Research Article

Smooth Decompositions of Triebel-Lizorkin and Besov Spaces on Product Spaces of Homogeneous Type

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We introduce Triebel-Lizorkin and Besov spaces by Calderón's reproducing formula on product spaces of homogeneous type. We also obtain smooth atomic and molecular decompositions for these spaces.

1. Introduction and Main Results

Atomic and molecular decompositions are significant tools in studying function spaces and operators in harmonic analysis. The atomic decomposition for Hardy spaces on $\mathbb{R}$ was first introduced by Coifman in [1] and was extended to $\mathbb{R}^n$ by Latter in [2]. Molecules in Hardy spaces were also introduced by Coifman in [3]. Coifman and Weiss in [4] extended molecules to more general setting in Hardy spaces.

The atomic decomposition of product Hardy spaces $H^p(\mathbb{R} \times \mathbb{R})$ was established by Chang and Fefferman in [5, 6]. Recently, general atomic decomposition on product Hardy spaces was constructed by Han et al. in [7] and Han et al. in [8].

The smooth atomic decomposition for the Triebel-Lizorkin spaces on $\mathbb{R}^n$ first is considered by Frazier and Jawerth in [9] and later extended to spaces of homogeneous type by Han and Sawyer in [10]; for similar results, see [11]. Smooth molecules of the Triebel-Lizorkin spaces on bidisc were given by Wang in [12].

In [13], Hardy spaces associated with different homogeneities were constructed. Wu in [14] introduced related Triebel-Lizorkin and Besov spaces similarly. In [7, 15], the authors studied the Hardy spaces on product spaces of homogeneous type and showed some properties of these spaces. A natural question arises: whether we introduce the Triebel-Lizorkin and Besov spaces on product spaces of homogeneous type and study some properties of these spaces.

The purpose of this paper is to answer the question. Namely, we introduce the Triebel-Lizorkin and Besov spaces on product spaces of homogeneous type. A theory of atomic and molecular decompositions for these spaces is also presented.

We begin with some necessary definitions and notation on spaces of homogeneous type.

A quasimetric ρ on a set X is a function $\rho: X \times X \rightarrow [0, \infty)$ satisfying that (i) $\rho(x, y) = 0$ if and only if $x = y$, (ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, and (iii) there exists a constant $A \in [1, \infty)$ such that for all x, y , and $z \in X$

$$\rho(x, y) \leq A [\rho(x, z) + \rho(z, y)]. \quad (1)$$

Any quasimetric defines a topology, for which the balls $B(x, r) = \{y \in X : \rho(y, x) < r\}$ with all $x \in X$ and all $r > 0$ form a basis.

We now state definitions of spaces of homogeneous type.

Definition 1. A space of homogeneous type (X, ρ, μ) is a set X with a quasimetric ρ and a nonnegative Borel regular measure μ on X such that $0 < \mu(B(x, r)) < \infty$, and there exists a positive constant $C < \infty$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad (2)$$

for all $x \in X$ and all $r > 0$, where μ is assumed to be defined on a σ -algebra which contains all Borel sets and all balls $B(x, r)$.

We suppose that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$. Further, suppose there exist constant $C > 0$ and regularity exponent $0 < \theta < 1$ such that, for all $0 < r < \infty$ and all $x, x', y \in X$, $\mu(B(x, r)) \sim r$ and $|\rho(x, y) - \rho(x', y)| \leq C\rho(x, x')^\theta[\rho(x, y) + \rho(x', y)]^{1-\theta}$.

Let us recall the definition of an approximation to the identity on spaces of homogeneous type.

Definition 2 (see [7]). Let X be a space of homogeneous type as in Definition 1 and constant A satisfying (1). A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0, \theta]$ if there exists $C > 0$ such that, for all x, x', y , and $y' \in X$, $S_k(x, y)$, the kernel of S_k , is a function from $X \times X$ into $\mathbb{C}$ satisfying

- (i) $|S_k(x, y)| \leq C(2^{-k\epsilon}/(2^{-k} + \rho(x, y)))^{1+\epsilon}$;
- (ii) $|S_k(x, y) - S_k(x', y)| \leq C(\rho(x, x')/(2^{-k} + \rho(x, y)))^\epsilon(2^{-k\epsilon}/(2^{-k} + \rho(x, y)))^{1+\epsilon}$
for $\rho(x, x') \leq (1/2A)(2^{-k} + \rho(x, y))$;
- (iii) $|S_k(x, y) - S_k(x, y')| \leq C(\rho(y, y')/(2^{-k} + \rho(x, y)))^\epsilon(2^{-k\epsilon}/(2^{-k} + \rho(x, y)))^{1+\epsilon}$
for $\rho(y, y') \leq (1/2A)(2^{-k} + \rho(x, y))$;
- (iv) $||S_k(x, y) - S_k(x, y')| - |S_k(x', y) - S_k(x', y')|| \leq C(\rho(x, x')/(2^{-k} + \rho(x, y)))^\epsilon(\rho(y, y')/(2^{-k} + \rho(x, y)))^\epsilon(2^{-k\epsilon}/(2^{-k} + \rho(x, y)))^{1+\epsilon}$
for $\rho(x, x') \leq (1/2A)(2^{-k} + \rho(x, y))$ and $\rho(y, y') \leq (1/2A)(2^{-k} + \rho(x, y))$;
- (v) $\int_X S_k(x, y)d\mu(y) = \int_X S_k(x, y)d\mu(x) = 1$.

Moreover, a sequence $\{S_k\}_{k \in \mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0, \theta]$ having compact support if there exists a constant $C > 0$ such that, for all $k \in \mathbb{Z}$ and all x, x', y , and $y' \in X$, $S_k(x, y)$, the kernel of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying (v) and

- (vi) $S_k(x, y) = 0$ if $\rho(x, y) \geq C2^{-k}$ and $\|S_k\|_{L^\infty(X \times X)} \leq C2^k$;
- (vii) $|S_k(x, y) - S_k(x', y)| \leq C2^{k(1+\epsilon)}\rho(x, x')^\epsilon$;
- (viii) $|S_k(x, y) - S_k(x, y')| \leq C2^{k(1+\epsilon)}\rho(y, y')^\epsilon$;
- (ix) $||S_k(x, y) - S_k(x, y')| - |S_k(x', y) - S_k(x', y')|| \leq C2^{k(1+2\epsilon)}\rho(x, x')^\epsilon\rho(y, y')^\epsilon$.

We present test functions on spaces of homogeneous type X before we give test functions on product spaces of homogeneous type $X_1 \times X_2$.

Definition 3 (see [15]). For fixed $x_0 \in X$, let $0 < \beta, \gamma < \theta$, and $r > 0$, where θ is the regularity exponent on X . A function $f(x, y)$ defined on $X \times X$ is said to be a test function of type (x_0, r, β, γ) , if, for all $x, y \in X$, f satisfies the following conditions:

- (i) $|f(x)| \leq C(r/(r + \rho(x, x_0)))^{1+\gamma}$;
- (ii) $|f(x) - f(y)| \leq C(\rho(x, y)/(r + \rho(x, x_0)))^\beta(r/(r + \rho(x, x_0)))^{1+\gamma}$ for $\rho(x, y) \leq (1/2A)(r + \rho(x, x_0))$.

If f is a test function of type (x_0, r, β, γ) , we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ and the norm of f is defined by

$$\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} = \inf \{C : \text{(i)-(ii) hold}\}. \quad (3)$$

We denote by $\mathcal{G}(\beta, \gamma)$ the class of $\mathcal{G}(x_0, r, \beta, \gamma)$ with $r = 1$ for fixed $x_0 \in X$. Set $f(x) \in \mathcal{G}_0(\beta, \gamma)$ if $f(x) \in \mathcal{G}(\beta, \gamma)$ with $\int_X f(x)d\mu(x) = 0$. It is easy to see that $\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$ with an equivalent norm for all $x_1 \in X$ and $r > 0$. Furthermore, we can check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Let $\mathcal{G}^\circ(\beta, \gamma)$ be the completion of the space $\mathcal{G}_0(\theta, \theta)$ in $\mathcal{G}(\beta, \gamma)$ with $0 < \beta, \gamma < \theta$. If $f \in \mathcal{G}(\beta, \gamma)$, we then define $\|f\|_{\mathcal{G}^\circ(\beta, \gamma)} = \|f\|_{\mathcal{G}(\beta, \gamma)}$.

We define the distribution space $(\mathcal{G}^\circ(\beta, \gamma))'$ by all linear functionals $\mathcal{L}$ from $\mathcal{G}^\circ(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that $|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{G}^\circ(\beta, \gamma)}$ for all $f \in \mathcal{G}^\circ(\beta, \gamma)$.

Let (X_i, ρ_i, μ_i) for $i = 1, 2$ be two spaces of homogeneous type as in Definition 1 and ρ_i satisfies (1) with A replaced by A_i for $i = 1, 2$. We define spaces of test functions on product space $X_1 \times X_2$ of spaces of homogeneous type.

Definition 4 (see [15]). Let $(x_0, y_0) \in X_1 \times X_2$, $0 < \beta_i, \gamma_i < \theta_i$, and $r_i > 0$, where θ_i is the regularity exponent on X_i for $i = 1, 2$. A function $f(x, y)$ defined on $X_1 \times X_2$ is said to be a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ if, for any fixed $y, y' \in X_2$, $f(x, y)$, as a function of variable of x , is a test function of $\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)$ on X_1 . Similarly, for any fixed $x, x' \in X_1$, $f(x, y)$, as a function of variable of y , is a test function of $\mathcal{G}(y_0, r_2, \beta_2, \gamma_2)$ on X_2 . Moreover the following conditions are satisfied:

- (i) $\|f(\cdot, y)\|_{\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)} \leq C(r_2/(r_2 + \rho_2(y, y_0)))^{1+\gamma_2}$;
- (ii) $\|f(\cdot, y) - f(\cdot, y')\|_{\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)} \leq C(\rho_2(y', y)/(r_2 + \rho_2(y, y_0)))^{\beta_2}(r_2/(r_2 + \rho_2(y, y_0)))^{1+\gamma_2}$ for all $y, y' \in X_2$ with $\rho_2(y, y') \leq (1/2A_2)(r_2 + \rho_2(y, y_0))$;
- (iii) properties (i)-(ii) also hold with x and y interchanged.

If f is a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$, we write

$$f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2), \quad (4)$$

and the norm of f is defined by

$$\|f\|_{\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} = \inf \{C : \text{(i) - (iii) hold}\}. \quad (5)$$

Similarly, we denote by $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $r_1 = r_2 = 1$ for fixed $(x_0, y_0) \in X_1 \times X_2$. Set $f(x, y) \in \mathcal{G}_0(\beta_1, \beta_2; \gamma_1, \gamma_2)$ if $f(x, y) \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $\int_{X_1} f(x, y)d\mu_1(x) = \int_{X_2} f(x, y)d\mu_2(y) = 0$. It is easy to see that

$$\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \quad (6)$$

with an equivalent norm for all $(x_0, y_0) \in X_1 \times X_2$ and $r_1, r_2 > 0$. Furthermore, we can check that $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$.

Let $\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2)$ be the completion of the space $\mathcal{G}_0(\theta_1, \theta_2; \theta_1, \theta_2)$ in $\mathcal{G}(\beta_1, \beta_2, \gamma_1, \gamma_2)$ with $0 < \beta_i, \gamma_i < \theta_i$ for $i = 1, 2$. If $f \in \mathcal{G}(\beta_1, \beta_2, \gamma_1, \gamma_2)$, we then define $\|f\|_{\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2)} = \|f\|_{\mathcal{G}(\beta_1, \beta_2, \gamma_1, \gamma_2)}$.

Also, we define the distribution space $(\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2))'$ by all linear functionals $\mathcal{L}$ from $\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists a constant $C > 0$ such that $|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ for all $f \in \mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2)$.

We give Calderón's reproducing formulas on product spaces of homogeneous type.

Lemma 5 (see [7, 15]). *Suppose that $i = 1, 2$. Let $\epsilon_i \in (0, \theta_i]$, let $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order ϵ_i on spaces of homogeneous type X_i , and let $D_{k_i} = S_{k_i} - S_{k_i-1}$ for $k_i \in \mathbb{Z}$. Then there are families of linear operators $\{\bar{D}_{k_i}\}_{k_i \in \mathbb{Z}}$ and $\{\bar{D}_{k_i}\}_{k_i \in \mathbb{Z}}$ on X_i such that, for all $f \in (\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2))'$ with $\beta_i, \gamma_i \in (0, \epsilon_i)$,*

$$f = \sum_{k_1, k_2 \in \mathbb{Z}} \bar{D}_{k_1} \bar{D}_{k_2} D_{k_1} D_{k_2} (f) = \sum_{k_1, k_2 \in \mathbb{Z}} D_{k_1} D_{k_2} \bar{D}_{k_1} \bar{D}_{k_2} (f), \quad (7)$$

where the series converges in the norm of both the spaces $(\mathcal{G}^\circ(\beta'_1, \beta'_2, \gamma'_1, \gamma'_2))'$ with $\beta'_i \in (0, \beta_i)$, $\gamma'_i \in (0, \gamma_i)$. Moreover, for $x_i, y_i \in X_i$ and all $k_i \in \mathbb{Z}$, $\bar{D}_{k_i}(x_i, y_i)$, the kernel of $\bar{D}_{k_i}$, satisfies conditions (i) and (ii) in Definition 2 with ϵ_i replaced by any $\epsilon'_i \in (0, \epsilon_i)$ and

$$\int_{X_i} \bar{D}_{k_i}(x_i, y_i) d\mu_i(y_i) = \int_{X_i} \bar{D}_{k_i}(x_i, y_i) d\mu_i(x_i) = 0. \quad (8)$$

$\bar{D}_k(x, y)$, the kernel of $\bar{D}_k$, satisfies conditions (i) and (iii) in Definition 2 with ϵ replaced by ϵ' for $0 < \epsilon' < \epsilon$ and (8).

We also need the following result, which gives an analogue of the grid Euclidean dyadic cubes on a space of homogeneous type.

Lemma 6 (see [15]). *Let X be a space of homogeneous type as in Definition 1. Then there exists a collection $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constants $C_1, C_2 > 0$ such that*

- (i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$, if $\alpha \neq \beta$;
- (ii) for any α, β, k, l with $l > k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;
- (iii) for each (k, α) and each $l < k$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;
- (iv) $\text{diam}(Q_\alpha^k) \leq C_1 2^{-k}$;
- (v) each Q_α^k contains some ball $B(y_\alpha^k, C_2 2^{-k})$, where $y_\alpha^k \in X$.

We think of Q_α^k as being a dyadic cube with diameter 2^{-k} and center y_α^k . When $k \in \mathbb{Z}$ and $\tau \in I_k$, we denote by $Q_\tau^{k, \nu}$,

where $\nu = 1, 2, \dots, N(\tau, k)$, the set of all cubes $Q_\tau^{k+j} \subset Q_\tau^k$, where j is a fixed large positive integer, and $y_\tau^{k, \nu}$ a point in $Q_\tau^{k, \nu}$.

Throughout the paper, we use C to denote positive constants, whose value may change from one occurrence to the next. We denote by $f \sim g$ that there exists a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. χ_A is the characteristic function of A . Let $a_+ = \max(0, a)$ for any $a \in \mathbb{R}$. We also denote M_1, M_2 are the Hardy-Littlewood maximal function on spaces of homogeneous type X_1, X_2 , respectively.

We introduce Triebel-Lizorkin space $\dot{F}_p^{s, q}$ and Besov space $\dot{B}_p^{s, q}$ on product spaces of homogeneous type.

Definition 7. Suppose that $i = 1, 2$, $s = (s_1, s_2)$, $s_i \in (-\epsilon_i, \epsilon_i)$ and let $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ be the same as in Lemma 5. Let $1 < p, q < \infty$ and $0 < \beta_i, \gamma_i < \epsilon_i < \theta_i$, $f \in (\mathcal{G}^\circ(\beta_1, \beta_2, \gamma_1, \gamma_2))'$. Then Triebel-Lizorkin space $\dot{F}_p^{s, q}(X_1 \times X_2)$ is defined by

$$\|f\|_{\dot{F}_p^{s, q}} := \left\| \left(\sum_{k_1, k_2 \in \mathbb{Z}} 2^{k_1 s_1 q} 2^{k_2 s_2 q} |D_{k_1} D_{k_2} (f)|^q \right)^{1/q} \right\|_p < \infty. \quad (9)$$

The Besov space $\dot{B}_p^{s, q}(X_1 \times X_2)$ is defined by

$$\|f\|_{\dot{B}_p^{s, q}} := \left(\sum_{k_1, k_2 \in \mathbb{Z}} 2^{k_1 s_1 q} 2^{k_2 s_2 q} \|D_{k_1} D_{k_2} (f)\|_p^q \right)^{1/q} < \infty. \quad (10)$$

In order to check that the definitions for $\dot{F}_p^{s, q}$ and $\dot{B}_p^{s, q}$ are independent of the choice of approximations to the identity, we recall almost orthogonality estimate.

Lemma 8 (see [15]). *Let $i = 1, 2$. Suppose that $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ and $\{P_{k_i}\}_{k_i \in \mathbb{Z}}$ are two approximations to the identity on spaces of homogeneous type, and $D_{k_i} = S_{k_i} - S_{k_i-1}$, $E_{k_i} = P_{k_i} - P_{k_i-1}$. Then for $\epsilon_i \in (0, \theta_i)$, there exists a positive constant C depending only on ϵ_i such that $D_{l_1} D_{l_2} E_{k_1} E_{k_2}(x_1, y_1; x_2, y_2)$, the kernel of $D_{l_1} D_{l_2} E_{k_1} E_{k_2}$, satisfies the following estimate:*

$$\begin{aligned} & |D_{l_1} D_{l_2} E_{k_1} E_{k_2}(x_1, y_1; x_2, y_2)| \\ & \leq C 2^{-|k_1 - l_1| \epsilon_1} 2^{-|k_2 - l_2| \epsilon_2} \frac{2^{-(k_1 \wedge l_1) \epsilon_1}}{(2^{-(k_1 \wedge l_1)} + \rho_1(x_1, y_1))^{1+\epsilon_1}} \\ & \quad \times \frac{2^{-(k_2 \wedge l_2) \epsilon_2}}{(2^{-(k_2 \wedge l_2)} + \rho_2(x_2, y_2))^{1+\epsilon_2}}. \end{aligned} \quad (11)$$

By Calderón's reproducing formula and almost orthogonality estimate, we can get the following product-type Plancherel-Pôlya inequalities. The proofs of the following theorems are similar to Proposition 4.1 in [10]. Here we omit the details.

Theorem 9. *Suppose that D_k and E_k are the same as in Lemma 8 for $k \in \mathbb{Z}$. Let $1 < p, q < \infty$, and $|s_i| < \epsilon_i$ for $i = 1, 2$.*

Then there exists a constant $C > 0$ such that for all $f \in (\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ with $0 < \beta_i, \gamma_i < \epsilon_i < \theta_i$ for $i = 1, 2$, and then

$$\begin{aligned} & \left\| \left\{ \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} (2^{k_1 s_1} 2^{k_2 s_2} |D_{k_1} D_{k_2}(f)|)^q \right\}^{1/q} \right\|_{L^p} \\ & \sim \left\| \left\{ \sum_{l_1 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} (2^{l_1 s_1} 2^{l_2 s_2} |E_{l_1} E_{l_2}(f)|)^q \right\}^{1/q} \right\|_{L^p}, \\ & \left\{ \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} (2^{k_1 s_1} 2^{k_2 s_2} \|D_{k_1} D_{k_2}(f)\|_{L^p})^q \right\}^{1/q} \\ & \sim \left\{ \sum_{l_1 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} (2^{l_1 s_1} 2^{l_2 s_2} \|E_{l_1} E_{l_2}(f)\|_{L^p})^q \right\}^{1/q}. \end{aligned} \quad (12)$$

Remark 10. Let $D_{l_1}, D_{l_2}, \bar{D}_{k_1}$, and $\bar{D}_{k_2}$ be as in Lemma 5. Then the kernel of $D_{l_1} D_{l_2}$ has compact support, but not of $\bar{D}_{k_1} \bar{D}_{k_2}$. Since (11) holds for D_{l_1} and D_{l_2} satisfying only the smoothness condition for the second variable, we conclude that Theorem 9 holds with $D_{l_1} D_{l_2}$ being replaced by $\bar{D}_{k_1} \bar{D}_{k_2}$.

2. Smooth Atomic Decomposition

In this section, smooth atomic decomposition is presented. We first give definitions of the smooth atoms for $\dot{B}_p^{s,q}$ and $\dot{F}_p^{s,q}$.

Definition 11. Suppose collections of open subset $\{Q_{\tau_1}^{k_1, \nu_1} \subset X_1 : k_1 \in \mathbb{Z}, \tau_1 \in I_{k_1}, \nu_1 = 1, \dots, N(k_1, \tau_1)\}$ and $\{Q_{\tau_2}^{k_2, \nu_2} \subset X_2 : k_2 \in \mathbb{Z}, \tau_2 \in I_{k_2}, \nu_2 = 1, \dots, N(k_2, \tau_2)\}$ satisfying the conditions in Lemma 6. Let $\epsilon = (\epsilon_1, \epsilon_2)$ and $R = Q_{\tau_1}^{k_1, \nu_1} \times Q_{\tau_2}^{k_2, \nu_2}$, and a function a_R defined on $X_1 \times X_2$ is said to be a ϵ -smooth atom for R if

(i)

$$\text{supp } a_R \subset B_1(z_{\tau_1}^{k_1, \nu_1}, A_1 C 2^{-k_1}) \times B_2(z_{\tau_2}^{k_2, \nu_2}, A_2 C 2^{-k_2}), \quad (13)$$

where $z_{\tau_i}^{k_i, \nu_i}$ is the center of $Q_{\tau_i}^{k_i, \nu_i}$ for $i = 1, 2$;

(ii)

$$\int_{X_2} a_R(x_1, x_2) d\mu_2(x_2) = \int_{X_1} a_R(x_1, x_2) d\mu_1(x_1) = 0; \quad (14)$$

(iii)

$$|a_R(x, y)| \leq C \mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2}, \quad (15)$$

$$\begin{aligned} & |a_R(x, y) - a_R(x', y)| \\ & \leq C \mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2} \rho_1(x, x')^{\epsilon_1}, \end{aligned} \quad (16)$$

$$\begin{aligned} & |a_R(x, y) - a_R(x, y')| \\ & \leq C \mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2-\epsilon_2} \rho_2(y, y')^{\epsilon_2}, \end{aligned} \quad (17)$$

$$\begin{aligned} & |a_R(x, y) - a_R(x', y) - a_R(x, y') - a_R(x', y')| \\ & \leq C \mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2-\epsilon_2} \\ & \quad \times \rho_1(x, x')^{\epsilon_1} \rho_2(y, y')^{\epsilon_2}. \end{aligned} \quad (18)$$

Now, we also define certain spaces of sequences of indexed by dyadic rectangles $\{Q_{\tau_1}^{k_1, \nu_1} \times Q_{\tau_2}^{k_2, \nu_2}\}$ in $X_1 \times X_2$ which will character the coefficients in our decomposition of $\dot{F}_p^{s,q}$ and $\dot{B}_p^{s,q}$. Suppose that $s = (s_1, s_2)$, $\epsilon = (\epsilon_1, \epsilon_2)$ with $-\epsilon_i < s_i < \epsilon_i$ for $i = 1, 2$, $1 < p, q < \infty$. Let R be as in Definition 11. Let $\dot{f}_p^{s,q}$ and $\dot{b}_p^{s,q}$ be the collection of sequences $S = \{s_{Q_1 \times Q_2}\}_{Q_1 \times Q_2 \in \{R\}}$ such that

$$\begin{aligned} & \|S\|_{\dot{f}_p^{s,q}} \\ & = \left\| \left\{ \sum_{k_1, k_2} \sum_{\tau_1, \tau_2} \sum_{\nu_1, \nu_2} \left(\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-s_1-1/2} \right. \right. \right. \\ & \quad \left. \left. \times \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-s_2-1/2} |s_R| \chi_R \right)^q \right\}^{1/q} \right\|_{L^p} \\ & < \infty, \\ & \|S\|_{\dot{b}_p^{s,q}} \\ & = \left\{ \sum_{k_1, k_2} \left(\sum_{\tau_1, \tau_2} \sum_{\nu_1, \nu_2} \left(\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-s_1} \right. \right. \right. \\ & \quad \left. \left. \times \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-s_2} \right)^{-1/2+1/p} |s_R|^p \right)^{q/p} \right\}^{1/q} \\ & < \infty. \end{aligned} \quad (19)$$

Smooth atomic decompositions for $\dot{B}_p^{s,q}$ and $\dot{F}_p^{s,q}$ can be stated as follows.

Theorem 12. Let R be as in Definition 11. Suppose that $1 < p, q < \infty$, $s = (s_1, s_2)$, $\epsilon = (\epsilon_1, \epsilon_2)$ with $-\epsilon_i < s_i < \epsilon_i$ for $i = 1, 2$.

(i) Then there exist sequence $S = \{s_R\} \in \dot{b}_p^{s,q}$ and ϵ -smooth atoms $\{a_R\}$ such that

$$f = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} s_R a_R \quad (20)$$

with convergence in $\dot{B}_p^{s,q}$, and

$$\|S\|_{\dot{b}_p^{s,q}} \leq C \|f\|_{\dot{B}_p^{s,q}}. \quad (21)$$

Similarly, there exist sequence $S = \{s_R\} \in \dot{f}_p^{s,q}$ and ϵ -smooth atoms $\{a_R\}$ such that

$$f = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{v_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{v_2=1}^{N(k_2, \tau_2)} s_R a_R \quad (22)$$

with convergence in $\dot{F}_p^{s,q}$ and

$$\|S\|_{\dot{F}_p^{s,q}} \leq C \|f\|_{\dot{F}_p^{s,q}}. \quad (23)$$

(ii) Conversely, if a_R are ϵ -smooth atoms and

$$f = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{v_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{v_2=1}^{N(k_2, \tau_2)} s_R a_R, \quad (24)$$

then

$$\|f\|_{\dot{B}_p^{s,q}} \leq C \|S\|_{\dot{B}_p^{s,q}}, \quad \|f\|_{\dot{F}_p^{s,q}} \leq C \|S\|_{\dot{F}_p^{s,q}}. \quad (25)$$

Proof. We first prove part (i). Let R be as in Definition 11. Suppose $f \in \dot{B}_p^{s,q}$; by Lemma 5, we have

$$\begin{aligned} f(x, y) &= \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{v_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{v_2=1}^{N(k_2, \tau_2)} \int_R D_{k_1}(x, s) \\ &\quad \times D_{k_2}(t, y) \overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t) d\mu_1(s) d\mu_2(t) \quad (26) \\ &:= \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{v_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{v_2=1}^{N(k_2, \tau_2)} s_R a_R, \end{aligned}$$

where

$$s_R = C \mu_1(Q_{\tau_1}^{k_1, v_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{-1/2} \quad (27)$$

$$\begin{aligned} &\times \int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t), \\ a_R &= C^{-1} \mu_1(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{1/2} \\ &\times \left(\int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^{-1} \quad (28) \\ &\times \int_R D_{k_1}(x, s) D_{k_2}(t, y) \overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t) \\ &\quad \times d\mu_1(s) d\mu_2(t). \end{aligned}$$

Similar to [7, 10], we can prove the convergence of the above series converges in $\dot{B}_p^{s,q}$ and $(\mathcal{G}^\circ(\beta'_1, \beta'_2, \gamma'_1, \gamma'_2))'$ with $\beta_i < \beta'_i < \epsilon_i$ and $\gamma_i < \gamma'_i < \epsilon_i$ for $i = 1, 2$. We now show that a_R satisfies conditions in Definition 11. Here we only show that a_R satisfies condition (iii) and we first prove that a_R satisfies

(15). By the size condition of D_{k_i} and $\mu_i(Q_{\tau_i}^{k_i, v_i}) \sim 2^{-k_i}$ for $i = 1, 2$, we have

$$\begin{aligned} |a_R(x, y)| &\leq C \mu_1(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{1/2} \\ &\quad \times \left(\int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^{-1} \\ &\quad \times \int_R \frac{2^{-k_1 \epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{1+\epsilon_1}} \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(y, t))^{1+\epsilon_2}} \\ &\quad \times |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \quad (29) \\ &\leq C \mu_1(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{1/2} 2^{k_1} 2^{k_2} \\ &\leq C \mu_1(Q_{\tau_1}^{k_1, v_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{-1/2}. \end{aligned}$$

We now verify (16); here, we only consider the case $\rho_1(x, x') \leq (1/2A)(2^{-k_1} + \rho_1(x, s))$ and similarly we estimate the case $\rho_1(x, x') > (1/2A)(2^{-k_1} + \rho_1(x, s))$; we get

$$\begin{aligned} |a_R(x, y) - a_R(x', y)| &\leq C \mu_1(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{1/2} \\ &\quad \times \left(\int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^{-1} \\ &\quad \times \int_R |D_{k_1}(x, s) - D_{k_1}(x', s)| |D_{k_2}(t, y)| \\ &\quad \times |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \\ &\leq C \mu_1(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, v_2})^{1/2} \\ &\quad \times \left(\int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^{-1} \quad (30) \\ &\quad \times \int_R \left(\frac{\rho_1(x, x')}{2^{-k_1} + \rho_1(x, s)} \right)^{\epsilon_1} \\ &\quad \times \frac{2^{-k_1 \epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{1+\epsilon_1}} \\ &\quad \times \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(y, t))^{1+\epsilon_2}} \\ &\quad \times |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \\ &\leq C \mu_1(Q_{\tau_1}^{k_1, v_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{k_2, v_2})^{-1/2} \rho_1(x, x')^{\epsilon_1}, \end{aligned}$$

which is a desired estimate. The proof of (17) is similar to that of (16) by symmetry. Here we omit the details.

We now estimate (18). If $\rho_1(x, x') \leq (1/2A)(2^{-k_1} + \rho_1(x, s))$ and $\rho_2(y, y') \leq (1/2A)(2^{-k_2} + \rho_2(y, t))$, then

$$\begin{aligned}
& |a_R(x, y) - a_R(x', y) - a_R(x, y') - a_R(x', y')| \\
& \leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{1/2} \\
& \quad \times \left(\int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^{-1} \\
& \quad \times \int_R |D_{k_1}(x, s) - D_{k_1}(x', s)| \\
& \quad \times |D_{k_2}(t, y) - D_{k_2}(t, y')| \\
& \quad \times |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \\
& \leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{1/2} \\
& \quad \times \left(\int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^{-1} \quad (31) \\
& \quad \times \int_R \left(\frac{\rho_1(x, x')}{2^{-k_1} + \rho_1(x, s)} \right)^{\epsilon_1} \left(\frac{\rho_2(y, y')}{2^{-k_2} + \rho_2(y, t)} \right)^{\epsilon_2} \\
& \quad \times \frac{2^{-k_1 \epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{1+\epsilon_1}} \\
& \quad \times \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(y, t))^{1+\epsilon_2}} \\
& \quad \times |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \\
& \leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2-\epsilon_2} \\
& \quad \times \rho_1(x, x')^{\epsilon_1} \rho_2(y, y')^{\epsilon_2},
\end{aligned}$$

which is the desired result. Similarly we can consider another three cases.

Then, by Hölder's inequality and Remark 10, we obtain

$$\begin{aligned}
& \|S\|_{\dot{b}_p^{s,q}} \\
& = \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} \right. \right. \\
& \quad \left(\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-s_1-1/2+1/p} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-s_2-1/2+1/p} \right. \\
& \quad \times C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2} \\
& \quad \left. \left. \times \int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \right)^p \right)^{q/p} \right\}^{1/q}
\end{aligned}$$

$$\begin{aligned}
& \leq C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} 2^{k_1 s_1 p} 2^{k_2 s_2 p} \right. \right. \\
& \quad \times \mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-p/p'} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-p/p'} \\
& \quad \times \left(\int_R \overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t) \right. \\
& \quad \left. \left. \times d\mu_1(s) d\mu_2(t) \right)^p \right)^{q/p} \right\}^{1/q} \\
& \leq C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(2^{k_1 s_1} 2^{k_2 s_2} \|\overline{D}_{k_1} \overline{D}_{k_2}(f)\|_p \right)^q \right\}^{1/q} \leq C \|f\|_{\dot{B}_p^{s,q}}. \quad (32)
\end{aligned}$$

The proof of the case $f \in \dot{F}_p^{s,q}$ goes by the analogous argument to the case that $f \in \dot{B}_p^{s,q}$, but using the vector-valued maximal inequality. Indeed, suppose that $f \in \dot{F}_p^{s,q}$; as in the proof of $\dot{B}_p^{s,q}$, we have

$$f = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} s_R a_R, \quad (33)$$

where s_R and a_R are as in (27) and (28), respectively. From [7, 10], we can also similarly obtain the series converges in the norm of $\dot{F}_p^{s,q}$ and in $(\mathcal{G}^o(\beta'_1, \beta'_2, \gamma'_1, \gamma'_2))'$ with $\beta_i < \beta'_i < \epsilon_i$ and $\gamma_i < \gamma'_i < \epsilon_i$ for $i = 1, 2$. To show the conclusion, using Fefferman-Stein vector-valued maximal inequality and Remark 10, we have

$$\begin{aligned}
& \|S\|_{\dot{F}_p^{s,q}} \\
& = \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} \left(\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-s_1-1/2} \right. \right. \right. \\
& \quad \times \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-s_2-1/2} C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2} \\
& \quad \left. \left. \times \int_R |\overline{D}_{k_1} \overline{D}_{k_2}(f)(s, t)| d\mu_1(s) d\mu_2(t) \chi_{Q_{\tau_1}^{k_1, \nu_1}} \chi_{Q_{\tau_2}^{k_2, \nu_2}} \right)^q \right\}^{1/q} \right\|_p \\
& \leq C \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(2^{k_1 s_1} 2^{k_2 s_2} M_2(M_1(\overline{D}_{k_1} \overline{D}_{k_2}(f))) \right)^q \right\}^{1/q} \right\|_p \\
& \leq C \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(2^{k_1 s_1} 2^{k_2 s_2} \overline{D}_{k_1} \overline{D}_{k_2}(f) \right)^q \right\}^{1/q} \right\|_p \leq C \|f\|_{\dot{F}_p^{s,q}}. \quad (34)
\end{aligned}$$

To prove part (ii), we need the discrete version of the Hardy-Littlewood maximal function estimate on one single factor X , which is an analogue to a result of [9]. Also see [10].

Lemma 13. Suppose that X is a space of homogeneous type as in Definition 1. Let $0 < a \leq r < \infty$ and $\lambda > r/a$. Fix μ, η with $\mu, \eta \in \mathbb{Z}$ and for any $x \in X$

$$\left(\sum_{\tau \in I_\mu} \sum_{v=1}^{N(\mu, \tau)} |s_{Q_\tau^{\mu, v}}|^r (1 + \min\{2^\eta, 2^\mu\} \rho(x, z_\tau^{\mu, v}))^{-\lambda} \right)^{1/r} \leq C 2^{(\mu-\eta)_+/a} \left(M \left(\sum_{\tau \in I_\mu} \sum_{v=1}^{N(\mu, \tau)} |s_{Q_\tau^{\mu, v}}|^a \chi_{Q_\tau^{\mu, v}} \right) (x) \right)^{1/a}, \quad (35)$$

where $z_\tau^{\mu, v}$ is the center of dyadic cube $Q_\tau^{\mu, v}$ and C depends only on λ, r, a .

We now turn to proof of part (ii). In this part, we denote $R = Q_{\tau_1}^{\mu_1, \nu_1} \times Q_{\tau_2}^{\mu_2, \nu_2}$. Suppose that $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ for $i = 1, 2$ is as in Lemma 5 and a_R is an ε -smooth atom. For $D_{k_1} D_{k_2}(a_R)(x, y)$, we consider it by four cases. We first estimate the case $k_1 > \mu_1, k_2 > \mu_2$. In this case, applying the cancellation conditions of D_{k_1} and D_{k_2} ,

$$\begin{aligned} & |D_{k_1} D_{k_2}(a_R)(x, y)| \\ &= \left| \int_{X_1 \times X_2} D_{k_1}(x, s) D_{k_2}(y, t) \right. \\ &\quad \times [a_R(s, t) - a_R(x, t) - a_R(s, y) + a_R(x, y)] \\ &\quad \times d\mu_1(s) d\mu_2(t) \Big| \\ &\leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2-\epsilon_2} \\ &\quad \times \int_{\substack{\rho_1(s, x) \leq 2C2^{-k_1} \\ \rho_2(t, y) \leq 2C2^{-k_2}}} \frac{2^{-k_1 \epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{1+\epsilon_1}} \\ &\quad \times \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(t, y))^{1+\epsilon_2}} \rho_1(x, s)^{\epsilon_1} \\ &\quad \times \rho_2(t, y)^{\epsilon_2} d\mu_1(s) d\mu_2(t) \\ &\leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2-\epsilon_2} \\ &\quad \times \int_{\substack{\rho_1(s, x) \leq 2C2^{-k_1} \\ \rho_2(t, y) \leq 2C2^{-k_2}}} d\mu_1(s) d\mu_2(t) \\ &\quad \times \frac{2^{k_1} 2^{-k_1 \epsilon_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \\ &\quad \times \frac{2^{k_2} 2^{-k_2 \epsilon_2}}{(1 + 2^{\mu_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}} \end{aligned}$$

$$\begin{aligned} &\leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \frac{2^{-(k_1 - \mu_1) \epsilon_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \\ &\quad \times \frac{2^{-(k_2 - \mu_2) \epsilon_2}}{(1 + 2^{\mu_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}}. \end{aligned} \quad (36)$$

To estimate the case $k_1 > \mu_1, k_2 \leq \mu_2$, we only estimate $\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq (1/2A)(2^{-k_2} + \rho_2(y, t))$, and afterwards we can consider the case $\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) > (1/2A)(2^{-k_2} + \rho_2(y, t))$ similarly. In this case, we have

$$\begin{aligned} & |D_{k_1} D_{k_2}(a_R)(x, y)| \\ &= \left| \int_{X_1 \times X_2} D_{k_1}(x, s) [D_{k_2}(y, t) - D_{k_2}(y, z_{\tau_2}^{\mu_2, \nu_2})] \right. \\ &\quad \times [a_R(s, t) - a_R(x, t)] d\mu_1(s) d\mu_2(t) \Big| \\ &\leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \int_{\substack{\rho_1(s, x) \leq 2C2^{-k_1} \\ \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 2C2^{-\mu_2}}} \frac{2^{-k_1 \epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{1+\epsilon_1}} \\ &\quad \times \left(\frac{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})}{2^{-k_2} + \rho_2(t, y)} \right)^{\epsilon_2} \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(t, y))^{1+\epsilon_2}} \\ &\quad \times \rho_1(x, s)^{\epsilon_1} d\mu_1(s) d\mu_2(t) \\ &\leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \int_{\substack{\rho_1(s, x) \leq 2C2^{-k_1} \\ \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 2C2^{-\mu_2}}} \frac{2^{k_1} 2^{-k_1 \epsilon_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \\ &\quad \times \frac{2^{-\mu_2 \epsilon_2} 2^{k_2} 2^{k_2 \epsilon_2}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}} d\mu_1(s) d\mu_2(t) \\ &\leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\epsilon_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \int_{\substack{\rho_1(s, x) \leq 2C2^{-k_1} \\ \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 2C2^{-\mu_2}}} d\mu_1(s) d\mu_2(t) \\ &\quad \times \frac{2^{k_1} 2^{-k_1 \epsilon_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \\ &\quad \times \frac{2^{-\mu_2 \epsilon_2} 2^{k_2} 2^{-k_2 \epsilon_2}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}} \end{aligned}$$

$$\begin{aligned}
&\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
&\quad \times \frac{2^{-(k_1 - \mu_1)\epsilon_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \\
&\quad \times \frac{2^{-(\mu_2 - k_2)(1+\epsilon_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}}.
\end{aligned} \tag{37}$$

The estimate of the case $k_1 \leq \mu_1, k_2 > \mu_2$ is similar to the case $k_1 > \mu_1, k_2 \leq \mu_2$ by symmetry and we obtain

$$\begin{aligned}
&|D_{k_1} D_{k_2}(a_R)(x, y)| \\
&\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
&\quad \times \frac{2^{-(\mu_1 - k_1)(1+\epsilon_1)}}{(1 + 2^{k_1} \rho_1(y, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \frac{2^{-(k_2 - \mu_2)\epsilon_2}}{(1 + 2^{\mu_2} \rho_2(x, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}}.
\end{aligned} \tag{38}$$

For $k_1 \leq \mu_1, k_2 \leq \mu_2$, when $\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}) \leq (1/2A)(2^{-k_1} + \rho_1(s, x))$ and $\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq (1/2A)(2^{-k_2} + \rho_2(y, t))$, then

$$\begin{aligned}
&|D_{k_1} D_{k_2}(a_R)(x, y)| \\
&= \left| \int_{X_1 \times X_2} [D_{k_1}(x, s) - D_{k_1}(x, z_{\tau_1}^{\mu_1, \nu_1})] \right. \\
&\quad \times [D_{k_2}(y, t) - D_{k_2}(y, z_{\tau_2}^{\mu_2, \nu_2})] \\
&\quad \times a_R(s, t) d\mu_1(s) d\mu_2(t) \Big| \\
&\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2}
\end{aligned}$$

$$\begin{aligned}
&\times \int_{\substack{\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}) \leq 2C2^{-\mu_1} \\ \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 2C2^{-\mu_2}}} \left(\frac{\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1})}{2^{-k_1} + \rho_1(x, s)} \right)^{\epsilon_1} \\
&\quad \times \frac{2^{-k_1\epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{1+\epsilon_1}} \left(\frac{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})}{2^{-k_2} + \rho_2(y, t)} \right)^{\epsilon_2} \\
&\quad \times \frac{2^{-k_2\epsilon_2}}{(2^{-k_2} + \rho_2(y, t))^{1+\epsilon_2}} d\mu_1(s) d\mu_2(t) \\
&\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
&\quad \times \int_{\substack{\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}) \leq 2C2^{-\mu_1} \\ \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 2C2^{-\mu_2}}} \frac{2^{-\mu_1\epsilon_1}}{(2^{-k_1} + \rho_1(x, s))^{\epsilon_1}} \\
&\quad \times \frac{2^{k_1}}{(1 + 2^{k_1} \rho_1(x, s))^{1+\epsilon_1}} \\
&\quad \times \frac{2^{-\mu_2\epsilon_2}}{(2^{-k_2} + \rho_2(y, t))^{\epsilon_2}} \\
&\quad \times \frac{2^{k_2}}{(1 + 2^{k_2} \rho_2(y, t))^{1+\epsilon_2}} d\mu_1(s) d\mu_2(t) \\
&\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
&\quad \times \frac{2^{-(\mu_1 - k_1)(1+\epsilon_1)}}{(1 + 2^{k_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\epsilon_1}} \\
&\quad \times \frac{2^{-(\mu_2 - k_2)(1+\epsilon_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\epsilon_2}}.
\end{aligned} \tag{39}$$

We can similarly get the same result for another three cases; here we omit the details.

Thus, if $f = \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1=1}^{N(\mu_1, \tau_1)} \sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2=1}^{N(\mu_2, \tau_2)} s_R a_R$, by Lemma 13 and above estimates, then

$$\begin{aligned}
&\left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(2^{k_1 s_1} 2^{k_2 s_2} \|D_{k_1} D_{k_2}(f)\|_p \right)^q \right\}^{1/q} \\
&= \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(2^{k_1 s_1} 2^{k_2 s_2} \left\| D_{k_1} D_{k_2} \left(\sum_{\mu_1, \mu_2} \sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1=1}^{N(\mu_1, \tau_1)} \sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2=1}^{N(\mu_2, \tau_2)} s_R a_R \right) \right\|_p \right)^q \right\}^{1/q} \\
&\leq C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1=-\infty}^{k_1} \sum_{\mu_2=-\infty}^{k_2} 2^{k_1 s_1} 2^{k_2 s_2} 2^{\mu_1/2} 2^{\mu_2/2} 2^{(\mu_1 - k_1)\epsilon_1} 2^{(\mu_2 - k_2)\epsilon_2} \right. \right. \\
&\quad \times \left. \left\| M_2 \left(\sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2=1}^{N(\mu_2, \tau_2)} M_1 \left(\sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1=1}^{N(\mu_1, \tau_1)} |s_R| \chi_{Q_{\tau_1}^{\mu_1, \nu_1}} \right) \chi_{Q_{\tau_2}^{\mu_2, \nu_2}} \right) \right\|_p \right)^q \right\}^{1/q}
\end{aligned}$$

$$\begin{aligned}
& + C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1 = -\infty}^{k_1} \sum_{\mu_2 = k_2 + 1}^{\infty} 2^{k_1 s_1} 2^{k_2 s_2} 2^{\mu_1/2} 2^{\mu_2/2} 2^{(\mu_1 - k_1)\epsilon_1} 2^{-(\mu_2 - k_2)} 2^{(1 + \epsilon_2)} \right. \right. \\
& \quad \times \left. \left\| 2^{(\mu_2 - k_2)} M_2 \left(\sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2 = 1}^{N(\mu_2, \tau_2)} M_1 \left(\sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1 = 1}^{N(\mu_1, \tau_1)} |s_R| \chi_{Q_{\tau_1}^{\mu_1, \nu_1}} \right) \chi_{Q_{\tau_2}^{\mu_2, \nu_2}} \right) \right\|_p \right)^q \left. \right\}^{1/q} \\
& + C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1 = k_1 + 1}^{\infty} \sum_{\mu_2 = -\infty}^{k_2} 2^{k_1 s_1} 2^{k_2 s_2} 2^{\mu_1/2} 2^{\mu_2/2} 2^{-(\mu_1 - k_1)(1 + \epsilon_1)} 2^{(\mu_2 - k_2)\epsilon_2} \right. \right. \\
& \quad \times \left. \left\| 2^{(\mu_1 - k_1)} M_2 \left(\sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2 = 1}^{N(\mu_2, \tau_2)} M_1 \left(\sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1 = 1}^{N(\mu_1, \tau_1)} |s_R| \chi_{Q_{\tau_1}^{\mu_1, \nu_1}} \right) \chi_{Q_{\tau_2}^{\mu_2, \nu_2}} \right) \right\|_p \right)^q \left. \right\}^{1/q} \\
& + C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1 = k_1 + 1}^{\infty} \sum_{\mu_2 = k_2 + 1}^{\infty} 2^{k_1 s_1} 2^{k_2 s_2} 2^{\mu_1/2} 2^{\mu_2/2} 2^{-(\mu_1 - k_1)(1 + \epsilon_1)} 2^{-(\mu_2 - k_2)(1 + \epsilon_2)} \right. \right. \\
& \quad \times \left. \left\| 2^{(\mu_1 - k_1)} 2^{(\mu_2 - k_2)} M_2 \left(\sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2 = 1}^{N(\mu_2, \tau_2)} M_1 \left(\sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1 = 1}^{N(\mu_1, \tau_1)} |s_R| \chi_{Q_{\tau_1}^{\mu_1, \nu_1}} \right) \chi_{Q_{\tau_2}^{\mu_2, \nu_2}} \right) \right\|_p \right)^q \left. \right\}^{1/q} \\
& := I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{40}$$

Let $A_\mu = (\sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1 = 1}^{N(\mu_1, \tau_1)} \sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2 = 1}^{N(\mu_2, \tau_2)} |s_R|^p \chi_R)^{1/p}$. We now consider I_1 . By Hölder's inequality and noting that $k_i \geq \mu_i$, $\epsilon_i > s_i$ for $i = 1, 2$, we have

$$\begin{aligned}
I_1 & \leq C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1 = -\infty}^{k_1} \sum_{\mu_2 = -\infty}^{k_2} 2^{k_1 s_1} 2^{k_2 s_2} 2^{\mu_1/2} 2^{\mu_2/2} 2^{(\mu_1 - k_1)\epsilon_1} 2^{(\mu_2 - k_2)\epsilon_2} A_\mu 2^{-\mu_1/p} 2^{-\mu_2/p} \right)^q \right\}^{1/q} \\
& \leq C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\mu_1 = -\infty}^{k_1} \sum_{\mu_2 = -\infty}^{k_2} 2^{-(k_1 - \mu_1)(\epsilon_1 - s_1)} 2^{-(k_2 - \mu_2)(\epsilon_2 - s_2)} \left(2^{\mu_1(1/2 + s_1 - 1/p)} 2^{\mu_2(1/2 + s_2 - 1/p)} A_\mu \right)^q \right\}^{1/q} \\
& \leq C \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \left(2^{\mu_1(1/2 + s_1 - 1/p)} 2^{\mu_2(1/2 + s_2 - 1/p)} A_\mu \right)^q \right\}^{1/q} \leq C \|S\|_{b_p^{s, q}}.
\end{aligned} \tag{41}$$

Similar to the proof of I_1 , we get desired results for I_2 , I_3 , and I_4 .

To show the second inequality of (25), let $a(x, y) = \{a_{\mu_1, \mu_2}(x, y)\}_{\mu_1, \mu_2 \in \mathbb{Z}}$ with

$$\begin{aligned} & a_{\mu_1, \mu_2}(x, y) \\ &= M_2 \left(\sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2=1}^{N(\mu_2, \tau_2)} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2-s_2} \right. \\ & \quad \times M_1 \left(\sum_{\tau_1 \in I_{\mu_1}} \sum_{\nu_1=1}^{N(\mu_1, \tau_1)} \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-s_1} |s_R| \chi_{Q_{\tau_1}^{\mu_1, \nu_1}} \right) \\ & \quad \left. \times (x) \chi_{Q_{\tau_2}^{\mu_2, \nu_2}} \right) (y). \end{aligned} \quad (42)$$

Applying Lemma 13 and Hölder's inequality, we get

$$\begin{aligned} & \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} (2^{k_1 s_1} 2^{k_2 s_2} |D_{k_1} D_{k_2}(f)|)^q \right\}^{1/q} \\ & \leq C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1=-\infty}^{k_1} \sum_{\mu_2=-\infty}^{k_2} 2^{-(k_1-\mu_1)(\epsilon_1-s_1)} 2^{-(k_2-\mu_2)(\epsilon_2-s_2)} \right. \right. \\ & \quad \left. \left. \times a_{\mu_1, \mu_2}(x, y) \right)^q \right\}^{1/q} \\ & + C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1=-\infty}^{k_1} \sum_{\mu_2=k_2+1}^{\infty} 2^{-(k_1-\mu_1)(\epsilon_1-s_1)} 2^{(k_2-\mu_2)(\epsilon_2+s_2)} \right. \right. \\ & \quad \left. \left. \times a_{\mu_1, \mu_2}(x, y) \right)^q \right\}^{1/q} \\ & + C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1=k_1+1}^{\infty} \sum_{\mu_2=-\infty}^{k_2} 2^{(k_1-\mu_1)(\epsilon_1+s_1)} 2^{-(k_2-\mu_2)(\epsilon_2-s_2)} \right. \right. \\ & \quad \left. \left. \times a_{\mu_1, \mu_2}(x, y) \right)^q \right\}^{1/q} \\ & + C \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{\mu_1=k_1+1}^{\infty} \sum_{\mu_2=k_2+1}^{\infty} 2^{(k_1-\mu_1)(\epsilon_1+s_1)} 2^{(k_2-\mu_2)(\epsilon_2+s_2)} \right. \right. \\ & \quad \left. \left. \times a_{\mu_1, \mu_2}(x, y) \right)^q \right\}^{1/q} \\ & \leq C \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \sum_{k_1=\mu_1}^{\infty} \sum_{k_2=\mu_2}^{\infty} 2^{-(k_1-\mu_1)(\epsilon_1-s_1)} 2^{-(k_2-\mu_2)(\epsilon_2-s_2)} \right. \\ & \quad \left. \times a_{\mu_1, \mu_2}(x, y) \right)^q \right\}^{1/q} \end{aligned}$$

$$\begin{aligned} & \left. \times a_{\mu_1, \mu_2}(x, y)^q \right\}^{1/q} \\ & + C \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \sum_{k_1=\mu_1}^{\infty} \sum_{k_2=-\infty}^{\mu_2} 2^{-(k_1-\mu_1)(\epsilon_1-s_1)} 2^{(k_2-\mu_2)(\epsilon_2+s_2)} \right. \\ & \quad \left. \times a_{\mu_1, \mu_2}(x, y)^q \right\}^{1/q} \\ & + C \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \sum_{k_1=-\infty}^{\mu_1} \sum_{k_2=\mu_2}^{\infty} 2^{(k_1-\mu_1)(\epsilon_1+s_1)} 2^{-(k_2-\mu_2)(\epsilon_2-s_2)} \right. \\ & \quad \left. \times a_{\mu_1, \mu_2}(x, y)^q \right\}^{1/q} \\ & + C \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \sum_{k_1=-\infty}^{\mu_1} \sum_{k_2=-\infty}^{\mu_2} 2^{(k_1-\mu_1)(\epsilon_1+s_1)} 2^{(k_2-\mu_2)(\epsilon_2+s_2)} \right. \\ & \quad \left. \times a_{\mu_1, \mu_2}(x, y)^q \right\}^{1/q} \\ & \leq C \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} a_{\mu_1, \mu_2}(x, y)^q \right\}^{1/q}. \end{aligned} \quad (43)$$

Hence, by Fefferman-Stein vector-valued maximal inequality, we obtain

$$\begin{aligned} & \|f\|_{\dot{F}_p^{s,q}} \\ &= \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} (2^{k_1 s_1} 2^{k_2 s_2} |D_{k_1} D_{k_2}(f)|)^q \right\}^{1/q} \right\|_p \\ & \leq C \left\| \left\{ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \left(\sum_{\tau_2 \in I_{\mu_1}} \sum_{\nu_1=1}^{N(\mu_1, \tau_1)} \sum_{\tau_2 \in I_{\mu_2}} \sum_{\nu_2=1}^{N(\mu_2, \tau_2)} \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-s_1} \right. \right. \right. \\ & \quad \left. \left. \times \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2-s_2} |s_R| \chi_R \right)^q \right\}^{1/q} \right\|_p \\ & \leq C \|S\|_{\dot{F}_p^{s,q}}, \end{aligned} \quad (44)$$

which completes the proof of Theorem 12. $\square$

3. Smooth Molecular Decomposition

In this section, we show smooth molecular decomposition of Triebel-Lizorkin and Besov spaces. We first give definitions of the smooth molecules as follows.

Definition 14. Suppose that collections of open subset $\{Q_{\tau_1}^{k_1, \nu_1} \subset X_1 : k_1 \in \mathbb{Z}, \tau_1 \in I_{k_1}, \nu_1 = 1, \dots, N(k_1, \tau_1)\}$ and $\{Q_{\tau_2}^{k_2, \nu_2} \subset X_2 : k_2 \in \mathbb{Z}, \tau_2 \in I_{k_2}, \nu_2 = 1, \dots, N(k_2, \tau_2)\}$ satisfy the conditions in Lemma 5. Let $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ and $R = Q_{\tau_1}^{k_1, \nu_1} \times Q_{\tau_2}^{k_2, \nu_2}$, and $z_{\tau_1}^{k_1, \nu_1}, z_{\tau_2}^{k_2, \nu_2}$ are the centers of $Q_{\tau_1}^{k_1, \nu_1}, Q_{\tau_2}^{k_2, \nu_2}$, respectively. A function m_R defined on $X_1 \times X_2$ is said to be a (β, γ) -smooth molecule for R if

(i)

$$\int_{X_2} m_R(x_1, x_2) d\mu_2(x_2) = \int_{X_1} m_R(x_1, x_2) d\mu_1(x_1) = 0; \quad (45)$$

(ii)

$$\begin{aligned} |m_R(x, y)| &\leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2} \\ &\quad \times (1 + 2^{-k_1} \rho_1(x, z_{\tau_1}^{k_1, \nu_1}))^{-(1+\gamma_1)} \\ &\quad \times (1 + 2^{-k_2} \rho_2(y, z_{\tau_2}^{k_2, \nu_2}))^{-(1+\gamma_2)}, \\ |m_R(x, y) - m_R(x', y)| \\ &\leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2-\beta_1} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2} \\ &\quad \times \rho_1(x, x')^{\beta_1} (1 + 2^{-k_2} \rho_2(y, z_{\tau_2}^{k_2, \nu_2}))^{-(1+\gamma_2)} \\ &\quad \times \left\{ (1 + 2^{-k_1} \rho_1(x, z_{\tau_1}^{k_1, \nu_1}))^{-(1+\gamma_1)} \right. \\ &\quad \left. + (1 + 2^{-k_1} \rho_1(x', z_{\tau_1}^{k_1, \nu_1}))^{-(1+\gamma_1)} \right\}, \\ |m_R(x, y) - m_R(x, y')| \\ &\leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2-\beta_2} \\ &\quad \times \rho_2(y, y')^{\beta_2} (1 + 2^{-k_1} \rho_1(x, z_{\tau_1}^{k_1, \nu_1}))^{-(1+\gamma_1)} \\ &\quad \times \left\{ (1 + 2^{-k_2} \rho_2(y, z_{\tau_2}^{k_2, \nu_2}))^{-(1+\gamma_2)} \right. \\ &\quad \left. + (1 + 2^{-k_2} \rho_2(y', z_{\tau_2}^{k_2, \nu_2}))^{-(1+\gamma_2)} \right\}, \\ |m_R(x, y) - m_R(x', y) - m_R(x, y') - m_R(x', y')| \\ &\leq C\mu_1(Q_{\tau_1}^{k_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{k_2, \nu_2})^{-1/2} \\ &\quad \times \rho_1(x, x')^{\beta_1} \rho_2(y, y')^{\beta_2} \\ &\quad \times \left\{ (1 + 2^{-k_1} \rho_1(x, z_{\tau_1}^{k_1, \nu_1}))^{-(1+\gamma_1)} \right. \\ &\quad \left. + (1 + 2^{-k_1} \rho_1(x', z_{\tau_1}^{k_1, \nu_1}))^{-(1+\gamma_1)} \right\} \end{aligned}$$

$$\begin{aligned} &\times \left\{ (1 + 2^{-k_2} \rho_2(y, z_{\tau_2}^{k_2, \nu_2}))^{-(1+\gamma_2)} \right. \\ &\quad \left. + (1 + 2^{-k_2} \rho_2(y', z_{\tau_2}^{k_2, \nu_2}))^{-(1+\gamma_2)} \right\}. \end{aligned} \quad (46)$$

Now we can state the smooth molecular decomposition for Triebel-Lizorkin and Besov spaces.

Theorem 15. Let R be as in Definition 11. Suppose that for $i = 1, 2$, $s = (s_1, s_2)$, $\epsilon = (\epsilon_1, \epsilon_2)$, $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$, $-\epsilon_i < s_i < \epsilon_i$, $1 < p, q < \infty$, m_R is a (β, γ) -smooth molecule with $\max(0, s_i) < \beta_i < \epsilon_i$ and $\max(0, -s_i) < \gamma_i < \epsilon_i$ for $i = 1, 2$. If

$$f = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} s_R m_R, \quad (47)$$

then $\|f\|_{\dot{B}_p^{s,q}} \leq C\|S\|_{\dot{b}_p^{s,q}}$ and $\|f\|_{\dot{F}_p^{s,q}} \leq C\|S\|_{\dot{f}_p^{s,q}}$.

Proof. Suppose that $R = Q_{\tau_1}^{\mu_1, \nu_1} \times Q_{\tau_2}^{\mu_2, \nu_2}$. From the proof of Theorem 12, we only need to claim the following estimates. If $k_1 \leq \mu_1, k_1 \leq \mu_2$, then

$$\begin{aligned} &|D_{k_1} D_{k_2}(m_R)(x, y)| \\ &\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \frac{2^{-(\mu_1-k_1)(1+\gamma_1)}}{(1 + 2^{k_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \frac{2^{-(\mu_2-k_2)(1+\gamma_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}. \end{aligned} \quad (48)$$

If $k_1 > \mu_1, k_1 \leq \mu_2$, then

$$\begin{aligned} &|D_{k_1} D_{k_2}(m_R)(x, y)| \\ &\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \frac{2^{-(k_1-\mu_1)\beta_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \frac{2^{-(\mu_2-k_2)(1+\gamma_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}. \end{aligned} \quad (49)$$

If $k_1 \leq \mu_1, k_1 > \mu_2$, then

$$\begin{aligned} &|D_{k_1} D_{k_2}(m_R)(x, y)| \\ &\leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ &\quad \times \frac{2^{-(\mu_1-k_1)(1+\gamma_1)}}{(1 + 2^{k_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \frac{2^{-(k_2-\mu_2)\beta_2}}{(1 + 2^{\mu_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}. \end{aligned} \quad (50)$$

If $k_1 > \mu_1, k_1 > \mu_2$, then

$$\begin{aligned} & \left| D_{k_1} D_{k_2} (m_R) (x, y) \right| \\ & \leq C \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ & \quad \times \frac{2^{-(k_1 - \mu_1) \beta_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1 + \gamma_1}} \frac{2^{-(k_2 - \mu_2) \beta_2}}{(1 + 2^{\mu_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1 + \gamma_2}}. \end{aligned} \quad (51)$$

We now prove the above estimates. Let $W_1 = \{x : \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}) \leq 4C_1 2^{-k_1}\}$, $W_2 = \{y : \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}) \leq 4C_2 2^{-k_2}\}$, $W_3 = \{x : \rho_1(x, s) \leq 4C_1 2^{-k_1}\}$, and $W_4 = \{y : \rho_2(y, t) \leq 4C_2 2^{-k_2}\}$, where $2C_i$ is the smallest constant which satisfies $\text{supp } D_{k_i}(x, y) \subset \{(x, y) \in X_i \times X_i : \rho_i(x, y) \leq 2C_i 2^{-k_i}\}$ for $i = 1, 2$, and W_1^C, W_2^C, W_3^C , and W_4^C are their complements, respectively. We first give the proof of (48). Note that $k_1 \leq \mu_1, k_2 \leq \mu_2$, and then

$$\begin{aligned} & \left| D_{k_1} D_{k_2} (m_R) (x, y) \right| \\ & \leq \int_{X_1 \times X_2} \left| D_{k_1} (x, s) - D_{k_1} (x, z_{\tau_1}^{\mu_1, \nu_1}) \right| \\ & \quad \times \left| D_{k_2} (y, t) - D_{k_2} (y, z_{\tau_2}^{\mu_2, \nu_2}) \right| \\ & \quad \times |m_R(s, t)| d\mu_1(s) d\mu_2(t) \\ & \leq \left(\chi_{W_1} \chi_{W_2} \int_{W_3 \times W_4} + \chi_{W_1^C} \chi_{W_2} \int_{W_3 \times W_4} \right. \\ & \quad + \chi_{W_1} \chi_{W_2^C} \int_{W_3 \times W_4} + \chi_{W_1} \chi_{W_2} \int_{W_3^C \times W_4} \\ & \quad + \chi_{W_1} \chi_{W_2} \int_{W_3 \times W_4^C} + \chi_{W_1^C} \chi_{W_2^C} \int_{W_3 \times W_4} \\ & \quad + \chi_{W_1^C} \chi_{W_2} \int_{W_3^C \times W_4} + \chi_{W_1^C} \chi_{W_2} \int_{W_3 \times W_4^C} \\ & \quad + \chi_{W_1} \chi_{W_2^C} \int_{W_3 \times W_4^C} + \chi_{W_1} \chi_{W_2^C} \int_{W_3^C \times W_4^C} \\ & \quad + \chi_{W_1} \chi_{W_2} \int_{W_3^C \times W_4^C} + \chi_{W_1^C} \chi_{W_2^C} \int_{W_3^C \times W_4^C} \\ & \quad \left. + \chi_{W_1} \chi_{W_2^C} \int_{W_3 \times W_4^C} + \chi_{W_1^C} \chi_{W_2^C} \int_{W_3^C \times W_4^C} \right) \\ & \quad \times \left| D_{k_1} (x, s) - D_{k_1} (x, z_{\tau_1}^{\mu_1, \nu_1}) \right| \\ & \quad \times \left| D_{k_2} (y, t) - D_{k_2} (y, z_{\tau_2}^{\mu_2, \nu_2}) \right| \\ & \quad \times |m_R(s, t)| d\mu_1(s) d\mu_2(t) \\ & := (48)_1 + (48)_2 + \dots + (48)_{16}. \end{aligned} \quad (52)$$

By the support of D_{k_i} and the definition of W_j for $i = 1, 2$ and $j = 1, 2, 3, 4$, we can easily get $(48)_7 = (48)_{10} = (48)_{12} = (48)_{13} = (48)_{14} = (48)_{15} = (48)_{16} = 0$. Here we only estimate the first five terms, and we can similarly estimate other terms. Now, we first prove $(48)_1$:

$$\begin{aligned} |(48)_1| & \leq C 2^{k_1(1+\epsilon_1)} 2^{k_2(1+\epsilon_2)} \\ & \quad \times \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \chi_{W_1} \chi_{W_2} \\ & \quad \times \int_{W_3 \times W_4} \frac{\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1})^{\epsilon_1}}{(1 + 2^{\mu_1} \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{(1+\gamma_1)}} \\ & \quad \times \frac{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2}}{(1 + 2^{\mu_2} \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}))^{(1+\gamma_2)}} d\mu_1(s) d\mu_2(t) \\ & \leq C \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ & \quad \times 2^{k_1(1+\epsilon_1)} 2^{k_2(1+\epsilon_2)} 2^{-\mu_1(1+\gamma_1)} 2^{\mu_2(1+\gamma_2)} \chi_{W_1} \chi_{W_2} \\ & \quad \times \int_{\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}) \leq 8C_1 2^{-k_1}} \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1})^{\epsilon_1 - \gamma_1 - 1} d\mu_1(s) \\ & \quad \times \int_{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 8C_2 2^{-k_2}} \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2 - \gamma_2 - 1} d\mu_2(t) \\ & \leq C \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ & \quad \times 2^{-(\mu_1 - k_1)(1+\gamma_1)} 2^{-(\mu_2 - k_2)(1+\gamma_2)} \chi_{W_1} \chi_{W_2} \\ & \leq C \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\ & \quad \times \frac{2^{-(\mu_1 - k_1)(1+\gamma_1)}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \\ & \quad \times \frac{2^{-(\mu_2 - k_2)(1+\gamma_2)}}{(1 + 2^{\mu_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}, \end{aligned} \quad (53)$$

which is the desired estimate.

We now estimate $(48)_2$ by

$$\begin{aligned} |(48)_2| & \leq C 2^{k_1} 2^{k_2(1+\epsilon_2)} \mu_1 (Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2 (Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \chi_{W_1^C} \chi_{W_2} \\ & \quad \times \int_{\substack{\rho_1(x, s) \\ \leq 2C_1 2^{-k_1} \\ \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}) \\ \geq \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1})/2}} \left(1 \times \left((1 + 2^{\mu_1} \right. \right. \\ & \quad \left. \left. \times \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1})^{(1+\gamma_1)} \right)^{-1} \right) d\mu_1(s) \end{aligned}$$

$$\begin{aligned}
& \times \int_{W_4} \frac{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2}}{(1 + 2^{\mu_2} \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}))^{(1+\gamma_2)}} d\mu_2(t) \\
& \leq C 2^{k_1} 2^{k_2(1+\epsilon_2)} \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \\
& \quad \times \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \chi_{W_1^C} \chi_{W_2} \\
& \quad \times \frac{2^{-k_1}}{2^{\mu_1(1+\gamma_1)} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1})^{(1+\gamma_1)}} 2^{-\mu_2(1+\gamma_2)} \\
& \quad \times \int_{W_4} \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2-1-\gamma} d\mu_2(t) \\
& \leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \frac{2^{-(\mu_1-k_1)(1+\gamma_1)}}{(1 + 2^{k_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \\
& \quad \times \frac{2^{-(\mu_2-k_2)(1+\gamma_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}, \tag{54}
\end{aligned}$$

which verifies (48)₂.

The verification of (48)₃ is similar to that of (48)₂ by symmetry.

Similarly we have

$$\begin{aligned}
|(48)_4| & \leq C 2^{k_1} 2^{k_2(1+\epsilon_2)} \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \chi_{W_1} \chi_{W_2} \\
& \quad \times \int_{\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}) \geq 2C_1 2^{-k_1}} \frac{1}{(2^{\mu_1} \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{(1+\gamma_1)}} d\mu_1(s) \\
& \quad \times \int_{W_4} \frac{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2}}{(1 + 2^{\mu_2} \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}))^{(1+\gamma_2)}} d\mu_2(t) \\
& \leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \chi_{W_1} \chi_{W_2} \\
& \quad \times 2^{-(\mu_1-k_1)(1+\gamma_1)} 2^{-(\mu_2-k_2)(1+\gamma_2)} \\
& \leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \frac{2^{-(\mu_1-k_1)(1+\gamma_1)}}{(1 + 2^{k_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \\
& \quad \times \frac{2^{-(\mu_2-k_2)(1+\gamma_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}, \tag{55}
\end{aligned}$$

which is also a desired estimate.

By an analogous argument to (48)₄, we can verify (48)₅. We now prove (49)₁

$$\begin{aligned}
& |D_{k_1} D_{k_2}(m_R)(x, y)| \\
& = \left| \int_{X_1 \times X_2} D_{k_1}(x, s) D_{k_2}(y, t) m_R(s, t) d\mu_1(s) d\mu_2(t) \right|
\end{aligned}$$

$$\begin{aligned}
& \leq \int_{X_1 \times X_2} |D_{k_1}(x, s)| |D_{k_2}(y, t) - D_{k_2}(y, z_{\tau_2}^{\mu_2, \nu_2})| \\
& \quad \times |m_R(s, t) - m_R(x, t)| d\mu_1(s) d\mu_2(t) \\
& = \left(\chi_{W_2} \int_{X_3 \times W_4} + \chi_{W_2} \int_{X_3 \times W_4^C} + \chi_{W_2}^C \int_{X_3 \times W_4} \right) \\
& \quad \times |D_{k_1}(x, s)| |D_{k_2}(y, t) - D_{k_2}(y, z_{\tau_2}^{\mu_2, \nu_2})| \\
& \quad |m_R(s, t) - m_R(x, t)| d\mu_1(s) d\mu_2(t) \\
& := (49)_1 + (49)_2 + (49)_3. \tag{56}
\end{aligned}$$

For (49)₁, we have

$$\begin{aligned}
& |(49)_1| \\
& \leq \chi_{W_2} 2^{k_1} 2^{k_2(1+\epsilon_2)} \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\beta_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \int_{W_3 \times W_4} \rho_1(x, s)^{\beta_1} \left(\frac{1}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right. \\
& \quad \left. + \frac{1}{(1 + 2^{\mu_1} \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right) \\
& \quad \times \frac{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2}}{(1 + 2^{\mu_2} \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}} d\mu_1(s) d\mu_2(t) \\
& \leq \chi_{W_2} 2^{k_1} 2^{k_2(1+\epsilon_2)} \\
& \quad \times \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\beta_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \int_{W_3} \left(\frac{\rho_1(x, s)^{\beta_1}}{(1 + 2^{\mu_1} \rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right. \\
& \quad \left. + \frac{\rho_1(x, s)^{\beta_1}}{(1 + 2^{\mu_1} \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right) d\mu_1(s) \\
& \quad \times \int_{\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}) \leq 8C_2 2^{-k_2}} 2^{-\mu_2(1+\gamma_2)} \\
& \quad \times \rho_2(t, z_{\tau_2}^{\mu_2, \nu_2})^{\epsilon_2-1-\gamma_2} d\mu_2(t) \\
& \leq C \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \frac{2^{-(\mu_2-k_2)(1+\gamma_2)}}{(1 + 2^{k_2} \rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}} \\
& \quad \times \left(2^{\mu_1 \beta_1 + k_1} \right. \\
& \quad \times \int_{\rho_1(x, s) \leq 4C_1 2^{-k_1}} \frac{\rho_1(x, s)^{\beta_1}}{(1 + 2^{\mu_1} \rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} d\mu_1(s)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2^{-(k_1-\mu_1)\beta_1}}{(1+2^{\mu_1}\rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \Bigg) \\
& \leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \frac{2^{-(k_1-\mu_1)\beta_1}}{(1+2^{\mu_1}\rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \\
& \quad \times \frac{2^{-(\mu_2-k_2)(1+\gamma_2)}}{(1+2^{\mu_2}\rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}, \tag{57}
\end{aligned}$$

since $1+2^{\mu_1}\rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}) \leq 2(2C_1+1)(1+2^{\mu_1}\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))$ by $k_1 > \mu_1$, which is the proof of (48)₁.

The verification of terms (49)₂ and (49)₃ is similar to that of (49)₁ and proof of (50) is similar to that of (49) by symmetry.

We now prove (51). In this case, we have

$$\begin{aligned}
& |D_{k_1} D_{k_2}(m_R)(x, y)| \\
& \leq \int_{X_1 \times X_2} |D_{k_1}(x, s)| |D_{k_2}(y, t)| \\
& \quad \times |m_R(s, t) - m_R(x, t) \\
& \quad - m_R(s, y) - m_R(x, y)| d\mu_1(s) d\mu_2(t) \\
& \leq 2^{k_1} 2^{k_2} \mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2-\beta_1} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2-\beta_2} \\
& \quad \times \int_{W_3} \rho_1(x, s)^{\beta_1} \left(\frac{1}{(1+2^{\mu_1}\rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right. \\
& \quad \left. + \frac{1}{(1+2^{\mu_1}\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right) d\mu_1(s) \\
& \quad \times \int_{W_4} \rho_2(y, t)^{\beta_2} \left(\frac{1}{(1+2^{\mu_2}\rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}} \right. \\
& \quad \left. + \frac{1}{(1+2^{\mu_2}\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}} \right) d\mu_2(t) \\
& \leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \left(\frac{2^{-(k_1-\mu_1)\beta_1}}{(1+2^{\mu_1}\rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \right. \\
& \quad \left. + 2^{\mu_1\beta_1+k_1} \int_{W_3} \frac{\rho_1(x, s)^{\beta_1}}{(1+2^{\mu_1}\rho_1(s, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} d\mu_1(s) \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{2^{-(k_2-\mu_2)\beta_2}}{(1+2^{\mu_2}\rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}} \right. \\
& \quad \left. + 2^{\mu_2\beta_2+k_2} \int_{W_4} \frac{\rho_2(y, t)^{\beta_2}}{(1+2^{\mu_2}\rho_2(t, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}} d\mu_2(t) \right) \\
& \leq C\mu_1(Q_{\tau_1}^{\mu_1, \nu_1})^{-1/2} \mu_2(Q_{\tau_2}^{\mu_2, \nu_2})^{-1/2} \\
& \quad \times \frac{2^{-(k_1-\mu_1)\beta_1}}{(1+2^{\mu_1}\rho_1(x, z_{\tau_1}^{\mu_1, \nu_1}))^{1+\gamma_1}} \\
& \quad \times \frac{2^{-(k_2-\mu_2)\beta_2}}{(1+2^{\mu_2}\rho_2(y, z_{\tau_2}^{\mu_2, \nu_2}))^{1+\gamma_2}}. \tag{58}
\end{aligned}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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