Positive Solutions of a Kind of Equations Related to the Laplacian and $p$-Laplacian

Fangfang Zhang and Zhanping Liang

School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China

Correspondence should be addressed to Zhanping Liang; lzp@sxu.edu.cn

Received 10 August 2014; Accepted 21 September 2014; Published 14 October 2014

1. Introduction

In this paper, we study the equation

$$-\Delta u - \Delta_p u = f(x, u), \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial \Omega,$$

(1)

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, $p > 2$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ for $N \geq 1$, and $f$ satisfies the following conditions:

$(f_1)$ $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x, t) \geq 0$ for any $x \in \overline{\Omega}, t > 0$ and $f(x, t) = 0$ for any $x \in \overline{\Omega}, t \leq 0$;

$(f_2)$ for $f_0, f_{\infty} < \infty$, the limits

$$\lim_{t \to 0^+} \frac{f(x, t)}{t} = f_0, \quad \lim_{t \to \infty} \frac{f(x, t)}{t^{p-1}} = f_{\infty}$$

(2)

exist uniformly for $x \in \overline{\Omega}$.

The asymptotic behaviors of $f$ near zero and infinity lead us to define

$$\lambda_1 = \inf \ left \{ \int_{\Omega} |\nabla u|^2 : u \in H^1_0(\Omega), \int_{\Omega} |u|^2 = 1 \right \},$$

$$\mu_1 = \inf \ left \{ \int_{\Omega} |\nabla u|^p : u \in W^{1,p}_0(\Omega), \int_{\Omega} |u|^p = 1 \right \},$$

(3)

where $H^1_0(\Omega)$ and $W^{1,p}_0(\Omega)$ are the usual Sobolev spaces defined as the completion of $C_0^{\infty}(\Omega)$ with respect to the norms $\|u\|_2 = (\int_{\Omega} |\nabla u|^2)^{1/2}$ and $\|u\|_p = (\int_{\Omega} |u|^p)^{1/p}$, respectively. Then it is well known that $\mu_1$ is the first eigenvalue of the nonlinear eigenvalue problem

$$-\Delta_p \phi = \mu |\phi|^{p-2} \phi, \quad \text{in } \Omega,$$

$$\phi = 0, \quad \text{on } \partial \Omega.$$

(4)

Moreover $\mu_1$ is a simple eigenvalue of (4), the associated eigenfunction $\phi_1$ can be chosen as positive in $\Omega$, and any eigenfunction corresponding to an eigenvalue larger than $\mu_1$ must change sign. The reader is referred to [1–3] for details.

By a solution $u$ of (1), we mean that $u$ solves (1) in the weak sense; that is, $u$ satisfies

$$\int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_{\Omega} f(x, u) v,$$

$$v \in W^{1,p}_0(\Omega).$$

Moreover, by a positive solution $u$ of (1), we mean that $u$ is a weak solution of (1), $u \neq 0$ and $u(x) \geq 0$ for $x \in \Omega$.

Our main result is the following theorem.

**Theorem 1.** Suppose that $f$ satisfies $(f_1)$ and $(f_2)$ with $f_0 < \lambda_1, f_{\infty} > \mu_1$. Then (1) has a positive solution.
Assume that $f = g + h$. Equation (1) can be viewed as combination of the following equations:
\begin{align*}
-\Delta u &= g(x,u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega, \\
-\Delta \rho u &= h(x,u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega.
\end{align*}
(6)
In the last decade or so, there was an extensive effort in studying the existence of solutions of (6); see [4–8].

Before concluding this section, we recall a theorem from [9], which will be used to prove our main theorem in this paper.

**Theorem 2.** Let $(E, \| \cdot \|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of $C_1$ functionals on $E$,
\begin{equation}
J_\gamma(u) = S(u) - \gamma T(u), \quad \gamma \in I,
\end{equation}
with $J_\gamma(0) = 0$, $\gamma \in I$, $T$ nonnegative and either $S(u) \to \infty$ or $T(u) \to \infty$ as $\|u\| \to \infty$. For any $\gamma \in I$, we set
\begin{equation}
\Gamma_\gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, J_\gamma(\gamma(1)) < 0 \}.
\end{equation}
If for every $\gamma \in I$ the set $\Gamma_\gamma$ is nonempty and
\begin{equation}
c_\gamma = \inf_{\gamma \in \Gamma_\gamma} \max_{t \in [0,1]} J_\gamma(\gamma(t)) > 0,
\end{equation}
then for almost every $\gamma \in I$ there exists a sequence $\{u^n_\gamma\} \subset E$ such that
\begin{enumerate}
\item \{u^n_\gamma\} is bounded;
\item $J_\gamma(u^n_\gamma) \to c_\gamma$ as $n \to \infty$;
\item $J_\gamma'(u^n_\gamma) \to 0$ in the dual $E^*$ as $n \to \infty$.
\end{enumerate}

Throughout this paper, we denote by $X$ the Sobolev space $W^{1,p}_0(\Omega)$ with the norm $\|u\|_X = \|u\|_p$, by $X^*$ the dual space of $X$, by weak convergence in $X$, and by $(\cdot, \cdot)$ the duality pairing between $X$ and $X^*$. The letters $C_1, C_2, \ldots$ will denote various positive constants whose exact values are not essential to the analysis of the problem. Let $P = \{u \in X : u(x) \geq 0, a.e. x \in \Omega \}$ and $p^* = \frac{np}{(N-p)}$ if $p < N$ or $p^* = \infty$ if $p \geq N$.

**2. Proof of Theorem 1**

In this section, we always assume $(f_1)$ and $(f_2)$ hold with $f_0 < \lambda_1$ and $f_\infty > \mu_1$. Hence, there exist $\epsilon_1 > 0, C_{\epsilon_1} > 0$ and $q \in (p, p^*)$ such that
\begin{equation}
F(x,t) \geq \frac{1}{p} (\mu_1 + \epsilon_1) t^p - C_{\epsilon_1}, \quad x \in \Omega, \quad t \geq 0,
\end{equation}
(10)
\begin{equation}
F(x,t) \leq \frac{1}{2} (1 - \epsilon_1) \lambda_1 t^2 + C_{\epsilon_1} t^q, \quad x \in \Omega, \quad t \in \mathbb{R},
\end{equation}
(11)
where $F(x,t) = \int_0^t f(x,s)ds$. In the following, we utilize Theorem 2 to complete the proof of Theorem 1. In the setting of Theorem 2 we have $E = X, I = [\delta, 1]$ with $\mu_1/(\mu_1 + \epsilon_1) < \delta < 1$, and
\begin{align*}
S(u) &= \frac{1}{2} \|u\|^2_2 + \frac{1}{p} \|u\|^p_X, \\
T(u) &= \int_\Omega F(x,u), \\
J_\gamma(u) &= \frac{1}{2} \|u\|^2_2 + \frac{1}{p} \|u\|^p_X - \gamma \int_\Omega F(x,u), \quad u \in X, \quad \gamma \in I.
\end{align*}
(12)
It is easy to verify that
\begin{equation}
\langle J_\gamma'(u), v \rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^p \nabla u \cdot \nabla v - \gamma \int_\Omega f(x,u) v, 
\end{equation}
(13)
\begin{equation}
u \in X, \quad \gamma \in I.
\end{equation}

Firstly, we show that $J_\gamma$ satisfies the conditions of Theorem 2 by proving several lemmas.

**Lemma 3.** $\Gamma_\gamma \neq \emptyset$ for any $\gamma \in I$.

**Proof.** Let $\phi_1 > 0$ be a $\mu_1$-eigenfunction. For $t > 0$, we have by (10) that
\begin{align*}
J_\gamma(t\phi_1) &= \frac{1}{2} t^2 \|\phi_1\|^2_2 + \frac{1}{p} t^p \|\phi_1\|^p_X - \gamma \int_\Omega F(x,t\phi_1) \\
&\leq \frac{1}{2} t^2 \|\phi_1\|^2_2 + \frac{1}{p} (\mu_1 + \epsilon_1) t^p \int_\Omega |\phi_1|^p \\
&\quad - \frac{1}{p} (\mu_1 + \epsilon_1) \delta t^p \int_\Omega |\phi_1|^p + C_1 \\
&= \frac{1}{2} t^2 \|\phi_1\|^2_2 - \frac{1}{p} C_2 t^p \int_\Omega |\phi_1|^p + C_1,
\end{align*}
(14)
where $C_2 = (\mu_1 + \epsilon_1) \delta - \mu_1$. Noting that $C_2 > 0$, we can choose $t_0 > 0$ large enough so that $J_\gamma(t_0 \phi_1) < 0$, where $t_0$ is independent of $\gamma \in I$. The proof is completed. \hfill $\Box$

**Lemma 4.** There exists a constant $c > 0$ such that $c_\gamma \geq c$ for any $\gamma \in I$.

**Proof.** For any $u \in X$, it follows from (II) that
\begin{align*}
J_\gamma(u) &= \frac{1}{2} \|u\|^2_2 + \frac{1}{p} \|u\|^p_X - \gamma \int_\Omega F(x,u) \\
&\geq \frac{1}{2} \|u\|^2_2 + \frac{1}{p} \|u\|^p_X - \frac{1}{2} (1 - \epsilon_1) \lambda_1 \int_\Omega |u|^2 - C_{\epsilon_1} \int_\Omega |u|^q \\
&\geq \frac{1}{2} \|u\|^2_2 + \frac{1}{p} \|u\|^p_X - \frac{1}{2} (1 - \epsilon_1) \|u\|^p_2 - C_{\epsilon_1} \int_\Omega |u|^q \\
&\geq \frac{1}{p} \|u\|^p_X - C_{\epsilon_1} \int_\Omega |u|^q.
\end{align*}
(15)

By Sobolev’s embedding theorem, we conclude that there exist $\rho > 0$ and $c > 0$ such that $J_\gamma(u) > 0$ for $\|u\| \in (0, \rho]$ and
\begin{equation}
J_\gamma(u) \geq c, \quad \|u\|_X = \rho.
\end{equation}
(16)
Fix \( v \in I \) and \( y \in \Gamma_v \). By the definition of \( \Gamma_v \), we have that \( \|y(1)\| > \rho \). Hence, there exists \( t_y \in (0, 1) \) such that \( \|y(t_y)\| = \rho \). So
\[
    c_y = \inf_{v \in \Gamma_v} \max_{t \in [0,1]} I_v(y(t)) \geq \inf_{v \in \Gamma_v} \int_0^1 y(t) \geq c. \quad (17)
\]

The proof is completed. \( \square \)

**Lemma 5.** For any \( v \in I \), if \( \{u_n\} \) is bounded and \( f'(u_n) \to 0 \) in \( X^* \) as \( n \to \infty \), then \( \{u_n\} \) admits a convergent subsequence.

**Proof.** Given \( v \in I \), assume that \( \{u_n\} \) is bounded, \( f'(u_n) \to 0 \) in \( X^* \) as \( n \to \infty \). By extracting a subsequence, we may suppose that there exists \( u \in X \) such that as \( n \to \infty \)
\[
    u_n \to u \quad \text{in } X, \quad u_n \rightharpoonup u \quad \text{in } L^s(\Omega), \quad s \in [1, p^*]. \quad (18)
\]

It follows from \((f_1)\) and \((f_2)\) that there exist \( C_1, C_2 > 0 \) such that
\[
    f(x,t) \leq C_1 |t| + C_2 |t|^{p-1}, \quad x \in \Omega, \quad t \in \mathbb{R}. \quad (19)
\]

Hence, by Hölder's inequality and Sobolev's embedding theorem, we have
\[
    \int_\Omega f(x,u_n)(u_n - u) \leq C_1 \int_\Omega |u_n| |u_n - u| + C_2 \int_\Omega |u_n|^{p-1} |u_n - u|
\]
\[
    \leq C_1 \left( \int_\Omega |u_n|^2 \right)^{1/2} \left( \int_\Omega |u_n - u|^2 \right)^{1/2}
\]
\[
    + C_2 \left( \int_\Omega |u_n|^p \right)^{(p-1)/p} \left( \int_\Omega |u_n - u|^p \right)^{1/p}
\]
\[
    \leq C_2 \left( \int_\Omega |u_n - u|^2 \right)^{1/2} + C_2 \left( \int_\Omega |u_n - u|^p \right)^{1/p} \to 0,
\]
\[
    n \to \infty. \quad (20)
\]

Similarly, we have
\[
    \int_\Omega f(x,u)(u_n - u) \rightharpoonup 0, \quad n \to \infty. \quad (21)
\]

Noting that
\[
    \langle f'(u_n) - f'(u), u_n - u \rangle = \langle f'(u_n) - f'(u), u_n - u \rangle
\]
\[
    = \int_\Omega \nabla u_n \cdot \nabla (u_n - u) + \int_\Omega |\nabla u_n|^p - |\nabla u|^p - \nabla u \cdot (u_n - u)
\]
\[
    - \int_\Omega |\nabla u|^p \cdot \nabla u \cdot (u_n - u) + \int_\Omega f(x,u_n) (u_n - u) - \int_\Omega f(x,u) (u_n - u)
\]
\[
    = \int_\Omega (|\nabla u_n|^p - |\nabla u|^p) \cdot \nabla (u_n - u)
\]
\[
    + \int_\Omega (|\nabla u_n|^p - |\nabla u|^p) \cdot \nabla u \cdot (u_n - u)
\]
\[
    - \int_\Omega f(x,u_n) (u_n - u) + \int_\Omega f(x,u) (u_n - u), \quad (22)
\]

and the inequality deduced from an inequality in Appendix of [3],
\[
    \int_\Omega (|\nabla u_n|^p - |\nabla u|^p) \cdot \nabla (u_n - u)
\]
\[
    \geq \frac{2}{p(2p-1)} \int_\Omega |\nabla (u_n - u)|^p, \quad (23)
\]

it follows from \((20)\) and \((21)\) that
\[
    \frac{2}{p(2p-1)} \int_\Omega |\nabla (u_n - u)|^p \leq \langle f'(u_n) - f'(u), u_n - u \rangle
\]
\[
    + \int_\Omega f(x,u_n) (u_n - u) - \int_\Omega f(x,u) (u_n - u)
\]
\[
    \to 0, \quad n \to \infty, \quad (24)
\]

where we have used the fact that
\[
    \langle f'(u_n) - f'(u), u_n - u \rangle \to 0, \quad n \to \infty. \quad (25)
\]

Hence \( u_n \to u \) in \( X \). The proof is completed. \( \square \)

**Lemma 6.** There exists a sequence \( \{v_n\} \subset I \) with \( v_n \to 1^- \) as \( n \to \infty \) and \( \{u_n\} \subset X \) such that \( f_n(v_n) = c_n, f_n'(u_n) = 0 \).

**Proof.** We only need to show that for almost every \( v \in I \) there exists \( u^* \in X \) such that \( f_v(u^*) = c_v \) and \( f_v'(u^*) = 0 \). By Theorem 2, for almost each \( v \in I \), there exists a bounded sequence \( \{u^*_n\} \subset X \) such that
\[
    f_v(u^*_n) \to c_v, \quad f_v'(u^*_n) \to 0, \quad n \to \infty. \quad (26)
\]
By Lemma 5, we may assume that $u_n^v \to u^v$ in $X$ as $n \to \infty$. Then the continuity of $J_v$ and $J'_v$ implies that $J_v(u^v) = c_v$ and $J'_v(u^v) = 0$. The proof is completed. 

Define $(Lu, v) = \int_{\Omega} f(x, u)v, (Ku, v) = \int_{\Omega} |u|^{p-2}uv$, $u, v \in X$. Then we have the following.

**Lemma 7.** Suppose $(f_1)$ and $(f_2)$ hold, then

$$
\lim_{\|u\|_X \to \infty, u \in P} \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}} = 0.
$$

Proof. By $(f_1)$, for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$
|f(x, t) - f_\infty t^{p-1}| \leq C_\varepsilon + \varepsilon t^{p-1}, \quad x \in \overline{\Omega}, \quad t \geq 0.
$$

For $u \in P \setminus \{0\}$, letting $w = u/\|u\|_X$, by Hölder’s inequality and Sobolev’s embedding theorem, we have

$$
\sup_{1 \leq \varepsilon \leq 1} \int_{\Omega} \left| \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}} \right| v
\leq \sup_{1 \leq \varepsilon \leq 1} \int_{\Omega} C_\varepsilon \|u\|_X^{-(p-1)} |v| + \varepsilon \|u\|_X^{-(p-1)} |v|
\leq C_5 \|u\|_X^{-(p-1)} + \varepsilon C_5,
$$

where $C_5$ is independent of $\varepsilon$. The proof is completed. 

**Proof of Theorem 1.** By Lemma 6, there exists a sequence $\{v_n\} \subset I$ with $v_n \to 1^-$ as $n \to \infty$ and $\{u_n\} \subset X$. Since $u_n \to 0$ and $\langle J'_v(u_n) \rangle = 0$, we have $c_n = c > 0$ and $J'_v(u_n) = 0$.\n
By Lemma 4 and (30), we have $c_n > c > 0$ and $\langle J'_v(u_n) \rangle = 0$. Hence $u_n \to 0$ in $X$. Suppose by contradiction that $\lim_{n \to \infty} \|u_n\|_X = \infty$. Let $w_n = u_n/\|u_n\|_X$.

Hence, we have, for $v \in X$,

$$
\frac{1}{\|u_n\|_X^{p-2}} \int_{\Omega} \nabla w_n \nabla v + \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla v = v_n f_\infty \int_{\Omega} \frac{u_n^{p-1}v - f_\infty u_n^{p-1}v}{\|u_n\|_X^{p-1}}.
$$

Since $\{w_n\}$ is bounded in $X$, we may assume that $w_n \to w_0 \in P \times X$, $w_n \to w_0$ in $L^p(\Omega)$ and $w_0(x) \to w_0(x)$ a.e. on $\Omega$ as $n \to \infty$. Letting $v = w_0 - w_0$ in (31) and $n \to \infty$, we get

$$
\lim_{n \to \infty} \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla (w_n - w_0) = 0.
$$

It follows from [5, Theorem 10] that $w_n \to w_0$ in $X$ as $n \to \infty$. Passing to limit $n \to \infty$ in (31), we obtain by Lemma 7 that

$$
\int_{\Omega} |\nabla w_0|^{p-2} \nabla w_0 \cdot \nabla v = f_\infty \int_{\Omega} w_0^{p-1}v, \quad v \in X.
$$

From (33) and the fact that $\|w_0\|_X = 1$, we know that $f_\infty = \mu_1$, which contradicts the assumption $f_\infty > \mu_1$. Since $\gamma_n \to 1^-$, we can show that

$$
J'_1(u_n) \to 0 \quad \text{in } X^*, \quad n \to \infty.
$$

In fact, for any $v \in X$, it follows from (19), Hölder’s inequality, and Sobolev’s embedding theorem that

$$
\left| \int_{\Omega} f(x, u_n)v \right| \leq C_1 \int_{\Omega} |u_n| |v| + C_2 \int_{\Omega} |u_n|^{p-1} |v|
\leq C_3 \|v\|_X.
$$

Furthermore, (30) implies that

$$
\langle J'_1(u_n), v \rangle + (1 - \gamma_n) \int_{\Omega} f(x, u_n)v = \langle J'_v(u_n), v \rangle = 0, \quad v \in X.
$$

Hence, $J'_1(u_n) \to 0$ in $X$ as $n \to \infty$. By Lemma 5, $\{u_n\}$ has a convergent subsequence. Without loss of generality, we may assume that $u_n \to u$ as $n \to \infty$. According to Lemma 4, and

$$
\left| \int_{\Omega} F(x, u_n) \right| \leq C_8,
$$

we have

$$
J_1(u) = \lim_{n \to \infty} J_1(u_n) = \lim_{n \to \infty} J'_v(u_n) \geq c > 0,
$$

$$
J'_1(u) = \lim_{n \to \infty} J'_1(u_n) = 0.
$$

The standard process shows that $u$ is a positive solution to (1). The proof is completed. 

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This paper is partially supported by National Natural Science Foundation of China (Grant nos. 11071149, 11301313) and Science Council of Shanxi Province (2012011004-2, 2013021001-4, and 2014021009-1).

**References**


