Research Article

Existence of Positive Solutions for Some Superlinear Fourth-Order Boundary Value Problems

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We are concerned with the following superlinear fourth-order equation $u^{(4)}(t) + u(t)\varphi(t, u(t)) = 0$, $t \in (0, 1)$; $u(0) = u(1) = 0$, $u'(0) = a$, $u'(1) = -b$, where a, b are nonnegative constants such that $a + b > 0$ and $\varphi(t, s)$ is a nonnegative continuous function that is required to satisfy some appropriate conditions related to a class $\mathcal{K}$ satisfying suitable integrability condition. Our purpose is to prove the existence, uniqueness, and global behavior of a classical positive solution to the above problem by using a method based on estimates on the Green function and perturbation arguments.

1. Introduction

A natural motivation for studying higher order BVPs lies in their applications. It is well-known (see for instance [1]) that the deformation of an elastic beam in equilibrium state, whose both ends clamped, can be described by fourth-order BVP

$$u^{(4)}(t) = f(t, u(t)), \quad t \in (0, 1), \quad (1)$$

subject to the boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0. \quad (2)$$

When the nonlinearity f is nonnegative such problems have been extensively investigated by many researchers, and various forms of the equation and boundary condition have been discussed; see, for example, [1–15] and references therein.

In particular, in 2005, Ma and Tisdell [1] studied the existence of positive solutions to the sublinear BVP

$$\begin{aligned} u^{(4)}(t) &= p(t)u^\sigma(t), \quad t \in (0, 1), \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (3)$$

where $\sigma \in (0, 1)$ and $p : (0, 1) \rightarrow [0, \infty)$ is continuous which may be singular at both ends $t = 0$ and $t = 1$ and satisfying some integrable conditions. Using the method of lower and upper solutions for fourth-order boundary value problems, they have given some necessary and sufficient conditions for the existence of regular positive solutions to the boundary value problem (3).

In [8], the authors used fixed-point index results, to prove the existence of a positive solution for the boundary value problem

$$\begin{aligned} u^{(4)}(t) &= p(t)f(u(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (4)$$

where $f : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing continuous function allowed to be superlinear and $p : (0, 1) \rightarrow [0, \infty)$ is continuous which may be singular at both ends $t = 0$ and $t = 1$ satisfying some adequate conditions.

Recently, in [4], the author considered problem (3), with $\sigma \in (-1, 1)$ and p is a nonnegative continuous function on $(0, 1)$ satisfying some hypotheses related to Karamata regular variation theory. Using the Schauder fixed-point theorem,

he established the existence and uniqueness of a positive solution to (3).

Here, by using a method based on estimates on the Green function and perturbation arguments, we show that, for negative nonlinearity f , problem (1) has a unique positive classical solution subject to some boundary conditions. More precisely, we are concerned with the following superlinear fourth order problem

$$\begin{aligned} u^{(4)}(t) + u(t)\varphi(t, u(t)) &= 0, \quad t \in (0, 1), \\ u(0) = u(1) &= 0, \quad u'(0) = a, \quad u'(1) = -b, \end{aligned} \quad (5)$$

where a, b are nonnegative constants such that $a + b > 0$. Our goal is to answer the questions of existence, uniqueness, and global behavior of a classical positive solution to problem (5), where the nonnegative nonlinear term $\varphi(t, s)$ is required to satisfy some appropriate conditions related to the following class $\mathcal{K}$.

Definition 1. A Borel measurable function q in $(0, 1)$ belongs to the class $\mathcal{K}$ if q satisfies the following condition:

$$\int_0^1 r^3(1-r)^3 |q(r)| dr < \infty. \quad (6)$$

We will often refer in this paper to

$$\omega(t) := at(1-t)^2 + bt^2(1-t), \quad (7)$$

the unique solution of the problem

$$\begin{aligned} u^{(4)}(t) &= 0, \quad t \in (0, 1), \\ u(0) = u(1) &= 0, \quad u'(0) = a, \quad u'(1) = -b. \end{aligned} \quad (8)$$

Observe that, for $t \in [0, 1]$,

$$\min(a, b)\rho(t) \leq \omega(t) \leq \max(a, b)\rho(t), \quad (9)$$

where $\rho(t) := t(1-t)$.

Also we denote by $G(t, s)$ the Green function of the operator $u \rightarrow u^{(4)}$, with the conditions $u(0) = u(1) = u'(0) = u'(1) = 0$, which can be explicitly given by

$$\begin{aligned} G(t, s) &= \frac{1}{6} \begin{cases} t^2(1-s)^2 [3s-t(1+2s)], & 0 \leq t \leq s \leq 1, \\ s^2(1-t)^2 [3t-s(1+2t)], & 0 \leq s \leq t \leq 1, \end{cases} \\ &= \frac{1}{6} (t \wedge s)^2 (1-t \vee s)^2 [3(t \vee s) - (t \wedge s)(1+2(t \vee s))], \end{aligned} \quad (10)$$

where $t \wedge s = \min(t, s)$ and $t \vee s = \max(t, s)$ (see [2]).

The outline of the paper is as follows. In Section 2, we give some sharp estimates on Green's function $G(t, s)$, including the following 3G-inequality: for each $t, s, r \in (0, 1)$,

$$\begin{aligned} \frac{G(t, r)G(r, s)}{G(t, s)} &\leq \frac{3}{2} \left[\left(\frac{\rho(r)}{\rho(t)} \right)^2 G(t, r) + \left(\frac{\rho(r)}{\rho(s)} \right)^2 G(r, s) \right], \\ \left(\frac{\rho(r)}{\rho(t)} \right)^2 G(t, r) &\leq \frac{1}{2} (\rho(r))^3, \end{aligned} \quad (11)$$

where $\rho(r) = r(1-r)$.

In particular, we derive from this 3G-inequality that, for each $q \in \mathcal{K}$, one has

$$\alpha_q := \sup_{t, s \in (0, 1)} \int_0^1 \frac{G(t, r)G(r, s)}{G(t, s)} |q(r)| dr < \infty. \quad (12)$$

In Section 3, our purpose is to study the superlinear fourth-order problem (5). The nonlinearity φ is required to satisfy a combination of the following assumptions.

- (H₁) φ is a nonnegative continuous function in $(0, 1) \times [0, \infty)$.
- (H₂) There exists a nonnegative function $q \in \mathcal{K} \cap C(0, 1)$ with $\alpha_q \leq 1/2$ such that for each $t \in (0, 1)$, the map $s \rightarrow s(q(t) - \varphi(t, s\omega(t)))$ is nondecreasing on $[0, 1]$.
- (H₃) For each $t \in (0, 1)$, the function $s \rightarrow s\varphi(t, s)$ is nondecreasing on $[0, \infty)$.

We will first exploit the 3G-inequality to prove that the inverse of fourth-order operators that are perturbed by a zero-order term are positivity preserving. That is, if ψ is a positive measurable function and q is a nonnegative function belonging to the class $\mathcal{K} \cap (C(0, 1))$ with $\alpha_q \leq 1/2$, then the following problem

$$\begin{aligned} u^{(4)}(t) + q(t)u(t) &= \psi(t), \quad t \in (0, 1), \\ u(0) = u(1) = u'(0) &= u'(1) = 0, \end{aligned} \quad (13)$$

has a positive solution. It turns out to prove that problem (13) admits a positive Green function $\mathcal{G}(t, s)$.

Based on the construction of this Green function and by using perturbation arguments, we prove the following.

Theorem 2. Assume (H₁)-(H₂), then problem (5) has a positive solution u in $C^4((0, 1)) \cap C([0, 1])$ satisfying

$$c_0\omega(t) \leq u(t) \leq \omega(t), \quad t \in [0, 1], \quad (14)$$

where c_0 is a constant in $(0, 1)$.

Moreover, if hypothesis (H₃) is also satisfied, then the solution u to problem (5) satisfying (14) is unique.

Corollary 3. Let f be a nonnegative function in $C^1([0, \infty))$ such that the map $s \rightarrow \theta(s) = sf(s)$ is nondecreasing on $[0, \infty)$. Let p be a nonnegative continuous function on $(0, 1)$ such that the function $t \rightarrow \bar{p}(t) := p(t) \max_{0 \leq \xi \leq \omega(t)} \theta'(\xi)$

belongs to the class $\mathcal{K}$. Then for $\lambda \in [0, 1/2\alpha_{\tilde{p}}]$, the following problem

$$\begin{aligned} u^{(4)}(t) + \lambda p(t) u(t) f(u(t)) &= 0, \quad t \in (0, 1), \\ u(0) = u(1) &= 0, \quad u'(0) = a, \quad u'(1) = -b, \end{aligned} \quad (15)$$

has a unique positive solution u in $C^4((0, 1)) \cap C([0, 1])$ satisfying

$$(1 - \lambda\alpha_{\tilde{p}})\omega(t) \leq u(t) \leq \omega(t), \quad t \in [0, 1]. \quad (16)$$

Observe that in Theorem 2, we obtain a positive classical solution u to problem (5) whose behavior is not affected by the perturbed term. That is, it behaves like the solution ω of the homogeneous problem (8). As typical example of nonlinearity satisfying (H_1) – (H_3) , we quote $\varphi(t, s) = \lambda p(t)s^\sigma$, where $\sigma \geq 0$, p is a positive continuous function on $(0, 1)$ such that

$$\int_0^1 r^{3+\sigma} (1-r)^{3+\sigma} p(r) dr < \infty, \quad (17)$$

and $q(t) = \lambda \tilde{p}(t) := \lambda(\sigma + 1)p(t)(\omega(t))^\sigma$, with $\lambda \in [0, 1/2\alpha_{\tilde{p}}]$. Note that by using (9) and (17) the function $\tilde{p}$ belongs to the class $\mathcal{K}$.

As usual, let $\mathcal{B}((0, 1))$ be the set of Borel measurable functions in $(0, 1)$ and let $\mathcal{B}^+((0, 1))$ be the set of nonnegative ones.

We define the kernel $V: \mathcal{B}^+((0, 1)) \rightarrow \mathcal{B}^+((0, 1))$ by

$$V\psi(t) := \int_0^1 G(t, r) \psi(r) dr, \quad t \in [0, 1]. \quad (18)$$

Let $q \in \mathcal{B}^+((0, 1))$. We define the kernel $V(q): \mathcal{B}^+((0, 1)) \rightarrow \mathcal{B}^+((0, 1))$ by

$$V(q)(\psi)(t) := \int_0^1 G(t, r) q(r) \psi(r) dr, \quad t \in [0, 1]. \quad (19)$$

Remark 4 (See [1]). Let $\psi \in \mathcal{B}^+((0, 1))$ such that the function $r \rightarrow (\rho(r))^2 \psi(r)$ is continuous and integrable on $(0, 1)$, then $V\psi$ is the unique solution in $C^4((0, 1)) \cap C([0, 1])$ of

$$\begin{aligned} u^{(4)}(t) &= \psi(t), \quad t \in (0, 1); \\ u(0) = u(1) &= u'(0) = u'(1) = 0. \end{aligned} \quad (20)$$

2. Estimates on the Green's Function

From the explicit expression of Green's function (10) we derive the following.

Proposition 5 (see [4, 7]). For $t, s \in [0, 1]$, one has

$$\begin{aligned} &\frac{1}{3}(t \wedge s)^2(1-t \vee s)^2(t \vee s)(1-t \wedge s) \\ &\leq G(t, s) \\ &\leq \frac{1}{2}(t \wedge s)^2(1-t \vee s)^2(t \vee s)(1-t \wedge s), \end{aligned} \quad (21)$$

$$\begin{aligned} &\frac{1}{3}(\rho(t))^2(\rho(s))^2 \leq G(t, s) \\ &\leq \frac{1}{2}\rho(t)\rho(s)(\rho(t) \wedge \rho(s)). \end{aligned} \quad (22)$$

Using (22), we deduce the following.

Corollary 6. For $s, t \in (0, 1)$, one has

(i)

$$\left(\frac{\rho(s)}{\rho(t)}\right)^2 G(t, s) \leq \frac{1}{2}(\rho(s))^3. \quad (23)$$

(ii) For $\psi \in \mathcal{B}^+((0, 1))$, the function $t \rightarrow V\psi(t)$ is continuous on $[0, 1]$ if and only if the integral $\int_0^1 (\rho(s))^2 \psi(s) ds$ converges.

Theorem 7 (3G inequality). For each $t, s, r \in (0, 1)$, one has

$$\frac{G(t, r)G(r, s)}{G(t, s)} \leq \frac{3}{2} \left[\left(\frac{\rho(r)}{\rho(t)}\right)^2 G(t, r) + \left(\frac{\rho(r)}{\rho(s)}\right)^2 G(r, s) \right], \quad (24)$$

where $\rho(t) = t(1-t)$.

Proof. To prove the inequality, we denote by $A(t, s) = (\rho(t)\rho(s))^2/G(t, s)$ and we claim that A is a quasi-metric, that is, for each $t, s, r \in (0, 1)$,

$$A(t, s) \leq \frac{3}{2} [A(t, r) + A(r, s)]. \quad (25)$$

By symmetry, we may assume that $t \leq s$.

Using (21), we deduce that

$$2s(1-t) \leq A(t, s) \leq 3s(1-t). \quad (26)$$

To show (25), we separate the proof into three cases.

Case 1 ($r \leq t \leq s$). In this case, one has

$$\begin{aligned} A(t, r) + A(r, s) &\geq 2(t+s)(1-r) \\ &\geq 2s(1-t) \geq \frac{2}{3}A(t, s). \end{aligned} \quad (27)$$

Case 2 ($t \leq r \leq s$). We obtain

$$\begin{aligned} A(t, r) + A(r, s) &\geq 2r(1-t) + 2s(1-r) \\ &\geq 2s(1-t) \geq \frac{2}{3}A(t, s). \end{aligned} \quad (28)$$

Case 3 ($t \leq s \leq r$). One has

$$\begin{aligned} A(t, r) + A(r, s) &\geq 2r(1-t) + 2r(1-s) \\ &\geq 2s(1-t) \geq \frac{2}{3}A(t, s). \end{aligned} \quad (29)$$

This completes the proof. $\square$

In the sequel, for any $q \in \mathcal{B}((0, 1))$, we recall that

$$\alpha_q := \sup_{t, s \in (0, 1)} \int_0^1 \frac{G(t, r)G(r, s)}{G(t, s)} |q(r)| dr, \quad (30)$$

and we denote by

$$h_1(t) = t(1-t)^2, \quad h_2(t) = t^2(1-t), \quad \text{for } t \in [0, 1]. \quad (31)$$

Proposition 8. *Let q be a function in $\mathcal{K}$; then*

(i)

$$\alpha_q \leq \frac{3}{2} \int_0^1 (\rho(r))^3 |q(r)| dr < \infty. \quad (32)$$

(ii) For $t \in [0, 1]$, one has

$$\int_0^1 G(t, s) h_1(s) |q(s)| ds \leq \alpha_q h_1(t). \quad (33)$$

(iii) For $t \in [0, 1]$, one has

$$\int_0^1 G(t, s) h_2(s) |q(s)| ds \leq \alpha_q h_2(t). \quad (34)$$

In particular, for $t \in [0, 1]$, one has

$$\int_0^1 G(t, s) \rho(s) |q(s)| ds \leq \alpha_q \rho(t). \quad (35)$$

Proof. Let q be a function in $\mathcal{K}$.

(i) Using (24) and (23), one has, for $t, s \in (0, 1)$,

$$\begin{aligned} \int_0^1 \frac{G(t, r)G(r, s)}{G(t, s)} |q(r)| dr &\leq \frac{3}{2} \int_0^1 \left(\frac{\rho(r)}{\rho(t)} \right)^2 G(t, r) \\ &\quad + \left(\frac{\rho(r)}{\rho(s)} \right)^2 G(r, s) |q(r)| dr \\ &\leq \frac{3}{2} \int_0^1 (\rho(r))^3 |q(r)| dr. \end{aligned} \quad (36)$$

Hence

$$\alpha_q \leq \frac{3}{2} \int_0^1 (\rho(r))^3 |q(r)| dr < \infty. \quad (37)$$

(ii) Since, for each $t, s \in (0, 1)$, one has $\lim_{r \rightarrow 0} (G(s, r)/G(t, r)) = h_1(s)/h_1(t)$, then we deduce by Fatou's lemma and (12), that

$$\begin{aligned} &\int_0^1 G(t, s) \frac{h_1(s)}{h_1(t)} |q(s)| ds \\ &\leq \liminf_{r \rightarrow 0} \int_0^1 G(t, s) \frac{G(s, r)}{G(t, r)} |q(s)| ds \leq \alpha_q; \end{aligned} \quad (38)$$

which implies that, for $t \in [0, 1]$,

$$\int_0^1 G(t, s) h_1(s) |q(s)| ds \leq \alpha_q h_1(t). \quad (39)$$

(iii) Similarly, we prove inequality (34) by observing that

$$\lim_{r \rightarrow 1} \frac{G(s, r)}{G(t, r)} = \frac{h_2(s)}{h_2(t)}. \quad (40)$$

Inequality (35) follows from Proposition 8 (ii)-(iii) and the fact that $\rho(t) = h_1(t) + h_2(t)$. This completes the proof. $\square$

3. Proofs of Main Results

In this section, we aim at proving Theorem 2 and Corollary 3. So, we need the following preliminaries results.

For a nonnegative function q in $\mathcal{K}$ such that $\alpha_q < 1$, we define the function $\mathcal{G}(t, s)$ on $[0, 1] \times [0, 1]$, by

$$\mathcal{G}(t, s) = \sum_{n=0}^{\infty} (-1)^n G_n(t, s), \quad (41)$$

where $G_0(t, s) = G(t, s)$ and

$$G_n(t, s) = \int_0^1 G(t, r) G_{n-1}(r, s) q(r) dr, \quad n \geq 1. \quad (42)$$

Next, we establish some inequalities on $G_n(t, s)$. In particular, we deduce that $\mathcal{G}(t, s)$ is well defined.

Lemma 9. *Let q be a nonnegative function in $\mathcal{K}$ such that $\alpha_q < 1$; then, for each $n \geq 0$ and $t, s \in [0, 1]$, one has*

$$(i) \quad G_n(t, s) \leq \alpha_q^n G(t, s).$$

In particular, $\mathcal{G}(t, s)$ is well defined in $[0, 1] \times [0, 1]$.

(ii)

$$L_n(\rho(t))^2(\rho(s))^2 \leq G_n(t, s) \leq R_n \rho(t) \rho(s) (\rho(t) \wedge \rho(s)); \quad (43)$$

where $L_n = (1/3^{n+1})(\int_0^1 (\rho(r))^4 q(r) dr)^n$ and $R_n = (1/2^{n+1})(\int_0^1 (\rho(r))^3 q(r) dr)^n$.

$$(iii) \quad G_{n+1}(t, s) = \int_0^1 G_n(t, r) G(r, s) q(r) dr \text{ and } G_n(t, s) = G_n(s, t).$$

$$(iv) \quad \int_0^1 \mathcal{G}(t, r) G(r, s) q(r) dr = \int_0^1 G(t, r) \mathcal{G}(r, s) q(r) dr.$$

Proof. (i) The assertion is clear for $n = 0$.

Assume that inequality in (i) holds for some $n \geq 0$; then by using (42) and (12), we obtain

$$G_{n+1}(t, s) \leq \alpha_q^n \int_0^1 G(t, r) G(r, s) q(r) dr \leq \alpha_q^{n+1} G(t, s). \quad (44)$$

Now, since $G_n(t, s) \leq \alpha_q^n G(t, s)$, it follows that $\mathcal{G}(t, s)$ is well defined in $[0, 1] \times [0, 1]$.

(ii) Using (22) and (42), we obtain (43) by induction.

(iii) Since $G(t, s) = G(s, t)$, then equalities in (iii) are clear for $n = 0$.

Assume that for a given integer $n \geq 1$ and $t, s \in [0, 1]$, one has

$$\begin{aligned} G_n(t, s) &= \int_0^1 G_{n-1}(t, r) G(r, s) q(r) dr, \\ G_{n-1}(t, s) &= G_{n-1}(s, t). \end{aligned} \quad (45)$$

Using (42) and Fubini-Tonelli theorem, we obtain

$$\begin{aligned} G_{n+1}(t, s) &= \int_0^1 G(t, r) \left(\int_0^1 G_{n-1}(r, \xi) G(\xi, s) q(\xi) d\xi \right) q(r) dr \\ &= \int_0^1 \left(\int_0^1 G(t, r) G_{n-1}(r, \xi) q(r) dr \right) G(\xi, s) q(\xi) d\xi \\ &= \int_0^1 G_n(t, \xi) G(\xi, s) q(\xi) d\xi. \end{aligned} \quad (46)$$

On the other hand, by (42) and (45), we deduce that

$$\begin{aligned} G_n(t, s) &= \int_0^1 G(t, r) G_{n-1}(r, s) q(r) dr \\ &= \int_0^1 G_{n-1}(s, r) G(r, t) q(r) dr \\ &= G_n(s, t). \end{aligned} \quad (47)$$

(iv) Let $n \geq 0$ and $t, r, s \in [0, 1]$. By Lemma 9 (i) and (22), one has

$$0 \leq G_n(t, r) G(r, s) q(r) \leq \alpha_q^n G(t, r) G(r, s) q(r). \quad (48)$$

Hence the series $\sum_{n \geq 0} \int_0^1 G_n(t, r) G(r, s) q(r) dr$ converges.

So we deduce by the dominated convergence theorem and Lemma 9 (iii) that

$$\begin{aligned} &\int_0^1 \mathcal{G}(t, r) G(r, s) q(r) dr \\ &= \sum_{n=0}^{\infty} \int_0^1 (-1)^n G_n(t, r) G(r, s) q(r) dr \\ &= \sum_{n=0}^{\infty} \int_0^1 (-1)^n G(t, r) G_n(r, s) q(r) dr \\ &= \int_0^1 G(t, r) \mathcal{G}(r, s) q(r) dr. \end{aligned} \quad (49)$$

□

Proposition 10. Let q be a nonnegative function in $\mathcal{K}$ such that $\alpha_q < 1$. Then the function $(t, s) \rightarrow \mathcal{G}(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

Proof. We claim that for $n \geq 0$, the function $G_n(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

The assertion is clear for $n = 0$.

Now assume that for a given integer $n \geq 1$, the function $(t, s) \rightarrow G_{n-1}(t, s)$ is continuous on $[0, 1] \times [0, 1]$. By (42), one has

$$G_n(t, s) = \int_0^1 G(t, r) G_{n-1}(r, s) q(r) dr. \quad (50)$$

Since the function $(t, s, r) \rightarrow G(t, r) G_{n-1}(r, s)$ is continuous on $[0, 1] \times [0, 1] \times [0, 1]$ and by Lemma 9 (i) and (22)

$$\begin{aligned} G(t, r) G_{n-1}(r, s) q(r) &\leq \alpha_q^{n-1} G(t, r) G(r, s) q(r) \\ &\leq (\rho(r))^3 q(r), \end{aligned} \quad (51)$$

we deduce by the dominated convergence theorem that $(t, s) \rightarrow G_n(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

This proves our claim.

Now by Lemma 9(i) and (22), one has, for each $t, s \in [0, 1]$,

$$G_n(t, s) \leq \alpha_q^n G(t, s) \leq \frac{1}{2} \rho(t) (\rho(s))^2 \alpha_q^n \leq \alpha_q^n. \quad (52)$$

This implies that the series $\sum_{n \geq 0} (-1)^n G_n(t, s)$ is uniformly convergent on $[0, 1] \times [0, 1]$ and therefore the function $(t, s) \rightarrow \mathcal{G}(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

The proof is completed. □

Lemma 11. Let q be a nonnegative function in $\mathcal{K}$ such that $\alpha_q \leq 1/2$. Then for t, s in $[0, 1]$, one has

$$(1 - \alpha_q) G(t, s) \leq \mathcal{G}(t, s) \leq G(t, s). \quad (53)$$

Proof. Since $\alpha_q \leq 1/2$, we deduce from Lemma 9 (i) that

$$|\mathcal{G}(t, s)| \leq \sum_{n=0}^{\infty} (\alpha_q)^n G(t, s) = \frac{1}{1 - \alpha_q} G(t, s). \quad (54)$$

On the other hand, from the expression of $\mathcal{G}$, one has

$$\mathcal{G}(t, s) = G(t, s) - \sum_{n=0}^{\infty} (-1)^n G_{n+1}(t, s). \quad (55)$$

Since the series $\sum_{n \geq 0} \int_0^1 G(t, r) G_n(r, s) q(r) dr$ is convergent, we deduce by (55) and (42) that

$$\begin{aligned} \mathcal{G}(t, s) &= G(t, s) - \sum_{n=0}^{\infty} (-1)^n \int_0^1 G(t, r) G_n(r, s) q(r) dr \\ &= G(t, s) - \int_0^1 G(t, r) \left(\sum_{n=0}^{\infty} (-1)^n G_n(r, s) \right) q(r) dr. \end{aligned} \quad (56)$$

That is,

$$\mathcal{G}(t, s) = G(t, s) - V(q\mathcal{G}(\cdot, s))(t). \quad (57)$$

Now from (54) and Lemma 9(i) (with $n = 1$), we obtain

$$\begin{aligned} V(q\mathcal{G}(\cdot, s))(t) &\leq \frac{1}{1 - \alpha_q} V(qG(\cdot, s))(t) \\ &= \frac{1}{1 - \alpha_q} G_1(t, s) \leq \frac{\alpha_q}{1 - \alpha_q} G(t, s). \end{aligned} \quad (58)$$

This implies that

$$\mathcal{G}(t, s) \geq G(t, s) - \frac{\alpha_q}{1 - \alpha_q} G(t, s) = \frac{1 - 2\alpha_q}{1 - \alpha_q} G(t, s) \geq 0. \quad (59)$$

So it follows that $\mathcal{G}(t, s) \leq G(t, s)$ and by (57) and Lemma 9(i) (with $n = 1$), one has

$$\mathcal{G}(t, s) \geq G(t, s) - V(q\mathcal{G}(\cdot, s))(t) \geq (1 - \alpha_q) G(t, s). \quad (60)$$

□

In the sequel, for a given nonnegative function $q \in \mathcal{K}$ such that $\alpha_q \leq 1/2$, we define the operator $V_q : \mathcal{B}^+((0, 1)) \rightarrow \mathcal{B}^+((0, 1))$ by

$$V_q \psi(t) = \int_0^1 \mathcal{G}(t, s) \psi(s) ds, \quad t \in [0, 1]. \quad (61)$$

Using (53) and (22), we obtain the following.

Corollary 12. *Let q be a nonnegative function in $\mathcal{K}$ such that $\alpha_q \leq 1/2$ and $\psi \in \mathcal{B}^+((0, 1))$; then the following statements are equivalent.*

- (i) *The function $t \rightarrow V_q \psi(t)$ is continuous on $[0, 1]$.*
- (ii) *The integral $\int_0^1 (\rho(s))^2 \psi(s) ds$ converges.*

Next, we will prove that the kernel V_q satisfies the following resolvent equation.

Lemma 13. *Let q be a nonnegative function in $\mathcal{K}$ such that $\alpha_q \leq 1/2$ and $\psi \in \mathcal{B}^+((0, 1))$. Then $V_q \psi$ satisfies the following resolvent equation:*

$$V\psi = V_q \psi + V_q(qV\psi) = V_q \psi + V(qV_q \psi). \quad (62)$$

In particular, if $V(q\psi) < \infty$, one has

$$\begin{aligned} (I - V_q(q\cdot))(I + V(q\cdot))\psi \\ = (I + V(q\cdot))(I - V_q(q\cdot))\psi = \psi. \end{aligned} \quad (63)$$

Proof. Let $t, s \in [0, 1]$, then by (57) one has

$$G(t, s) = \mathcal{G}(t, s) + V(q\mathcal{G}(\cdot, s))(t), \quad (64)$$

which implies by Fubini-Tonelli theorem that, for $\psi \in \mathcal{B}^+((0, 1))$,

$$\begin{aligned} V\psi(t) &= \int_0^1 (\mathcal{G}(t, s) + V(q\mathcal{G}(\cdot, s))(t)) \psi(s) ds \\ &= V_q \psi(t) + V(qV_q \psi)(t). \end{aligned} \quad (65)$$

On the other hand, by Lemma 9 (iii) and Fubini-Tonelli theorem, we obtain for $\psi \in \mathcal{B}^+((0, 1))$ and $t \in [0, 1]$

$$\begin{aligned} \int_0^1 \int_0^1 \mathcal{G}(t, r) G(r, s) \psi(s) q(r) dr ds \\ = \int_0^1 \int_0^1 G(t, r) \mathcal{G}(r, s) \psi(s) q(r) dr ds; \end{aligned} \quad (66)$$

that is,

$$V_q(qV\psi)(t) = V(qV_q \psi)(t). \quad (67)$$

So we obtain

$$V\psi = V_q \psi + V(qV_q \psi) = V_q \psi + V_q(qV\psi)(t). \quad (68)$$

This completes the proof. □

Proposition 14. *Let q be a nonnegative function in $\mathcal{K} \cap \mathcal{E}(0, 1)$ such that $\alpha_q \leq 1/2$ and let $\psi \in \mathcal{B}^+((0, 1))$ such that $t \rightarrow (\rho(t))^2 \psi(t)$ is continuous and integrable on $(0, 1)$. Then $V_q \psi$ is the unique nonnegative solution in $C^4((0, 1)) \cap C([0, 1])$ of the perturbed fourth-order equation (13) satisfying*

$$(1 - \alpha_q) V\psi \leq V_q \psi \leq V\psi. \quad (69)$$

Proof. It is clear by Corollary 12 that the function $t \rightarrow q(t)V_q \psi(t)$ is continuous on $(0, 1)$. Using (62) and (22), there exists a nonnegative constant c such that

$$V_q \psi(t) \leq V\psi(t) \leq \frac{1}{2} \int_0^1 \rho(t) (\rho(s))^2 \psi(s) ds \leq c \rho(t). \quad (70)$$

So we deduce that

$$\int_0^1 (\rho(s))^2 q(s) V_q \psi(s) ds \leq c \int_0^1 (\rho(s))^3 q(s) ds < \infty. \quad (71)$$

Hence by using Remark 4, the function $u = V_q \psi = V\psi - V(qV_q \psi)$ satisfies the equation

$$\begin{aligned} u^{(4)}(t) &= \psi(t) - q(t)u(t), \quad t \in (0, 1), \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \quad (72)$$

and by integration inequalities (53), we obtain (69).

It remains to prove the uniqueness. Assume that v is another nonnegative solution in $C^4((0, 1)) \cap C([0, 1])$ of problem (13) satisfying (69).

Since the function $t \rightarrow q(t)v(t)$ is continuous on $(0, 1)$ and by (69) and (70), the function $t \rightarrow (\rho(t))^2 q(t)v(t)$ is

integrable on $(0, 1)$; then it follows by Remark 4 that the function $\tilde{v} := v + V(qv)$ satisfies

$$\begin{aligned}\tilde{v}^{(4)}(t) &= \psi(t), \quad t \in (0, 1), \\ \tilde{v}(0) &= \tilde{v}(1) = \tilde{v}'(0) = \tilde{v}'(1) = 0.\end{aligned}\quad (73)$$

From the uniqueness in Remark 4, we deduce that

$$\tilde{v} := v + V(qv) = V\psi. \quad (74)$$

Hence

$$(I + V(q\cdot))(v - u) = 0. \quad (75)$$

Now since by (69), (70), and (35), one has

$$V(q|v - u|) \leq 2cV(q\rho) \leq 2c\alpha_q\rho < \infty, \quad (76)$$

then by (63), we deduce that $u = v$. This completes the proof. $\square$

Proof of Theorem 2. Let $a \geq 0$ and $b \geq 0$ with $a + b > 0$ and recall that

$$\omega(t) = at(1-t)^2 + bt^2(1-t) = ah_1(t) + bh_2(t). \quad (77)$$

Since φ satisfies (H_2) , then there exists a positive function q in $\mathcal{K} \cap C(0, 1)$ such that $\alpha_q \leq 1/2$ and for each $t \in (0, 1)$, the map $s \rightarrow s(q(t) - \varphi(t, s\omega(t)))$ is nondecreasing on $[0, 1]$. Let

$$\Lambda := \{u \in \mathcal{B}^+((0, 1)) : (1 - \alpha_q)\omega \leq u \leq \omega\}, \quad (78)$$

and define the operator T on Λ by

$$Tu = \omega - V_q(q\omega) + V_q((q - \varphi(\cdot, u))u). \quad (79)$$

By (62) and Proposition 8, one has

$$V_q(q\omega) \leq V(q\omega) \leq \alpha_q\omega \leq \omega, \quad (80)$$

and by (H_2) , we obtain

$$0 \leq \varphi(\cdot, u) \leq q, \quad \forall u \in \Lambda. \quad (81)$$

So we claim that Λ is invariant under T . Indeed, using (81) and (80), one has for $u \in \Lambda$

$$\begin{aligned}Tu &\leq \omega - V_q(q\omega) + V_q(qu) \leq \omega, \\ Tu &\geq \omega - V_q(q\omega) \\ &\geq (1 - \alpha_q)\omega.\end{aligned}\quad (82)$$

Next, we will prove that the operator T is nondecreasing on Λ . Indeed, let $u, v \in \Lambda$ be such that $u \leq v$. Since the map $s \rightarrow s(q(t) - \varphi(t, s\omega(t)))$ is nondecreasing on $[0, 1]$, for $t \in (0, 1)$, we obtain

$$Tv - Tu = V_q([v(q - \varphi(\cdot, v)) - u(q - \varphi(\cdot, u))]) \geq 0. \quad (83)$$

Now, we consider the sequence (u_n) defined by $u_0 = (1 - \alpha_q)\omega$ and $u_{n+1} = Tu_n$, for $n \in \mathbb{N}$. Since Λ is invariant under T , one has $u_1 = Tu_0 \geq u_0$ and by the monotonicity of T , we deduce that

$$(1 - \alpha_q)\omega = u_0 \leq u_1 \leq \dots \leq u_n \leq u_{n+1} \leq \omega. \quad (84)$$

Hence by dominated convergence theorem and hypotheses (H_1) - (H_2) , we conclude that the sequence (u_n) converges to a function $u \in \Lambda$ satisfying

$$u = (I - V_q(q\cdot))\omega + V_q((q - \varphi(\cdot, u))u). \quad (85)$$

That is,

$$(I - V_q(q\cdot))u = (I - V_q(q\cdot))\omega - V_q(u\varphi(\cdot, u)). \quad (86)$$

On the other hand, since by (80), one has $V(qu) \leq V(q\omega) \leq \omega < \infty$, then by applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (62) and (63), we conclude that u satisfies

$$u = \omega - V(u\varphi(\cdot, u)). \quad (87)$$

Next we aim at proving that u is a solution of problem (5). To this end, we remark by (81) and (9) that

$$u\varphi(\cdot, u) \leq q\omega \leq \max(a, b)q\rho. \quad (88)$$

This implies by Corollary 6 (ii) that the function $t \rightarrow V(u\varphi(\cdot, u))(t)$ is continuous on $[0, 1]$ and so by (87), u is continuous on $[0, 1]$.

Now, since by (H_1) and (88), the function $t \rightarrow (\rho(t))^2 u(t)\varphi(t, u(t))$ is continuous and integrable on $(0, 1)$, we conclude by Remark 4 that u is the required solution.

It remains to prove that under condition (H_3) , u is the unique solution to problem (5) satisfying (14). Assume that v is another nonnegative solution in $C^4((0, 1)) \cap C([0, 1])$ to problem (5) satisfying (14). Since $v \leq \omega$, we deduce by (88) that

$$0 \leq v\varphi(\cdot, v) \leq q\omega \leq \max(a, b)q\rho. \quad (89)$$

So the function $t \rightarrow (\rho(t))^2 v(t)\varphi(t, v(t))$ is continuous and integrable on $(0, 1)$ and by Remark 4, we conclude that the function $\tilde{v} := v + V(v\varphi(\cdot, v))$ satisfies

$$\begin{aligned}\tilde{v}^{(4)}(t) &= 0, \quad t \in (0, 1), \\ \tilde{v}(0) &= \tilde{v}(1) = 0, \quad \tilde{v}'(0) = a, \quad \tilde{v}'(1) = -b.\end{aligned}\quad (90)$$

From the uniqueness in problem (8), we deduce that

$$v = \omega - V(v\varphi(\cdot, v)). \quad (91)$$

Now let h be the function defined on $(0, 1)$ by

$$h(t) = \begin{cases} \frac{v(t)\varphi(t, v(t)) - u(t)\varphi(t, u(t))}{v(t) - u(t)} & \text{if } v(t) \neq u(t), \\ 0 & \text{if } v(t) = u(t). \end{cases} \quad (92)$$

Then by (H_3) , $h \in \mathcal{B}^+((0, 1))$ and by (87) and (91), one has

$$(I + V(h \cdot))(v - u) = 0. \quad (93)$$

On the other hand, by (H_2) , we remark that $h \leq q$ and by (80) we deduce that

$$V(h|v - u|) \leq 2V(q\omega) \leq 2\alpha_q \omega < \infty. \quad (94)$$

Hence by (63), we conclude that $u = v$. This completes the proof. $\square$

Proof of Corollary 3. Let $\varphi(t, s) = \lambda p(t)f(s)$, $\theta(s) = sf(s)$, and $\tilde{p}(t) = p(t)\max_{0 \leq \xi \leq \omega(t)} \theta'(\xi)$. It is clear that hypotheses (H_1) and (H_3) are satisfied. Since the function $q(t) := \lambda \tilde{p}(t)$ belongs to the class $\mathcal{K}$, one has $\alpha_q \leq 1/2$ for $\lambda \in [0, 1/2\alpha_{\tilde{p}}]$. Moreover, by a simple computation, for $s \in [0, 1]$ and $t \in (0, 1)$, we obtain

$$\frac{d}{ds} [s(q(t) - \varphi(t, s\omega(t)))] = q(t) - \lambda p(t)\theta'(s\omega(t)) \geq 0. \quad (95)$$

This implies that the function φ satisfies hypothesis (H_2) . So the result follows by Theorem 2. $\square$

Example 15. Let $a \geq 0$ and $b \geq 0$ with $a + b > 0$. Let $\sigma \geq 0$, and p be a positive continuous function on $(0, 1)$ such that

$$\int_0^1 r^{3+\sigma}(1-r)^{3+\sigma} p(r) dr < \infty. \quad (96)$$

Since the function $\tilde{p}(t) := (\sigma + 1)p(t)(\omega(t))^\sigma$ belongs to the class $\mathcal{K}$, then, for $\lambda \in [0, 1/2\alpha_{\tilde{p}}]$, the problem

$$\begin{aligned} u^{(4)}(t) + \lambda p(t)u^{\sigma+1}(t) &= 0, \quad t \in (0, 1), \\ u(0) = u(1) &= 0, \quad u'(0) = a, \quad u'(1) = -b \end{aligned} \quad (97)$$

has a unique positive solution u in $C^4((0, 1)) \cap C([0, 1])$ satisfying

$$(1 - \lambda\alpha_{\tilde{p}})\omega(t) \leq u(t) \leq \omega(t), \quad t \in [0, 1]. \quad (98)$$

Example 16. Let $a \geq 0$ and $b \geq 0$ with $a + b > 0$. Let $\sigma \geq 0$, $\gamma > 0$, and p be a positive continuous function on $(0, 1)$ such that

$$\int_0^1 r^{3+\sigma+\gamma}(1-r)^{3+\sigma+\gamma} p(r) dr < \infty. \quad (99)$$

Let $\theta(s) = s^{\sigma+1} \log(1 + s^\gamma)$. Since the function $\tilde{p}(t) := p(t)\max_{0 \leq \xi \leq \omega(t)} \theta'(\xi) \in \mathcal{K}$, then, for $\lambda \in [0, 1/2\alpha_{\tilde{p}}]$, the problem

$$\begin{aligned} u^{(4)}(t) + \lambda p(t)u^{\sigma+1}(t) \log(1 + u^\gamma(t)) &= 0, \quad t \in (0, 1), \\ u(0) = u(1) &= 0, \quad u'(0) = a, \quad u'(1) = -b \end{aligned} \quad (100)$$

has a unique positive solution u in $C^4((0, 1)) \cap C([0, 1])$ satisfying

$$(1 - \lambda\alpha_{\tilde{p}})\omega(t) \leq u(t) \leq \omega(t), \quad t \in [0, 1]. \quad (101)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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