m \in A_{\infty}$, and $(\vec{\varphi}_1, \vec{\varphi}_2)$ satisfies the condition*

$$\begin{aligned} &\int_s^\infty \left(1 + \ln \frac{r}{s}\right)^m \frac{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^m \omega_i(B(x, r))^{1/p_i}} \frac{dr}{r} \\ &\leq C \varphi_2(x, s). \end{aligned} \quad (15)$$

If $b_i \in BMO$, $\vec{b} = (b_1, \dots, b_m)$, then there exist constants $C > 0$ independent of $\vec{f} = (f_1, \dots, f_m)$ such that

$$\begin{aligned} \|T_{\Sigma b}(\vec{f})\|_{M_{\varphi_2}^p(\nu_{\vec{\omega}})} &\leq C \prod_{i=1}^m \|b_i\|_* \cdot \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}, \\ \|T_{\Pi b}(\vec{f})\|_{M_{\varphi_2}^p(\nu_{\vec{\omega}})} &\leq C \prod_{i=1}^m \|b_i\|_* \cdot \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}, \end{aligned} \quad (16)$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ and $\varphi_2 = \prod_{i=1}^m \varphi_{2i}$.

Example 4. Let $\varphi_{1i}(x, t) = \varphi_{2i}(x, t) = \omega_i(B(x, t))^{(\kappa-1)/p_i}$ and $\omega_i \in A_{\infty}$ for $i = 1, \dots, m$, $0 < \kappa < 1$. Then, $(\vec{\varphi}_1, \vec{\varphi}_2)$ satisfies conditions (8) and (15).

In fact, from (34), in Section 2, we have a constant $\delta > 0$ such that

$$\omega_i(B(x, 2^k s)) \geq C 2^{\delta n k} \omega_i(B(x, s)) \quad (17)$$

for all i . Then,

$$\begin{aligned} &\int_s^\infty \frac{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^m \omega_i(B(x, r))^{1/p_i}} \frac{dr}{r} \\ &\leq \int_s^\infty \left(1 + \ln \frac{r}{s}\right)^m \\ &\quad \times \frac{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^m \omega_i(B(x, r))^{1/p_i}} \frac{dr}{r} \\ &= \sum_{k=0}^\infty \int_{2^k s}^{2^{k+1} s} \left(1 + \ln \frac{r}{s}\right)^m \prod_{i=1}^m \omega_i(B(x, r))^{(\kappa-1)/p_i} \frac{dr}{r} \\ &\leq C \sum_{k=0}^\infty k^m \prod_{i=1}^m \omega_i(B(x, 2^{k+1} s))^{(\kappa-1)/p_i} \\ &\leq C \sum_{k=0}^\infty k^m 2^{(k+1)n\delta((\kappa-1)/p)} \prod_{i=1}^m \omega_i(B(x, s))^{(\kappa-1)/p_i} \\ &\leq C \prod_{i=1}^m \omega_i(B(x, s))^{(\kappa-1)/p_i} = C \prod_{i=1}^m \varphi_{2i}(x, s). \end{aligned} \quad (18)$$

Hence, we get the following corollaries.

Corollary 5 (see [6]). Let $m \geq 2$ and let T be an m -linear Calderón-Zygmund operator. Suppose that $1/p = \sum_{i=1}^m 1/p_i$, $0 < \kappa < 1$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfies the $A_{\vec{p}}$ condition with $\omega_1, \dots, \omega_m \in A_{\infty}$, and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$.

(1) If $1 < p_i < \infty$, $i = 1, \dots, m$, then

$$\|T(\vec{f})\|_{L^{p, \kappa}(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i, \kappa}(\omega_i)}. \quad (19)$$

(2) If $1 \leq p_i < \infty$, $i = 1, \dots, m$, and at least one of the $p_i = 1$, then

$$\|T(\vec{f})\|_{WL^{p, \kappa}(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i, \kappa}(\omega_i)}. \quad (20)$$

Corollary 6. Let $m \geq 2$ and let T be an m -linear Calderón-Zygmund operator. Suppose that $1/p = \sum_{i=1}^m 1/p_i$, $0 < \kappa < 1$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfies the $A_{\vec{p}}$ condition with $\omega_1, \dots, \omega_m \in A_{\infty}$, and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. If $b_i \in BMO$, $\vec{b} = (b_1, \dots, b_m)$, then there exist constants $C > 0$ independent of $\vec{f} = (f_1, \dots, f_m)$ such that

$$\begin{aligned} \|T_{\Sigma b}(\vec{f})\|_{L^{p, \kappa}(\nu_{\vec{\omega}})} &\leq C \prod_{i=1}^m \|b_i\|_* \cdot \prod_{i=1}^m \|f_i\|_{L^{p_i, \kappa}(\omega_i)}, \\ \|T_{\Pi b}(\vec{f})\|_{L^{p, \kappa}(\nu_{\vec{\omega}})} &\leq C \prod_{i=1}^m \|b_i\|_* \cdot \prod_{i=1}^m \|f_i\|_{L^{p_i, \kappa}(\omega_i)}. \end{aligned} \quad (21)$$

2. Definitions and Preliminaries

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B . For any ball B and $\lambda > 0$, λB denotes the ball concentric with B whose radius is λ times as long. For a given weight function ω and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set the weighted measure $\omega(E) = \int_E \omega(x) dx$.

Let $1 \leq p < \infty$, let φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$, and let ω be a nonnegative measurable function on $\mathbb{R}^n$. Following [14], we denote by $M_{\varphi}^p(\omega)$ the generalized weighted Morrey space and the space of all functions $f \in L_{\text{loc}}^p(\omega)$ with finite norm

$$\|f\|_{M_{\varphi}^p(\omega)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x, r)} \left(\frac{1}{\omega(B(x, r))} \|f\|_{L^p(\omega, B(x, r))}^p \right)^{1/p}, \quad (22)$$

where

$$\|f\|_{L^p(\omega, B(x, r))} = \left(\int_{B(x, r)} |f(y)|^p \omega(y) dy \right)^{1/p}. \quad (23)$$

Furthermore, by $WM_{\varphi}^p(\omega)$ we denote the weak generalized weighted Morrey space of all functions $f \in WM_{\varphi}^p(\omega)$ for which

$$\|f\|_{WM_{\varphi}^p(\omega)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x, r)} \left(\frac{1}{\omega(B(x, r))} \|f\|_{WL^p(\omega, B(x, r))}^p \right)^{1/p}, \quad (24)$$

where

$$\|f\|_{WL^p(\omega, B(x, r))} = \sup_{t > 0} t \omega(\{y \in B(x, r) : |f(y)| > t\})^{1/p}. \quad (25)$$

(1) If $\omega = 1$ and $\varphi(x, r) = r^{(\lambda-n)/p}$ with $0 < \lambda < n$, then $M_{\varphi}^p(\omega) = L^{p, \lambda}$ is the classical Morrey space.

- (2) If $\varphi(x, r) = \omega(B(x, r))^{(\kappa-1)/p}$, then $M_\varphi^p(\omega) = L^{p,\kappa}(\omega)$ is the weighted Morrey space.
- (3) If $\varphi(x, r) = \nu(B(x, r))^{\kappa/p} \omega(B(x, r))^{-1/p}$, then $M_\varphi^p(\omega) = L^{p,\kappa}(\nu, \omega)$ is the two weighted Morrey space.
- (4) If $\omega = 1$, then $M_\varphi^p(\omega) = M_\varphi^p$ is the generalized Morrey space.
- (5) If $\varphi(x, r) = \omega(B(x, r))^{-1/p}$, then $M_\varphi^p(\omega) = L^p(\omega)$.

A weight ω is said to belong to A_p , for $1 < p < \infty$, if there exists a constant C such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-p'} dx \right)^{p-1} \leq C, \quad (26)$$

where s' is the dual of s such that $1/s + 1/s' = 1$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \cdot \operatorname{ess\,inf}_{x \in B} \omega(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \quad (27)$$

A weight ω is said to belong to A_∞ if there are positive numbers C and $0 < \delta < 1$ so that

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta \quad (28)$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p. \quad (29)$$

By (26), we have

$$\left(\int_B \omega(x) dx \right) \left(\int_B \omega(x)^{1-p'} dx \right)^{p-1} \leq C |B|^p \quad (30)$$

for $1 < p < \infty$. Note that

$$\left(\operatorname{ess\,inf}_{x \in E} f(x) \right)^{-1} = \operatorname{ess\,sup}_{x \in E} \frac{1}{f(x)} \quad (31)$$

is true for any real-valued nonnegative function f and measurable on E (see [18], p.143) and, by (27), we get

$$\begin{aligned} \|\omega^{-1}\|_{L^\infty(B)} &= \operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} = \frac{1}{\operatorname{ess\,inf}_{x \in B} \omega(x)} \\ &\leq C |B| \omega(B)^{-1}. \end{aligned} \quad (32)$$

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal function in [19].

Lemma 7 (see [19, 20]). *Suppose that $\omega \in A_p$ and the following statements hold.*

- (i) *For any $1 \leq p < \infty$, there are positive numbers C and $0 < \delta < 1$ such that*

$$\omega(2B) \leq C 2^{n\delta} \omega(B), \quad (33)$$

$$\omega(2B) \geq C 2^{n\delta} \omega(B); \quad (34)$$

- (ii) $A_{p_1} \subset A_{p_2}$ for any $1 \leq p_1 < p_2 \leq \infty$;
- (iii) *for any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}$.*

Now, let us recall the definition of multiple weights. For m exponents $p_1, \dots, p_m$, we write $\vec{p} = (p_1, \dots, p_m)$. Let $p_1, \dots, p_m \in [1, \infty)$, and let $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$. Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if it satisfies

$$\begin{aligned} \sup_B \left(\frac{1}{|B|} \int_B \nu_{\vec{\omega}}(x) dx \right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|B|} \int_B \omega_i(x)^{1-p'_i} dx \right)^{1/p'_i} \\ < \infty. \end{aligned} \quad (35)$$

When $p_i = 1$, $(1/|B|) \int_B \omega_i(x)^{1-p'_i} dx$ is understood as $(\inf_{x \in B} \omega_i(x))^{-1}$.

Lemma 8 (see [5]). *Let $p_1, \dots, p_m \in [1, \infty)$, and let $1/p = \sum_{k=1}^m 1/p_k$. Then, $\vec{\omega} = (\omega_1, \dots, \omega_m) \in A_{\vec{p}}$ if and only if*

$$\nu_{\vec{\omega}} \in A_{pm}, \quad \omega_i^{1-p'_i} \in A_{mp'_i} \quad \text{for } i = 1, \dots, m, \quad (36)$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ and the condition $\omega_i^{1-p'_i} \in A_{mp'_i}$ in the case $p_i = 1$ is understood as $\omega_i^{1/p_i} \in A_1$.

Lemma 9 (see [6]). *Let $p_1, \dots, p_m \in [1, \infty)$, and let $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$. Assume that $\omega_1, \dots, \omega_m \in A_\infty$ and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. Then, for any ball B , there exists a constant $C > 0$ such that*

$$\prod_{i=1}^m \left(\int_B \omega_i(x) dx \right)^{p/p_i} \leq C \int_B \nu_{\vec{\omega}}(x) dx. \quad (37)$$

Let us recall the definition and some properties of BMO. Following [21], a locally integrable function b is said to be in BMO if

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| dx = \|b\|_* < \infty, \quad (38)$$

where $b_B = |B|^{-1} \int_B b(y) dy$.

Lemma 10 (see [22]). *Suppose that $\omega \in A_\infty$ and $b \in \text{BMO}$. Then, for any $p \geq 1$, one has*

$$\left(\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x) dx \right)^{1/p} \leq C \|b\|_*. \quad (39)$$

Lemma 11 (see [23]). *Let $b \in \text{BMO}$, $1 \leq p < \infty$, and $r_1, r_2 > 0$. Then,*

$$\begin{aligned} \left(\frac{1}{|B(x_0, r_1)|} \int_{B(x_0, r_1)} |b(y) - b_{B(x_0, r_2)}|^p dy \right)^{1/p} \\ \leq C \|b\|_* \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right), \end{aligned} \quad (40)$$

where $C > 0$ is independent of f , x_0 , r_1 , and r_2 .

By Lemmas 10 and 11, it is easy to prove the following results.

Lemma 12. Suppose that $\omega \in A_\infty$ and $b \in BMO$. Then, for any $1 \leq p < \infty$ and $r_1, r_2 > 0$, one has

$$\left(\frac{1}{\omega(B(x_0, r_1))} \int_{B(x_0, r_1)} |b(x) - b_{B(x_0, r_2)}|^p \omega(x) dx \right)^{1/p} \leq C \|b\|_* \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right). \quad (41)$$

3. Proof of Theorem 2

We first prove the following conclusions.

Theorem 13. Let $m \geq 2$, let T be an m -linear Calderón-Zygmund operator, and let $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfy the $A_{\vec{p}}$ condition with $\omega_1, \dots, \omega_m \in A_\infty$. If $p_1, \dots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, then, for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \|T(\vec{f})\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} &\leq C \prod_{i=1}^m \omega_i(B(x_0, s))^{1/p_i} \\ &\times \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (42)$$

If $p_1, \dots, p_m \in [1, \infty)$, $\min\{p_1, \dots, p_m\} = 1$, and $p \in (0, \infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, then, for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \|T(\vec{f})\|_{WLP(\nu_{\vec{\omega}}, B(x_0, s))} &\leq C \prod_{i=1}^m \omega_i(B(x_0, s))^{1/p_i} \\ &\times \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (43)$$

Proof. We represent f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{B(x_0, 2s)}$, $i = 1, \dots, m$, and $\chi_{B(x_0, 2s)}$ denotes the characteristic function of $B(x_0, 2s)$. Then,

$$\begin{aligned} \prod_{i=1}^m f_i(y_i) &= \prod_{i=1}^m (f_i^0(y_i) + f_i^\infty(y_i)) \\ &= \sum_{\alpha_1, \dots, \alpha_m \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m) \\ &= \prod_{i=1}^m f_i^0(y_i) + \sum' f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m), \end{aligned} \quad (44)$$

where each term of $\sum'$ contains at least one $\alpha_i \neq 0$. Since T is an m -linear operator, then

$$\begin{aligned} \|T(\vec{f})\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} &\leq C \|T(f_1^0, \dots, f_m^0)\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} \\ &\quad + C \sum' \|T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} \\ &= I^{0, \dots, 0} + \sum' I^{\alpha_1, \dots, \alpha_m}, \\ \|T(\vec{f})\|_{WLP(\nu_{\vec{\omega}}, B(x_0, s))} &\leq C \|T(f_1^0, \dots, f_m^0)\|_{WLP(\nu_{\vec{\omega}}, B(x_0, s))} \\ &\quad + C \sum' \|T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})\|_{WLP(\nu_{\vec{\omega}}, B(x_0, s))} \\ &= J^{0, \dots, 0} + \sum' J^{\alpha_1, \dots, \alpha_m}. \end{aligned} \quad (45)$$

Then, by Theorem 1, if $1 < p_i < \infty$, $i = 1, \dots, m$, we get

$$I^{0, \dots, 0} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))}. \quad (46)$$

If $1 \leq p_i < \infty$, $\min\{p_1, \dots, p_m\} = 1$, then

$$J^{0, \dots, 0} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))}. \quad (47)$$

Applying Hölder's inequality, for $1 \leq p_i < \infty$, $i = 1, \dots, m$, we have

$$|B(x_0, 2s)| \leq \omega_i(B(x_0, 2s))^{1/p_i} \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, 2s))}. \quad (48)$$

Thus, for $1 \leq p_i < \infty$,

$$\begin{aligned} &\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))} \\ &\leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))} \cdot |B(x_0, 2s)|^m \int_{2s}^\infty \frac{dr}{r^{mn+1}} \\ &\leq C \prod_{i=1}^m \left(\omega_i(B(x_0, 2s))^{1/p_i} \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))} \right. \\ &\quad \times \left. \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, 2s))} \right) \int_{2s}^\infty \frac{dr}{r^{mn+1}} \\ &\leq C \prod_{i=1}^m \omega_i(B(x_0, 2s))^{1/p_i} \\ &\quad \cdot \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, r))} \right) \\ &\quad \times \frac{dr}{r^{mn+1}}. \end{aligned} \quad (49)$$

From (35) and Lemma 9, we get

$$\begin{aligned} & \prod_{i=1}^m \left\| \omega_i^{-1/p_i} \right\|_{L^{p'_i}(B(x_0, r))} \\ & \leq C |B(x_0, r)|^{1/p + \sum_{i=1}^m (1/p'_i)} \left(\int_{B(x_0, r)} v_{\tilde{\omega}}(x) dx \right)^{-1/p} \\ & \leq C |B(x_0, r)|^m \prod_{i=1}^m \omega_i(B(x_0, r))^{-1/p_i}. \end{aligned} \quad (50)$$

Then, for $1 \leq p_i < \infty, i = 1, \dots, m$,

$$\begin{aligned} & \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))} \\ & \leq C \prod_{i=1}^m \omega_i(B(x_0, s))^{1/p_i} \\ & \cdot \int_{2s}^{\infty} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (51)$$

This gives that $I^{0, \dots, 0}$ and $J^{0, \dots, 0}$ are majored by

$$\begin{aligned} & C \prod_{i=1}^m \omega_i(B(x_0, s))^{1/p_i} \\ & \cdot \int_{2s}^{\infty} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (52)$$

For the other term, let us first consider the case when $\alpha_1 = \dots = \alpha_m = \infty$. By the size condition (1), for any $x \in B(x_0, s)$, we obtain

$$\begin{aligned} & |T(f_1^{\infty}, \dots, f_m^{\infty})(x)| \\ & \leq C \int_{(\mathbb{R}^n \setminus B(x_0, 2s))^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(|x - y_1| + \dots + |x - y_m|)^{mn}} dy_1 \cdots dy_m \\ & \leq C \sum_{j=1}^{\infty} \int_{(B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s))^m} (|f_1(y_1) \cdots f_m(y_m)|) \\ & \quad \times ((|x - y_1| + \dots + |x - y_m|)^{mn})^{-1} dy_1 \cdots dy_m \\ & \leq C \sum_{j=1}^{\infty} \prod_{i=1}^m \int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i \\ & \leq C \sum_{j=1}^{\infty} \prod_{i=1}^m \left((2^{j+1}s)^{-n} \int_{B(x_0, 2^{j+1}s)} |f_i(y_i)| dy_i \right). \end{aligned} \quad (53)$$

Applying Hölder's inequality, it can be found that $\sup_{x \in B(x_0, s)} |T(f_1^{\infty}, \dots, f_m^{\infty})(x)|$ is less than

$$C \sum_{j=1}^{\infty} \prod_{i=1}^m \left((2^{j+1}s)^{-n} \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2^{j+1}s))} \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, 2^{j+1}s))} \right). \quad (54)$$

Hence,

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |T(f_1^{\infty}, \dots, f_m^{\infty})(x)| \\ & \leq C \sum_{j=1}^{\infty} \int_{2^{j+1}s}^{2^{j+2}s} (2^{j+2}s)^{-nm-1} \\ & \quad \times \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2^{j+1}s))} \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, 2^{j+1}s))} \right) dr \\ & \leq C \sum_{j=1}^{\infty} \int_{2^{j+1}s}^{2^{j+2}s} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2^{j+1}s))} \right. \\ & \quad \times \left. \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, 2^{j+1}s))} \right) \frac{dr}{r^{mn+1}} \\ & \leq C \int_{2s}^{\infty} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2^{j+1}s))} \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, 2^{j+1}s))} \right) \frac{dr}{r^{mn+1}}. \end{aligned} \quad (55)$$

Substituting (50) into above, we obtain

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |T(f_1^{\infty}, \dots, f_m^{\infty})(x)| \\ & \leq C \int_{2s}^{\infty} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (56)$$

Using Hölder's inequality,

$$\left(\int_{B(x_0, s)} v_{\tilde{\omega}}(x) dx \right)^{1/p} \leq C \prod_{i=1}^m \omega_i(B(x_0, s))^{1/p_i}. \quad (57)$$

From (56) and (57), we know that $I^{\infty, \dots, \infty}$ and $J^{\infty, \dots, \infty}$ are not greater than (52) for $1 \leq p_i < \infty, i = 1, \dots, m$.

Now, we consider the case where exactly τ of the α_i are ∞ for some $1 \leq \tau < m$. We only give the arguments for one of the cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. Applying the size condition again, we deduce that, for any $x \in B(x_0, s)$,

$$\begin{aligned} & |T(f_1^{\infty}, \dots, f_{\tau}^{\infty}, f_{\tau+1}^0, \dots, f_m^0)(x)| \\ & \leq C \int_{(\mathbb{R}^n \setminus B(x_0, 2s))^{\tau}} \int_{(B(x_0, 2s))^{m-\tau}} (|f_1(y_1) \cdots f_m(y_m)|) \\ & \quad \times ((|x - y_1| + \dots + |x - y_m|)^{mn})^{-1} dy_1 \cdots dy_m \end{aligned}$$

$$\begin{aligned}
&\leq C \prod_{i=\tau+1}^m \int_{B(x_0, 2s)} |f_i(y_i)| dy_i \\
&\quad \times \sum_{j=1}^{\infty} \frac{1}{|B(x_0, 2^{j+1}s)|^m} \\
&\quad \times \int_{(B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s))^\tau} |f_1(y_1) \cdots f_\tau(y_\tau)| dy_1 \cdots dy_\tau \\
&\leq C \prod_{i=\tau+1}^m \int_{B(x_0, 2s)} |f_i(y_i)| dy_i \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{1}{|B(x_0, 2^{j+1}s)|^m} \prod_{i=1}^{\tau} \int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} |f_i(y_i)| dy_i \\
&\leq C \sum_{j=1}^{\infty} \prod_{i=1}^m \frac{1}{|B(x_0, 2^{j+1}s)|} \int_{B(x_0, 2^{j+1}s)} |f_i(y_i)| dy_i.
\end{aligned} \tag{58}$$

Similar to the estimates for $I^{\infty, \dots, \infty}$, we get

$$\begin{aligned}
&\sup_{x \in B(x_0, s)} |T(f_1^\infty, \dots, f_\tau^\infty, f_{\tau+1}^0, \dots, f_m^0)(x)| \\
&\leq C \int_{2s}^{\infty} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{59}$$

Then, $I^{\infty, \dots, \infty, 0, \dots, 0}$ and $J^{\infty, \dots, \infty, 0, \dots, 0}$ are all less than

$$\begin{aligned}
&C \prod_{i=1}^m \omega_i(B(x_0, s))^{1/p_i} \\
&\quad \cdot \int_{2s}^{\infty} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{60}$$

Combining the above estimates, we complete the proof of Theorem 13. $\square$

Now, we can give the proof of Theorem 2. From the definition of generalized Morrey space, the norm of $T(\vec{f})$ on $M_{\varphi_2}^p(\nu_{\vec{\omega}})$ equals

$$\begin{aligned}
&\sup_{x \in R^n, r > 0} \left(\prod_{i=1}^m \varphi_{2i}(x, s) \right)^{-1} \left(\frac{1}{\nu_{\vec{\omega}}(B(x, s))} \right. \\
&\quad \times \left. \int_{B(x, s)} |T(\vec{f})(y)|^p \nu_{\vec{\omega}}(y) dy \right)^{1/p}.
\end{aligned} \tag{61}$$

By (37), we get

$$\left(\int_{B(x, s)} \nu_{\vec{\omega}}(x) dx \right)^{-1/p} \leq C \prod_{i=1}^m \left(\int_{B(x, s)} \omega_i(x) dx \right)^{-1/p_i}. \tag{62}$$

Combining (42) and (62),

$$\begin{aligned}
&\left(\frac{1}{\nu_{\vec{\omega}}(B(x, s))} \int_{B(x, s)} |T(\vec{f})(y)|^p \nu_{\vec{\omega}}(y) dy \right)^{1/p} \\
&\leq C \int_s^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))} \omega_i(B(x, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{63}$$

Since $f_i \in M_{\varphi_{1i}}^{p_i}(\omega_i)$, then, by (31) and the fact that $\|f_i\|_{L^{p_i}(B(x, r))}$ are all nondecreasing function of r , we get

$$\begin{aligned}
&\frac{\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))}}{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}} \\
&\leq \text{ess sup}_{0 < r < t < \infty} \frac{\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))}}{\prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}} \\
&\leq \text{ess sup}_{0 < t < \infty} \frac{\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, t))}}{\prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}} \\
&\leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}.
\end{aligned} \tag{64}$$

Then,

$$\begin{aligned}
&\int_s^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))} \omega_i(B(x, r))^{-1/p_i} \right) \frac{dr}{r} \\
&= \int_s^\infty \frac{\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))}}{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}} \\
&\quad \times \frac{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^m \omega_i(B(x, t))^{1/p_i}} \frac{dr}{r} \\
&\leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)} \\
&\quad \times \int_s^\infty \frac{\text{ess inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^m \omega_i(B(x, t))^{1/p_i}} \frac{dr}{r}.
\end{aligned} \tag{65}$$

By condition (8), we get

$$\begin{aligned}
&\int_s^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))} \omega_i(B(x, r))^{1/p_i} \right) \frac{dr}{r} \\
&\leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)} \prod_{i=1}^m \varphi_{2i}(x, s).
\end{aligned} \tag{66}$$

Combining (61), (63), and (66), the proof of the first part of Theorem 2 is completed.

Similarly, the norm of $T(\vec{f})$ on $WM_{\varphi_2}^p(\nu_{\vec{\omega}})$ equals

$$\begin{aligned}
&\sup_{x \in R^n, r > 0} \left(\prod_{i=1}^m \varphi_{2i}(x, s) \right)^{-1} \\
&\quad \times \left(\frac{1}{\nu_{\vec{\omega}}(B(x, s))} \|T(\vec{f})\|_{WLP(\nu_{\vec{\omega}}, B(x, s))}^p \right)^{1/p}.
\end{aligned} \tag{67}$$

Combining (43) and (62),

$$\left(\frac{1}{\nu_{\vec{\omega}}(B(x, s))} \|T(\vec{f})\|_{WL^p(\nu_{\vec{\omega}}, B(x, s))}^p \right)^{1/p} \leq C \int_s^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))} \omega_i(B(x, r))^{-1/p_i} \right) \frac{dr}{r}. \quad (68)$$

Substituting (66) into (68),

$$\left(\frac{1}{\nu_{\vec{\omega}}(B(x, s))} \|T(\vec{f})\|_{WL^p(\nu_{\vec{\omega}}, B(x, s))}^p \right)^{1/p} \leq C \varphi_2(x, s) \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}. \quad (69)$$

Then,

$$\|T(\vec{f})\|_{WM_{\varphi_2}^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}. \quad (70)$$

This completes the proof of the second part of Theorem 2.

4. Proof of Theorem 3

Theorem 14. Let $m \geq 2$, let T be an m -linear Calderón-Zygmund operator, and let $\vec{\omega} = (\omega_1, \dots, \omega_m)$ satisfy the $A_{\vec{p}}$ condition with $\omega_1, \dots, \omega_m \in A_\infty$. If $p_1, \dots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, then, for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} & \|T_{\Sigma b}(\vec{f})\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} \\ & \leq C \prod_{i=1}^m \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\ & \quad \times \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right)^m \\ & \quad \times \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}, \end{aligned} \quad (71)$$

$$\begin{aligned} & \|T_{\Pi b}(\vec{f})\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} \\ & \leq C \prod_{i=1}^m \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\ & \quad \times \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right)^m \\ & \quad \times \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (72)$$

Proof. We will give the proof for $T_{\Pi b}$ because the proof for $T_{\Sigma b}$ is very similar but easier. Moreover, for simplicity of the expansion, we only present the case $m = 2$.

We represent f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{B(x_0, 2s)}$, $i = 1, 2$, and $\chi_{B(x_0, 2s)}$ denotes the characteristic function of $B(x_0, 2s)$. Then,

$$\begin{aligned} & \|T_{\Pi b}(\vec{f})\|_{L^p(\nu_{\vec{\omega}}, B(x_0, s))} \\ & \leq C \left(\int_{B(x_0, s)} |T_{\Pi b}(f_1^0, f_2^0)(x)|^p \nu_{\vec{\omega}}(x) dx \right)^{1/p} \\ & \quad + C \left(\int_{B(x_0, s)} |T_{\Pi b}(f_1^0, f_2^\infty)(x)|^p \nu_{\vec{\omega}}(x) dx \right)^{1/p} \\ & \quad + C \left(\int_{B(x_0, s)} |T_{\Pi b}(f_1^\infty, f_2^0)(x)|^p \nu_{\vec{\omega}}(x) dx \right)^{1/p} \\ & \quad + C \left(\int_{B(x_0, s)} |T_{\Pi b}(f_1^\infty, f_2^\infty)(x)|^p \nu_{\vec{\omega}}(x) dx \right)^{1/p} \\ & = I + II + III + IV. \end{aligned} \quad (73)$$

Since $T_{\Pi b}$ bounded from $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2)$ to $L^p(\nu_{\vec{\omega}})$, we get

$$\begin{aligned} & \left(\int_{B(x_0, s)} |T_{\Pi b}(f_1^0, f_2^0)(x)|^p \nu_{\vec{\omega}}(x) dx \right)^{1/p} \\ & \leq C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{L^{p_i}(\omega_i, B(x_0, 2s))}. \end{aligned} \quad (74)$$

Then, by (51), we get

$$\begin{aligned} I & \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\ & \quad \cdot \int_{2s}^\infty \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (75)$$

Owing to the symmetry of II and III , we only estimate II . Taking $\lambda_i = (b_i)_{B(x_0, s)}$, then

$$\begin{aligned} & T_{\Pi b}(f_1^0, f_2^\infty)(x) \\ & = (b_1(x) - \lambda_1)(b_2(x) - \lambda_2) T(f_1^0, f_2^\infty)(x) \\ & \quad - (b_1(x) - \lambda_1) T(f_1^0, (b_2 - \lambda_2) f_2^\infty)(x) \\ & \quad - (b_2(x) - \lambda_2) T((b_1 - \lambda_1) f_1^0, f_2^\infty)(x) \\ & \quad + T((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty)(x) \\ & = II_1 + II_2 + II_3 + II_4. \end{aligned} \quad (76)$$

Similar to the estimate of (59), for any $x \in B(x_0, s)$, we can deduce

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |T(f_1^0, f_2^\infty)(x)| \\ & \leq C \int_{2s}^\infty \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (77)$$

By Lemma 8, we know that $\nu_{\tilde{\omega}} \in A_{\infty}$. Applying Hölder's inequality and Lemma 10, we have

$$\begin{aligned} & \left(\int_{B(x_0, s)} |(b_1(x) - \lambda_1)(b_2(x) - \lambda_2)|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\ & \leq C \prod_{i=1}^2 \left(\int_{B(x_0, s)} |b_i(x) - \lambda_i|^{2p} \nu_{\tilde{\omega}}(x) dx \right)^{1/2p} \\ & \leq C \prod_{i=1}^2 \|b_i\|_* \cdot (\nu_{\tilde{\omega}}(B(x_0, s)))^{1/p}. \end{aligned} \quad (78)$$

Then, by (77), (78), and (57), we have

$$\begin{aligned} & \left(\int_{B(x_0, s)} |II_1|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\ & \leq \left(\int_{B(x_0, s)} |(b_1(x) - \lambda_1)(b_2(x) - \lambda_2)|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\ & \quad \times \sup_{x \in B(x_0, s)} |T(f_1^0, f_2^{\infty})(x)| \\ & \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\ & \quad \cdot \int_{2s}^{\infty} \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (79)$$

Applying the size condition, we deduce that, for any $x \in B(x_0, s)$,

$$\begin{aligned} & |T(f_1^0, (b_2 - \lambda_2)f_2^{\infty})(x)| \\ & \leq C \int_{B(x_0, 2s)} \int_{\mathbb{R}^n \setminus B(x_0, 2s)} (|f_1(y_1)(b_2(y_2) - \lambda_2)f_2(y_2)|) \\ & \quad \times ((|x - y_1| + |x - y_2|)^{2n})^{-1} dy_1 dy_2 \\ & \leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-2n} \int_{B(x_0, 2^{j+1}s)} |f_1(y_1)| dy_1 \\ & \quad \times \int_{B(x_0, 2^{j+1}s)} |(b_2(y_2) - \lambda_2)f_2(y_2)| dy_2. \end{aligned} \quad (80)$$

Note that

$$\begin{aligned} & \int_{B(x_0, 2^{j+1}s)} |f_1(y_1)| dy_1 \\ & \leq C \|f_1\|_{L^{p_1}(\omega_1, B(x_0, 2^{j+1}s})} \|\omega_1^{-1/p_1}\|_{L^{p'_1}(B(x_0, 2^{j+1}s})}, \\ & \int_{B(x_0, 2^{j+1}s)} |(b_2(y_2) - \lambda_2)f_2(y_2)| dy_2 \\ & \leq C \|f_2\|_{L^{p_2}(\omega_2, B(x_0, 2^{k+1}s})} \|b_2(\cdot) - \lambda_2\|_{L^{p'_2}(\omega_2^{1-p'_2}, B(x_0, 2^{k+1}s})}. \end{aligned} \quad (81)$$

Then,

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |T(f_1^0, (b_2 - \lambda_2)f_2^{\infty})(x)| \\ & \leq C \int_{2s}^{\infty} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \\ & \quad \times \|\omega_1^{-1/p_1}\|_{L^{p'_1}(B(x_0, r))} \|b_2(\cdot) - \lambda_2\|_{L^{p'_2}(\omega_2^{1-p'_2}, B(x_0, r))} \frac{dr}{r^{2n+1}}. \end{aligned} \quad (82)$$

Since $\omega_2^{1-p'_2} \in A_{2p'_2}$, then, from Lemma 12, we get

$$\begin{aligned} & \|b_2(\cdot) - \lambda_2\|_{L^{p'_2}(\omega_2^{1-p'_2}, B(x_0, r))} \\ & \leq C \left(\int_{B(x_0, r)} |b_2(z) - \lambda_2|^{p'_2} \omega_2^{1-p'_2}(z) dz \right)^{1/p'_2} \\ & \leq C \left(1 + \left| \ln \frac{r}{s} \right| \right) \|b_2\|_* \left(\omega_2^{1-p'_2}(B(x_0, r)) \right)^{1/p'_2} \\ & = C \left(1 + \left| \ln \frac{r}{s} \right| \right) \|b_2\|_* \|\omega_2^{-1/p_2}\|_{L^{p'_2}(B(x_0, r))}. \end{aligned} \quad (83)$$

By (50), we have

$$\begin{aligned} & \prod_{i=1}^2 \|\omega_i^{-1/p_i}\|_{L^{p'_i}(B(x_0, r))} \\ & \leq C |B(x_0, r)|^2 \prod_{i=1}^2 \omega_i(B(x_0, r))^{-1/p_i}. \end{aligned} \quad (84)$$

From (82), (83), and (84), we can deduce

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |T(f_1^0, (b_2 - \lambda_2)f_2^{\infty})(x)| \\ & \leq C \|b_2\|_* \int_{2s}^{\infty} \left(1 + \ln \frac{r}{s} \right) \\ & \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (85)$$

Applying Lemma 10 and (57), we have

$$\begin{aligned} & \left(\int_{B(x_0, s)} |b_1(x) - \lambda_1|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\ & \leq C \|b_1\|_* (\nu_{\tilde{\omega}}(B(x_0, s)))^{1/p} \\ & \leq C \|b_1\|_* \prod_{i=1}^2 \omega_i(B(x_0, r))^{1/p_i}. \end{aligned} \quad (86)$$

Then, by (85),

$$\begin{aligned}
& \left(\int_{B(x_0, s)} |II_2|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\
& \leq \left(\int_{B(x_0, s)} |b_1(x) - \lambda_1|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\
& \quad \times \sup_{x \in B(x_0, s)} |T(f_1^0, (b_2 - \lambda_2) f_2^\infty)(x)| \\
& \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\
& \quad \times \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right) \\
& \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{87}$$

Similarly, we also have that

$$\begin{aligned}
& \left(\int_{B(x_0, s)} |II_3|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\
& \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\
& \quad \times \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right) \\
& \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{88}$$

Applying the size condition again, for any $x \in B(x_0, s)$, with the same method of estimate for (85), we have

$$\begin{aligned}
& |T((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty)(x)| \\
& \leq C \sum_{j=1}^\infty (2^{j+1}s)^{-2n} \prod_{i=1}^2 \int_{B(x_0, 2^{j+1}s)} |(b_i(y_i) - \lambda_i) f_i(y_i)| dy_i \\
& \leq C \int_{2s}^\infty \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \right. \\
& \quad \cdot \left. \|b_i(\cdot) - \lambda_i\|_{L^{p'_i}(\omega_i^{1-p'_i}, B(x_0, r))} \right) \frac{dr}{r^{2n+1}} \\
& \leq C \prod_{i=1}^2 \|b_i\|_* \\
& \quad \cdot \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right)^2 \\
& \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{89}$$

Then,

$$\begin{aligned}
& \left(\int_{B(x_0, s)} |II_4|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\
& \leq C (\nu_{\tilde{\omega}}(B(x_0, s)))^{1/p} \\
& \quad \times \sup_{x \in B(x_0, s)} |((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty)(x)| \\
& \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\
& \quad \cdot \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right)^2 \\
& \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{90}$$

Then, combining (79), (87), (88), and (90), we get

$$\begin{aligned}
& \left(\int_{B(x_0, s)} |II|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\
& \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\
& \quad \times \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right)^2 \\
& \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{91}$$

Finally, we still decompose $T_{\Pi b}(f_1^\infty, f_2^\infty)(x)$ as follows:

$$\begin{aligned}
& T_{\Pi b}(f_1^\infty, f_2^\infty)(x) \\
& = (b_1(x) - \lambda_1)(b_2(x) - \lambda_2) T(f_1^\infty, f_2^\infty)(x) \\
& \quad - (b_1(x) - \lambda_1) T(f_1^\infty, (b_2 - \lambda_2) f_2^\infty)(x) \\
& \quad - (b_2(x) - \lambda_2) T((b_1 - \lambda_1) f_1^\infty, f_2^\infty)(x) \\
& \quad + T((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty)(x) \\
& = IV_1 + IV_2 + IV_3 + IV_4.
\end{aligned} \tag{92}$$

Because each term IV_j is completely analogous to II_j , $j = 1, 2, 3, 4$ with a bit difference, we get the following estimate without details:

$$\begin{aligned}
& \left(\int_{B(x_0, s)} |IV|^p \nu_{\tilde{\omega}}(x) dx \right)^{1/p} \\
& \leq C \prod_{i=1}^2 \left(\|b_i\|_* \omega_i(B(x_0, s))^{1/p_i} \right) \\
& \quad \times \int_{2s}^\infty \left(1 + \ln \frac{r}{s} \right)^2 \\
& \quad \times \left(\prod_{i=1}^2 \|f_i\|_{L^{p_i}(\omega_i, B(x_0, r))} \omega_i(B(x_0, r))^{-1/p_i} \right) \frac{dr}{r}.
\end{aligned} \tag{93}$$

Summing up the above estimates, (72) is proved. $\square$

We give the proof of Theorem 3 in the following. From (62) and (72),

$$\begin{aligned} & \left(\frac{1}{v_{\vec{\omega}}(B(x, s))} \int_{B(x, s)} |T_{\Pi b}(\vec{f})(y)|^p v_{\vec{\omega}}(y) dy \right)^{1/p} \\ & \leq C \prod_{i=1}^m \|b_i\|_* \\ & \quad \cdot \int_s^\infty \left(1 + \ln \frac{r}{s} \right)^m \\ & \quad \times \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))} \omega_i(B(x, r))^{-1/p_i} \right) \frac{dr}{r}. \end{aligned} \quad (94)$$

Since $(\vec{\varphi}_1, \vec{\varphi}_2)$ satisfies condition (15), $\|f_i\|_{L^{p_i}(B(x, r))}$ are all nondecreasing function of r , and $f_i \in M_{\varphi_{1i}}^{p_i}(\omega_i)$, then

$$\begin{aligned} & \int_s^\infty \left(1 + \ln \frac{r}{s} \right)^m \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))} \omega_i(B(x, r))^{-1/p_i} \right) \frac{dr}{r} \\ & = \int_s^\infty \frac{\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i, B(x, r))}}{\operatorname{ess\,inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}} \\ & \quad \times \left(1 + \ln \frac{r}{s} \right)^m \\ & \quad \times \frac{\operatorname{ess\,inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^n \omega_i(B(x, t))^{1/p_i}} \frac{dr}{r} \\ & \leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)} \\ & \quad \times \int_s^\infty \left(1 + \ln \frac{r}{s} \right)^m \\ & \quad \times \frac{\operatorname{ess\,inf}_{r < t < \infty} \prod_{i=1}^m \varphi_{1i}(x, t) \omega_i(B(x, t))^{1/p_i}}{\prod_{i=1}^n \omega_i(B(x, t))^{1/p_i}} \frac{dr}{r} \\ & \leq C \varphi_2(x, s) \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}. \end{aligned} \quad (95)$$

Combining (94) and (95), we have proved that

$$\|T_{\Pi b}(\vec{f})\|_{M_{\varphi_2}^p(v_{\vec{\omega}})} \leq C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i)}. \quad (96)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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