Molecular Characterization of Hardy Spaces Associated with Twisted Convolution

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1. Introduction

In this paper, we consider the $2n$ linear differential operators

$$
Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4} z_j, \quad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{4} \overline{z}_j,
$$

on $\mathbb{C}^n$, $j = 1, 2, \ldots, n$. (1)

Together with the identity they generate a Lie algebra $\mathfrak{h}^n$ which is isomorphic to the $2n + 1$ dimensional Heisenberg algebra. The only nontrivial commutation relations are

$$
[Z_j, \overline{Z}_j] = -\frac{1}{2} I, \quad j = 1, 2, \ldots, n. \quad (2)
$$

The operator $L$ defined by

$$
L = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \overline{Z}_j + \overline{Z}_j Z_j)
$$

is nonnegative, self-adjoint, and elliptic. Therefore, it generates a diffusion semigroup $\{T^L_t\}_{t \geq 0} = \{e^{-tL}\}_{t \geq 0}$. The operators in (1) generate a family of “twisted translations” $\tau_w$ on $\mathbb{C}^n$ defined on measurable functions by

$$
(\tau_w f) (z) = \exp \left( \frac{1}{2} \sum_{j=1}^{n} (w_j z_j + \overline{w}_j \overline{z}_j) \right) f (z)
= f (z + w) \exp \left( \frac{i}{2} \text{Im} (z \cdot \overline{w}) \right).
$$

We give a molecular characterization of the Hardy space associated with twisted convolution. As an application, we prove the boundedness of the local Riesz transform on the Hardy space.
We first give some basic notations about $H^p_L(C^n)$. Let $\mathcal{B}$ denote the class of $C^\infty$ functions $\varphi$ on $C^n$, supported on the ball $B(0,1)$ such that $\|\varphi\|_{C^n} \leq 1$ and $|\nabla \varphi|_{C^n} \leq 2$. For $t > 0$, let $\varphi_t(z) = t^{-n} \varphi(z/t)$. Given $\sigma > 0$, $0 < \sigma \leq +\infty$, and a tempered distribution $f$, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi \in \mathcal{B}, \varphi \geq 0} \sup_{0 < t < \sigma} |\varphi_t \times f(z)|.$$  

(6)

Then, the Hardy space $H^p_L(C^n)$ can be defined by

$$H^p_L(C^n) = \{ f \in \mathcal{L}'(C^n) : M_{\infty} f \in L^p(C^n) \}.$$  

(7)

For any $f \in H^p_L(C^n)$, define $\|f\|_{H^p_L(C^n)} = \|M_{\infty} f\|_{L^p}$.  

**Definition 1.** Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$. A function $a(z)$ is a $H^p_L$-atom for the Hardy space $H^p_L(C^n)$ associated to a ball $B(z_0, r)$ if

(i) $\sup \sigma a \subset B(z_0, r)$;

(ii) $\|a\|_q \leq |B(z_0, r)|^{1/q-1/p}$;

(iii) $\int_{C^n} a(w)\mathcal{W}(z_0, w)dw = 0$.

We define the atomic Hardy space $H^p_{L, a}(C^n)$ to be the set of all tempered distributions of the form $\sum_j \lambda_j \mathcal{A}_j$ (the sum converges in the topology of $\mathcal{L}'(C^n)$), where $\mathcal{A}_j$ are $H^p_{L, a}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.

The atomic quasinorm in $H^p_{L, a}(C^n)$ is defined by

$$\|f\|_{H^p_{L, a}} = \inf \left\{ \left( \sum_j |\lambda_j|^p \right)^{1/p} \right\},$$  

(8)

where the infimum is taken over all decompositions $f = \sum_j \lambda_j \mathcal{A}_j$ and $\mathcal{A}_j$ are $H^p_{L, a}$-atoms.

The following result has been proved in [4, 7].

**Proposition 2.** Let $2n/(2n+1) < p \leq 1$. Then, for a tempered distribution $f$ on $C^n$, the following are equivalent:

(i) $M_{\infty} f \in L^p(C^n)$;

(ii) for some $\sigma$, $0 < \sigma < +\infty$, $M_{\sigma} f \in L^p(C^n)$;

(iii) for some radial function $\varphi \in \mathcal{L}'$, such that $\int_{C^n} \varphi(z)dz \neq 0$, we have

$$\sup_{0 < t < \sigma} |\varphi_t \times f(z)| \in L^p(C^n);$$  

(9)

(iv) $f$ can be decomposed as $f = \sum \lambda_j \mathcal{A}_j$, where $\mathcal{A}_j$ are $H^p_{L, a}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.

**Corollary 3.** Let $2n/(2n+1) < p \leq 1$ and $1 < q \leq \infty$. Then, $H^p_{L, a}(C^n) = H^p_L(C^n)$ with equivalent norms.

In order to give the main result of this paper, we need the dual space of Hardy space $H^p_L(C^n)$.

**Definition 4.** Let $0 \leq \alpha < 1/2n$; a locally integrable function $f$ is said to be in the Campanato type space $\Lambda^\alpha_{L, a}$ if there exists a constant $K > 0$ such that, for every ball $B = B(z, r)$,

$$|B|^{-\alpha} \left( \int_B \left| \frac{1}{|B|} \int_B f(u) \overline{\varphi}(z, u)du \right| \omega(z, v) \right)^{1/2} \leq K.$$  

(10)

The norm $\|f\|_{\Lambda^\alpha_{L, a}}$ of $f$ is the least value of $K$ for which the above inequality holds.

The dual space of $H^p_L(C^n)$ is the BMO type space $\text{BMO}_L(C^n)$ (cf. [4]). Note that $\Lambda^\alpha_{L, a}$ is identified with $\text{BMO}_L$. Let $\mathcal{H}^{p,2,a}_L$ denote the space of finite linear combinations of $H^p_{L, 2, a}$-atoms, which coincides with $L^2_L(C^n)$, the space of square integrable functions with compact support. By Proposition 2, $\mathcal{H}^{p,2,a}_L$ is a dense subspace of $H^p_L(C^n)$. Set

$$\mathcal{L}_g(f) = \int_{C^n} f(z) \overline{g}(z)dz, \quad f \in \mathcal{H}^{p,2,a}_L, \quad g \in L^1_{\text{loc}}(C^n).$$  

(11)

Similar to the classical case in [10], we immediately obtain the following theorem which proves that $\Lambda^\alpha_{L, p-1}$ is the dual space of $H^p_L(C^n)$ for $2n/(2n+1) < p < 1$.

**Theorem 5.** Let $2n/(2n+1) < p < 1$. Then

(a) suppose $g \in \Lambda^\alpha_{L, (p-1)^{-1}}$; then $\mathcal{L}_g$ given by (11) extends to a bounded linear functional on $H^p_L(C^n)$ and satisfies

$$\|\mathcal{L}_g\| \leq C \|g\|_{\Lambda^\alpha_{L, (p-1)^{-1}}};$$  

(12)

(b) conversely, every bounded linear functional $\mathcal{L}$ on $H^p_L(C^n)$ can be realized as $\mathcal{L} = \mathcal{L}_g$ with $g \in \Lambda^\alpha_{L, (1/p)^{-1}}$ and

$$\|g\|_{\Lambda^\alpha_{L, (1/p)^{-1}}} \leq C \|\mathcal{L}\|.$$  

(13)

**Remark 6.** We may define the space $\Lambda^\alpha_{L, (p-1)^{-1}, q'}$, $2n/(2n+1) < p < 1$, $1 \leq q' \leq \infty$, by

$$|B|^{-1/p} \left( \int_B \left| \frac{1}{|B|} \int_B f(u) \overline{\varphi}(z, u)du \right| \omega(z, v) \right)^{1/q'} \leq K,$$  

(14)

where $B = B(z, r)$. The norm $\|f\|_{\Lambda^\alpha_{L, (p-1)^{-1}, q'}}$ of $f$ is the least value of $K$ for which the above inequality holds. Due to Theorem 5, $\Lambda^\alpha_{L, (p-1)^{-1}, q'}$ is also identified with the dual space of $H^p_L(C^n)$. The proof is almost the same as that of Theorem 5. Thus, the space $\Lambda^\alpha_{L, (p-1)^{-1}, q'}$ coincides with $\Lambda^\alpha_{L, (p-1)^{-1}}$ and $\|f\|_{\Lambda^\alpha_{L, (p-1)^{-1}, q'}} \sim \|f\|_{\Lambda^\alpha_{L, (p-1)^{-1}}}$.  

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Definition 7. Let \(2n/(2n+1) < p \leq 1 \leq q \leq \infty, p \neq q, \) and \(\epsilon > (1/p) - 1.\) Set \(a = 1 - (1/p) + \epsilon, b = 1 - (1/p) + \epsilon.\) A function \(M \in L^q\) is called a \(H^{p,q}_{L}\)-molecule with the center \(z_0\) if

\[
\begin{align*}
(1) & \ |z|^{-2b} M(z) \in L^q, \\
(2) & \mathcal{N}(M) = \|M\|_{L^q} \|\cdot - z_0\|^{-2b} M\|_{L^q}^{1-a/b} < \infty, \\
(3) & \int_{C^n} M(z)\overline{w}(z_0, z) \, dz = 0.
\end{align*}
\]

Then, we can obtain a molecular characterization of \(H_{L}^{p}(C^n)\) as follows.

Theorem 8. Given \(p, q, \epsilon\) as in Definition 7, then \(f \in H_{L}^{p}(C^n)\) if and only if \(f\) can be written as \(f = \sum f_j M_j\), where \(M_j\) are \(H^{p,q}_{L}\)-molecules and \(\sum |\lambda_j|^p < \infty\). The sum converges in \(H_{L}^{p}\) norm and also in \((L^q_{(1/p)-1})^*\) when \(2n/(2n+1) < p < 1\). Moreover,

\[
\|f\|_{H_{L}^{p}} \sim \|f\|_{H_{L}^{p,q,M}} = \inf \left\{ \left( \sum |\lambda_j|^p \right)^{1/p} \right\},
\]

where the infimum is taken over all decompositions of \(f\) into \(H^{p,q}_{L}\)-molecules.

Lemma 11. If \(a\) is a \(H^{p,q}_{L}\)-atom for \(2n/(2n+1) < p \leq 1 \leq q \leq +\infty\) supported in \(B(z_0, r)\), then \(a\) is a \(H^{p,q}_{L}\)-molecule centered at \(z_0\) and

\[
\mathcal{N}(a) \leq C,
\]

where \(\epsilon > 0\) and \(C\) is a positive constant that is independent of \(a\).

Proof. Since

\[
\|a\|_q \leq |B|^{1-a/b} = |B|^{a-b},
\]

we get

\[
\left\| \cdot - z_0 \right\|^{-2b} |a(\cdot)|_q \leq \sigma^{2b} \|a\|_q \leq C |B|^{a-b} - B|B|^{a}.
\]

Therefore,

\[
\mathcal{N}(a) \leq C |B|^{a-b} |B|^{(1-a/b)} = C.
\]

This proves that \(a\) is a molecule with center at \(z_0\).

The following lemma is the key step for the proof of Theorem 8.

Lemma 12. If \(M\) is a \(H^{p,q}_{L}\)-molecule with center at \(z_0\), then \(M \in H^{p,q}_{L}(C^n)\) and

\[
\|M\|_{H_{L}^{p,q}} \leq C \mathcal{N}(M),
\]

where \(C\) is independent of \(M\).

Proof. When \(p = 1\), Theorem 9 is proved by the connection between \(H_{L}^{p}(C^n)\) and Hardy space on the Heisenberg group \(H_{1}(H)\) (cf. Lemma 4.9 in [4]).

Throughout the paper, we will use \(C\) to denote a positive constant, which is independent of main parameters and may be different at each occurrence. By \(B_1 \sim B_2,\) we mean that there exists a constant \(C > 1\) such that \(1/C \leq B_1/B_2 \leq C.\)

2. Molecule Characterization of \(H_{L}^{p}(C^n)\)

In this section, we prove the main result of this paper. Firstly, we have the following lemma.
In the following, we will prove

\[ \frac{1}{|B_k|} \int_{B_k} |M_k - P_k|^2 \, dz \leq \frac{C}{|E_k|} \int_{E_k} |M_k(z)|^2 \, dz. \]  \hspace{1cm} (27)

In fact, by

\[ \int_{E_k} (M_k(z) - P_k(z)) \bar{P}_k(z) \, dz = 0, \]  \hspace{1cm} (28)

we have

\[ \frac{1}{|B_k|} \int_{B_k} |M_k(z) - P_k(z)|^2 \, dz = \frac{1}{|B_k|} \int_{B_k} (M_k(z) - P_k(z))(M_k(z) - \bar{P}_k(z)) \, dz \]
\[ = \frac{1}{|B_k|} \int_{B_k} (M_k(z) - P_k(z)) \bar{\mathcal{M}}_k(z) \, dz. \]  \hspace{1cm} (29)

Since

\[ \int_{E_k} \bar{\mathcal{M}}_k(z) P_k(z) \, dz \]
\[ = \int_{E_k} \bar{\mathcal{M}}_k(z) \int_{E_k} M(u) \bar{\omega}(z_0, u) \, du \, dz \]
\[ = \int_{E_k} M(u) \bar{\omega}(z_0, u) \, du \int_{E_k} \bar{\mathcal{M}}_k(z) \omega(z_0, z) \, dz \]
\[ = |E_k| |P_k(z)|^2, \]  \hspace{1cm} (30)

we get

\[ \frac{1}{|B_k|} \int_{B_k} |M_k(z) - P_k(z)|^2 \, dz \]
\[ = \frac{1}{|B_k|} \int_{E_k} |M_k(z)|^2 \, dz - \int_{E_k} \frac{|E_k|}{|B_k|} |P_k(z)|^2 \, dz \]
\[ \leq \frac{1}{|B_k|} \int_{E_k} |M_k(z)|^2 \, dz \leq \frac{C}{|E_k|} \int_{E_k} |M_k(z)|^2 \, dz. \]  \hspace{1cm} (31)

Therefore, (27) holds true. In particular, we have

\[ \frac{1}{|B_0|} \int_{B_0} |M_0(z) - P_0(z)|^2 \, dz \]
\[ \leq \frac{C}{|E_0|} \int_{E_0} |M_0(z)|^2 \, dz \]
\[ \leq C \sigma^{-2n} \sigma^{4n((1/2)-(1/p))} = C |B_0|^{-2/p}. \]  \hspace{1cm} (32)

For \( k \geq 1 \),

\[ \frac{1}{|B_k|} \int_{B_k} |M_k(z) - P_k(z)|^2 \, dz \]
\[ \leq \frac{C}{|E_k|} \int_{E_k} |M_k(z)|^2 \, dz \]
\[ = \frac{C}{|E_k|} \int_{E_k} |M_k(z)|^2 |z - z_0|^{2n(1+2\epsilon)} \times |z - z_0|^{-2n(1+2\epsilon)} \, dz \]
\[ \leq C (2^k \sigma)^{-2n} (2^{k-1} \sigma)^{-2n(1+2\epsilon)} \times \int_{E_k} |M_k(z)|^2 |z - z_0|^{2n(1+2\epsilon)} \, dz \]
\[ \leq C \sigma^{-4(n+1+2\epsilon)} 4^{k-1} 4^{2n} \sigma^{4n \epsilon} \]
\[ \leq C \sigma^{-4n+4k-4kn+4kn/p} (2^k \sigma)^{-2n/p} \]
\[ = C 2^{-4kn+4kn/p}|B_k|^{-2/p}, \]

where \( C \) depends on \( n, \epsilon \). This proves that \( M_k - P_k = \lambda_k a_k \), where \( a_k \) is a \( H^1_0 \)-atom supported on \( B_k \) and \( |\lambda_k| \leq C 2^{-2kn}. \)

Now, we prove that \( \sum_{k=1}^{\infty} P_k(z) \) has atomic decomposition. For \( k \geq 1 \),

\[ |P_k(z)| \leq \frac{1}{|E_k|} \int_{E_k} |M(u)| \, du \]
\[ = \frac{1}{|E_k|} \int_{E_k} |u - z_0|^{2b} |M(u)| |u - z_0|^{-2b} \, du \]
\[ \leq C (2^k \sigma)^{-2n+b} \frac{1}{|E_k|} \left\| |z_0 - z_0|^{2b} M \cdot |E_k| \right\|_{L^2} \|
\[ \leq C 2^{-4n+2b-n_\sigma^{2n} \sigma^{2n} \sigma^{2b}}. \]

Therefore,

\[ P_k(z) = \sum_{l=0}^{\infty} (P_{l-1}(z) - P_l(z)). \]  \hspace{1cm} (35)

Let

\[ N^k = \sum_{l=0}^{\infty} \int_{E_l} M(u) \bar{\omega}(z_0, u) \, du. \]  \hspace{1cm} (36)

Then,

\[ N^0 = \sum_{l=0}^{\infty} \int_{E_l} M(u) \bar{\omega}(z_0, u) \, du \]
\[ = \int_{C^n} M(u) \bar{\omega}(z_0, u) \, du = 0. \]  \hspace{1cm} (37)
Thus, by Abel transform,
\[
\sum_{k=0}^{\infty} P_k(z) = \sum_{k=0}^{\infty} \sum_{k=0}^{k+1} (P_{k+1}(z) - P_k(z)) \\
= \sum_{k=0}^{\infty} N^{k+1} \left[ |E_k|^{-1} \omega(z_0, z) \chi_k(z) \right] \\
\quad - |E_{k+1}|^{-1} \omega(z_0, z) \chi_{k+1}(z). 
\] (38)

Following from (34), we obtain
\[
|N^{k+1} \left[ |E_k|^{-1} \omega(z_0, z) \chi_k(z) \right] \\
- |E_{k+1}|^{-1} \omega(z_0, z) \chi_{k+1}(z)| \\
\leq C2^{-2n(k+1)} \sigma^{2n-2np} |b_k|^{-1/p} \\
= C2^{-2n(k+1)} \sigma^{2n-2np} \left( \frac{1}{2} \right)^{k+1} \\
= C2^{-2n(k+1)} \left[ |b_k|^{-1/p} \right]. 
\] (39)

Let \( \mu_k = C2^{-2n(k+1)} \) and
\[
b_k(z) = C^{-1} 2^{2n(k+1)} N^{k+1} \left[ |E_k|^{-1} \omega(z_0, z) \chi_k(z) \right] \\
- |E_{k+1}|^{-1} \omega(z_0, z) \chi_{k+1}(z). 
\] (40)

Then, \( b_k \) are \( H^p_{L} \)-atoms, \( \sum_{k=0}^{\infty} |\mu_k|^p < \infty \), and
\[
\sum_{k=0}^{\infty} P_k(z) = \sum_{k=0}^{\infty} \mu_k b_k(z). 
\] (41)

Therefore,
\[
M(z) = \sum_{k=0}^{\infty} \lambda_k a_k(z) + \sum_{k=0}^{\infty} \mu_k b_k(z) 
\] (42)

holds pointwise, where \( a_k \) are \( H^{p,2}_{L} \)-atoms and \( b_k \) are \( H^{p,\infty}_{L} \)-atoms, and
\[
\sum_{k=0}^{\infty} \left( |\lambda_k|^p + |\mu_k|^p \right) < \infty. 
\] (43)

When \( p = 1 \), it is easy to see that the sum in (42) converges in \( L^1 \).

To prove \( M \in H^p_{L} \) for \( 2n/(2n+1) < p < 1 \), we need to show that, for every \( g \in \mathcal{N}_{L}^{1/p-1} \),
\[
\int_{C^n} M(z) g(z) dz \\
= \lim_{m \to \infty} \sum_{k=0}^{m} \int_{C^n} [\lambda_k a_k(z) + \mu_k b_k(z)] g(z) dz. 
\] (44)

In fact, (44) implies that (42) holds in \( \delta'(C^n) \).

For any \( z \in C^n \), there exists \( k \geq 0 \) such that \( z \in E_k \). If \( k = 0 \), then
\[
M(z) = \lambda_0 a_0(z) + \mu_0 b_0(z) . 
\] (45)

If \( k \geq 1 \), then
\[
M(z) = \lambda_k a_k(z) + \sum_{j=k-1}^{k} \mu_j b_j(z). 
\] (46)

Therefore, when \( |z - z_0| \leq 2^m \sigma \),
\[
\sum_{k=0}^{m} (\lambda_k a_k(z) + \mu_k b_k(z)) \\
= \sum_{k=0}^{m} (\lambda_k a_k(z) + \mu_k b_k(z)) = M(z). 
\] (47)

Thus,
\[
\int_{|z| \leq 2^m \sigma} M(z) g(z) dz \\
= \int_{|z| \leq 2^m \sigma} M(z) g(z) dz. 
\] (48)

Let \( m \to \infty \); the right side is \( \int_{C^n} M(z) g(z) dz \). The left side is
\[
\lim_{m \to \infty} \sum_{k=0}^{m} \int_{|z| \leq 2^m \sigma} (\lambda_k a_k(z) + \mu_k b_k(z)) g(z) dz \\
= \lim_{m \to \infty} \sum_{k=0}^{m} \int_{C^n} (\lambda_k a_k(z) + \mu_k b_k(z)) g(z) dz. 
\] (49)

This proves (42) and the case of \( q = 2 \) for Lemma 12 is proved. Similarly, the case of \( q \neq 2 \) can be proved as the case of \( q = 2 \). Lemma 12 is proved.

\textbf{Proof of Theorem 8.} Theorem 8 follows from Lemmas 11 and 12.

\section{3. The Boundedness of Local Riesz Transform on $H^p_{L}(C^n)$}

In this section, we prove the boundedness of local Riesz transform on \( H^p_{L}(C^n) \) by using Theorem 8.

\textbf{Proof of Theorem 9.} By Theorem 8, it is sufficient to prove that, for any \( H^{p,2}_{L} \)-atom \( a, R_j(a) \) is a \( H^{p,2,x}_{L} \)-molecule and the norm \( \mathcal{M}(R_j(a)) \leq C \), where \( C \) is independent of \( a \).

Assume that \( \text{supp} \ a \subset B(z_0, r) \); then
\[
\int_{C^n} R_j(a)(z) \bar{w}(z_0, z) dz \\
= \int_{C^n} \left( \int_{C^n} a(z - u) \frac{u_i}{|u|^{2n+1}} \psi(u) \bar{w}(z, u) du \right) \\
\quad \times \bar{w}(z_0, z) dz 
\]
where the last equality is valid because $a(\cdot - u)$ is an atom supported on $B(u + z_0, r)$. This proves that $R_j(a)$ satisfies moment condition.

Denote $M(z) = R_j(a)(z)$. Then, we have

$$\|M\|_2 = \|R_j(a)\|_2 \leq C\|a\|_2$$

(51)

Let $B^* = \{z \in \mathbb{C}^n : |z - z_0| \leq 2r\}$. Then,

$$\int_{B^*} |z - z_0|^{2n(1 + 2\epsilon)} |M(z)|^2 dz$$

$$= \int_{B} |z - z_0|^{2n(1 + 2\epsilon)} |M(z)|^2 dz$$

(52)

$$+ \int_{(B^*)^c} |z - z_0|^{2n(1 + 2\epsilon)} |M(z)|^2 dz = I + II.$$  

For I,

$$I \leq C|B|^{1 + 2\epsilon} \int_{C^*} |M(z)|^2 dz \leq C|B|^{2 + 2\epsilon - (2/\epsilon)} = C|B|^{2\epsilon}.$$  

(53)

For II, since

$$|R_j(a)(z)|$$

$$= \int_{C^*} a(u) |z - u| \frac{z - u_j}{|z - u_j|^{2n+1+\epsilon}} \psi(z - u) \bar{\omega}(z, u) du$$

$$= \int_{C^*} a(u) \bar{\omega}(z_0, u) \left( \frac{z_j - u_j}{|z_j - u_j|^{2n+1+\epsilon}} \psi(z - u) \right.$$  

$$\times \omega(z_0 - z, u)$$

$$- \frac{z_j - z_0, j}{|z - z_0|^{2n+1+\epsilon}} \psi(z - z_0)$$

$$\times \omega(z_0 - z, z_0) \bigg) du$$

$$\leq C \int_{C^*} \left( \frac{|u - z_0|}{|z - z_0|^{2n+1+\epsilon}} |a(\cdot)| \right) du$$

$$\leq Cr|B|^{1 - \epsilon/p} \frac{1}{|z - z_0|^{2n+1+\epsilon}}$$

we get

$$II \leq C r|B|^{2 - 2\epsilon/p} \int_{(B^*)^c} \frac{|z - z_0|^{2n(1 + 2\epsilon)}}{|z - z_0|^{4n+2}} dz.$$  

(55)

Let $0 < \epsilon < 1/2n$. Then

$$II \leq C|B|^{2 + 2\epsilon - 2/p} = C|B|^{2\epsilon}.$$  

(56)

Therefore,

$$\mathcal{N}(M) = \|M\|^{\alpha(b)}_{L^2([-z_0, a(1 + 2\epsilon))} M(\cdot)_{L^2(\mathbb{R}^n)}^{1 - (\alpha/b)}$$

$$\leq C|B|^{(\alpha/b)(\alpha - b)} = C.$$  

(57)

This completes the proof of Theorem 9. \hfill \Box

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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