Research Article

Sobolev Spaces on Locally Compact Abelian Groups: Compact Embeddings and Local Spaces

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We continue our research on Sobolev spaces on locally compact abelian (LCA) groups motivated by our work on equations with infinitely many derivatives of interest for string theory and cosmology. In this paper, we focus on compact embedding results and we prove an analog for LCA groups of the classical Rellich lemma and of the Rellich-Kondrachov compactness theorem. Furthermore, we introduce Sobolev spaces on subsets of LCA groups and study its main properties, including the existence of compact embeddings into L^p -spaces.

1. Introduction

In this paper we continue our research on Sobolev spaces on locally compact abelian groups [1, 2], and we examine analogs of the Rellich lemma and the Rellich-Kondrachov compactness theorem. Sobolev spaces are well understood on domains of $\mathbb{R}^n$; see [3, 4], compact Riemannian manifolds [5, 6], and metric measure spaces [7–9]. There are also some works on Sobolev spaces in the p -adic context; see [10, 11] and references therein and in special cases of locally compact groups such as the Heisenberg group [12].

We are interested in Sobolev spaces in this general context due to our work on nonlinear equations “in infinitely many derivatives” of interest for contemporary physical theories: in [13–15], two of the present authors in collaboration with H. Prado have investigated the existence of regular solutions to the *generalized Euclidean Bosonic string equation*

$$\Delta e^{-c\Delta} \phi = U(x, \phi), \quad c > 0 \quad (1)$$

and some of its generalizations, and, in [16, 17], the same researchers have developed a functional calculus appropriate for the study of the initial value problem for “ordinary” equations of the form

$$f(\partial_t) \phi = J(t). \quad (2)$$

Equations such as (1) and (2) are specially interesting for string theory and cosmology; see [18–21] and references therein. These two areas are undergoing such a fast development that it seems important to understand (1) and (2) in contexts beyond the usual geometric arena of analysis on (Riemannian) manifolds. We think that topological groups are a natural testing ground for gathering a better understanding of (1) and (2). For instance, this setting would allow us to consider (1) for functions ϕ on finite spaces with group structure (see, e.g., [22]), or for functions depending on an infinite number of independent variables. On the other hand, this generalization makes it necessary to develop a theory of Sobolev spaces on LCA groups appropriate for the study of nonlocal equations along the lines of [13–15]. It is indeed possible to do so, essentially because of the existence of group structure and the availability of Fourier transform.

We introduced Sobolev spaces on LCA groups in [1]. In that reference, we proved analogs of the Sobolev embedding and Rellich-Kondrachov theorems, and we used these results to prove the existence of regular solutions to (1) on compact abelian groups. Then in [2], we considered a version of the classical Rellich lemma and presented another theorem on regular solutions to (1). Now, our version of the Rellich lemma appearing in [2] relies on a technical assumption on the structure of the group of characters of the given

group G which limits its applicability. In this paper, we remove this assumption and prove a version of the Rellich lemma which can be applied in great generality, and we also improve our original Rellich-Kondrachov theorem proven in [1]. Moreover, we introduce Sobolev spaces on subsets of LCA groups, in analogy with the Sobolev spaces defined on domains of $\mathbb{R}^n$. As in this classical case, we expect these spaces to be useful in the study of differential equations and other applications [23].

We organize this paper as follows. In Section 2, we recall our definition of Sobolev spaces and our previous embedding and compactness results. In Section 3, we state and prove our new compactness results, and in Section 4, we discuss Sobolev spaces on subsets of LCA groups.

We use standard notations from harmonic analysis [24, 25]. Let us fix a locally compact abelian group G . We denote by dx the unique Haar measure of G and by $G^\wedge$ the dual group of the group G ; that is, $G^\wedge$ is the locally compact abelian group of all continuous group homomorphisms from G to the circle group T . The L^p spaces over G are defined as usual:

$$L^p(G) = \left\{ f : G \rightarrow \mathbb{C} : \int_G |f(x)|^p dx < \infty \right\}, \quad (3)$$

and we set

$$\|f\|_{L^p(G)} = \left(\int_G |f(x)|^p dx \right)^{1/p}. \quad (4)$$

We also recall that the Fourier transform on G is defined as follows: if $f \in L^1(G)$, then its Fourier transform is the function $\hat{f} : G^\wedge \rightarrow \mathbb{C}$ given by

$$\hat{f}(\xi) = \int_G \overline{\xi(x)} f(x) dx. \quad (5)$$

We consider general LCA groups in Section 2, but we restrict ourselves to compact abelian groups when proving compactness results in Section 3.

2. Sobolev Spaces

We introduce Sobolev spaces following our previous papers [1, 2]. Our definition uses essentially the Fourier transform for LCA groups and, as explained in [1], it generalizes naturally the standard notions of Sobolev spaces in the case of $\mathbb{T}^n$ and $\mathbb{R}^n$; see [26] and [4, Chapter 4].

We denote by Γ the set

$$\Gamma = \left\{ \gamma : G^\wedge \rightarrow [0, \infty) : \exists c_\gamma \forall \alpha, \beta \in G^\wedge \gamma(\alpha\beta) \leq c_\gamma [\gamma(\alpha) + \gamma(\beta)] \right\}. \quad (6)$$

Definition 1. Let us fix a map $\gamma \in \Gamma$ and a nonnegative real number s . We will say that $f \in L^2(G)$ belongs to the Sobolev space $H_\gamma^s(G)$ if the following integral is finite:

$$\int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 d\xi. \quad (7)$$

Moreover, for $f \in H_\gamma^s(G)$, its norm $\|f\|_{H_\gamma^s(G)}$ is defined as follows:

$$\|f\|_{H_\gamma^s(G)} = \left(\int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}. \quad (8)$$

Remark 2. A particular instance of Definition 1 appears in the paper [26] by Feichtinger and Werther. Another particular case of our definition is in [27]. We also note that in p -adic analysis, Sobolev spaces are defined in a way analogous to our Definition 1: if we take $\gamma(\xi) = \|\xi\|_p$, where $\|\cdot\|_p$ is a p -adic norm on $\mathbb{Q}_p^n \simeq \mathbb{Q}_p^{n^*}$, then (7) and (8) allow us to recover the p -adic Sobolev spaces considered in [11].

Remark 3. The fact that $\gamma \in \Gamma$ implies that our spaces $H_\gamma^s(G)$ are Banach algebras under some assumptions on s ; see our previous paper [1].

We recall the following results proven in [1].

Proposition 4. Let G be a locally compact abelian group. One has the following.

- (1) Consider $H_\gamma^s(G) \hookrightarrow L^2(G)$. Moreover, for each $f \in H_\gamma^s(G)$, the following inequality holds:

$$\|f\|_{L^2(G)} \leq \|f\|_{H_\gamma^s(G)}. \quad (9)$$

- (2) If $s > \sigma$, then $H_\gamma^s(G) \hookrightarrow H_\gamma^\sigma(G)$. Moreover, the following inequality holds:

$$\|f\|_{H_\gamma^\sigma(G)} \leq \|f\|_{H_\gamma^s(G)}. \quad (10)$$

Theorem 5. Let G be a locally compact abelian group. One has the following.

- (1) If $1/(1 + \gamma(\cdot)^2) \in L^s(G^\wedge)$, then

$$H_\gamma^s(G) \hookrightarrow C(G), \quad (11)$$

in which $C(G)$ denotes the space of continuous complex-valued functions on G . Moreover, there exists a constant $C(\gamma, s)$ such that for each $f \in H_\gamma^s(G)$, the following inequality holds:

$$\|f\|_{C(G)} \leq C(\gamma, s) \|f\|_{H_\gamma^s(G)}. \quad (12)$$

- (2) If $\alpha > s$ and $1/(1 + \gamma(\cdot)^2) \in L^\alpha(G^\wedge)$, then

$$H_\gamma^s(G) \hookrightarrow L^{\alpha^*}(G), \quad (13)$$

where $\alpha^* = 2\alpha/(\alpha - s)$. Moreover, there exists a constant $D(\gamma, s)$ such that for each $f \in H_\gamma^s(G)$, the following inequality holds:

$$\|f\|_{L^{\alpha^*}(G)} \leq D(\gamma, s) \|f\|_{H_\gamma^s(G)}. \quad (14)$$

Theorem 5 is our version of the classical Sobolev embedding theorem appearing, for instance, in [3] for the case $G = \mathbb{R}^n$.

3. Compact Embedding

We recall that the notation $A \hookrightarrow B$ means that the space A is compactly embedded in B . We begin with our refined version of the classical Rellich lemma.

Theorem 6. *Let G be a compact group. If $\lim_{\xi \rightarrow \infty} \gamma(\xi) = \infty$, that is, for each $M > 0$ there exists finite set U such that for any $\xi \in G^\wedge \setminus U$, one has $\gamma(\xi) \geq M$, then for all $s > \delta$,*

$$H_\gamma^s(G) \hookrightarrow H_\gamma^\delta(G). \quad (15)$$

Proof. We begin the proof stating a classical fact (see, e.g., [28]) on the characterization of precompact sets in $L^p(H)$ spaces.

Theorem 7 (see [28]). *Let Φ be a family of functions in $L^p(H)$, $1 \leq p < \infty$. Then Φ is compact in $L^p(H)$ if and only if the following conditions hold.*

- (i) *There exists $M > 0$ such that $\|\phi\|_{L^p(H)} \leq M$ for all $\phi \in \Phi$.*
- (ii) *For every $\varepsilon > 0$, there exists compact set K in G such that for each $\phi \in \Phi$,*

$$\|\phi\|_{L^p(H \setminus K)} \leq \varepsilon. \quad (16)$$

- (iii) *For all $\varepsilon > 0$, there exists unit neighborhood V such that for all $x \in V$ and all $\phi \in \Phi$,*

$$\|L_x \phi - \phi\|_{L^p(H)} \leq \varepsilon. \quad (17)$$

Now we return to the proof of Theorem 6. Let f_n be a bounded sequence in $H_\gamma^s(G)$, $\|f_n\|_{H_\gamma^s(G)} \leq M$. We need to show that there exists subsequence that converges strongly in $H_\gamma^\delta(G)$. We will prove this fact by showing that the following sequence

$$g_n(\xi) = |\widehat{f}_n(\xi)| (1 + \gamma^2(\xi))^{\delta/2} \quad (18)$$

is compact in $L^2(G^\wedge)$. We use Weil's theorem: since $\delta < s$, we get

$$\|g_n\|_{L^2(G^\wedge)} \leq \|f_n\|_{H_\gamma^s(G)} \leq M, \quad (19)$$

and so (i) holds. Now we consider condition (ii). Let us fix $\varepsilon > 0$. We consider two cases: if $G^\wedge$ is finite group, then we can take simply $K = G^\wedge$ and condition (ii) is satisfied. On the other hand, if the dual group $G^\wedge$ is infinite, it is enough to recall that if G is a compact group, then by the Pontryagin duality theorem, its dual $G^\wedge$ is discrete and therefore every compact set must be finite. From our assumption, we can find a compact set K such that

$$\frac{1}{(1 + \gamma^2(\xi))^{s-\delta}} \leq \frac{\varepsilon}{M^2}, \quad \text{for all } \xi \in G^\wedge \setminus K. \quad (20)$$

Hence,

$$\begin{aligned} & \int_{G^\wedge \setminus K} |\widehat{f}_n(\xi)|^2 (1 + \gamma^2(\xi))^\delta d\xi \\ &= \int_{G^\wedge \setminus K} |\widehat{f}_n(\xi)|^2 \frac{(1 + \gamma^2(\xi))^s}{(1 + \gamma^2(\xi))^{s-\delta}} d\xi \\ &\leq \sup_{\xi \in G^\wedge \setminus K} \frac{1}{(1 + \gamma^2(\xi))^{s-\delta}} \int_{G^\wedge \setminus K} |\widehat{f}_n(\xi)|^2 (1 + \gamma^2(\xi))^s d\xi \\ &\leq \|f_n\|_{H_\gamma^s(G)}^2 \sup_{\xi \in G^\wedge \setminus K} \frac{1}{(1 + \gamma^2(\xi))^{s-\delta}} \leq \varepsilon, \end{aligned} \quad (21)$$

and so (ii) holds. It remains to check condition (iii). Since $G^\wedge$ is discrete and each set is open, we can take $V = \{e\}$, where e is unit in $G^\wedge$. Thus, condition (iii) is satisfied and Theorem 6 follows from Weil's result. $\square$

Theorem 6 appears in our previous paper [2] under the additional assumption that the dual group $G^\wedge$ is countable.

Now we consider embeddings of $H_\gamma^s(G)$ into $C(G)$ and $L^p(G)$. We proved in [1] that $H_\gamma^s(G)$ is continuously embedded in $C(G)$. We prove in Theorem 10 below that if G is compact, then $H_\gamma^s \hookrightarrow C(G)$, and finally in Theorem 11, we consider a version of the Rellich-Kondrachov which refines an analogous result from [1]. We need the following lemma.

Lemma 8. *Let H be a discrete group and $f \in L^1(H)$. Then for every $\varepsilon > 0$, there exists a finite set U such that for any $h \in H \setminus U$, one has $|f(h)| \leq \varepsilon$.*

Proof. Let us suppose that the theorem is not true, so that there is a number $\varepsilon > 0$ such that for every finite set U , there exists $h \in H \setminus U$ with $|f(h)| \geq \varepsilon$. Let U_1 be a finite set and let $h_1 \in H \setminus U_1$ be such that $|f(h_1)| \geq \varepsilon$. We define $U_2 = U_1 \cup \{h_1\}$. Since U_2 is finite, there exists $h_2 \in H \setminus U_2$ such that $|f(h_2)| \geq \varepsilon$. By induction, we get a sequence of sets $\{U_i\}$ and $\{h_i\}$ for $i = 1, 2, \dots$ such that $U_{i+1} = U_i \cup \{h_i\}$ and for each i , we have $|f(h_i)| \geq \varepsilon$. Since Haar measure on discrete groups is a multiple of the counting measure, we get

$$\begin{aligned} \int_H |f(h)| dh &= \int_{H \setminus U} |f(h)| dh + \int_U |f(h)| dh \\ &\geq \sum_{i=1}^{\infty} |f(h_i)| \geq \sum_{i=1}^{\infty} \varepsilon = +\infty, \end{aligned} \quad (22)$$

where $U = \bigcup_{i=1}^{\infty} U_i$, and this is a contradiction. $\square$

Remark 9. We note that if H is countable, the proof of Lemma 8 is elementary: $\int_H |f(h)| dh = a \sum_{i=1}^{\infty} |f(h_i)|$ for some $a \in \mathbb{C}$ and $\sum_{i=1}^{\infty} |f(h_i)| < \infty$ implies that given $\varepsilon > 0$, there exists $K > 0$ such that $\sum_{i=K}^{\infty} |f(h_i)| < \varepsilon$. The result then follows.

Theorem 10. *If G is compact, $1/(1 + \gamma^2(\cdot)) \in L^\alpha(G^\wedge)$, and $s < \alpha$ then*

$$H_\gamma^s(G) \hookrightarrow C(G). \quad (23)$$

Proof. Using Lemma 8 for the function $1/(1 + \gamma^2(\cdot))^\alpha$, we see that γ satisfies the assumptions of Theorem 6, so for $s < \alpha$, we get that

$$H_\gamma^s(G) \hookrightarrow H_\gamma^\alpha(G). \quad (24)$$

Moreover, thanks to the first part of Theorem 5, we have

$$H_\gamma^\alpha(G) \hookrightarrow C(G), \quad (25)$$

and the proof follows. $\square$

Theorem 11. *If G is compact, $1/(1 + \gamma^2(\cdot)) \in L^\alpha(G^\wedge)$, and $s < \alpha$, then*

$$H_\gamma^s(G) \hookrightarrow L^p(G) \quad (26)$$

for all $p < \alpha^* = 2\alpha/(\alpha - s)$.

Proof. Let us take $\delta = \alpha - 2\alpha/p$; then, $\delta < s$. Thus, by Theorem 6, we have the compact embedding

$$H_\gamma^s(G) \hookrightarrow H_\gamma^\delta(G). \quad (27)$$

Next, using the second part of Theorem 5, we have the continuous embedding

$$H_\gamma^\delta(G) \hookrightarrow L^{\alpha^*}(G), \quad (28)$$

where $\alpha^* = 2\alpha/(\alpha - \delta) = p$. $\square$

4. Sobolev Spaces on Subsets of LCA Groups

In this section, we deal with Sobolev spaces defined on subsets of locally compact abelian groups. As mentioned in Section 1, we are motivated by analogous studies of function spaces on domains of $\mathbb{R}^n$ (see, e.g., [29]) by the fact that interesting applications exist, [23], and by the possibility of using them as tools for the study of differential equations on subsets of LCA groups. We start with the following definition.

Definition 12. Let U be a subset of a LCA group G . We define the Sobolev space $H_\gamma^s(U) \subset L^2(U)$ as the space of all $f \in L^2(U)$ such that there exists $F \in H_\gamma^s(G)$ with $F|_U = f$ and we equip it with the norm

$$\|f\|_{H_\gamma^s(U)} = \inf \left\{ \|F\|_{H_\gamma^s(G)} : F \in H_\gamma^s(G) \text{ and } F|_U = f \right\}. \quad (29)$$

An analogous definition (of spaces B_{pq}^s on domains of $\mathbb{R}^n$) appears in [29]; see his Definition 2.3. It can be easily shown that $H_\gamma^s(U)$ is a Banach space. We will say that it is a *local* Sobolev space.

Using appropriate embeddings for $H_\gamma^s(G)$ and the definition of $H_\gamma^s(U)$, we can prove the following.

Theorem 13. *Let G be a locally compact abelian group and let $U \subset G$. Then we have.*

(1) *The continuous inclusion $H_\gamma^s(U) \hookrightarrow L^2(U)$ holds. Moreover, for each $f \in H_\gamma^s(U)$, the following inequality holds:*

$$\|f\|_{L^2(U)} \leq \|f\|_{H_\gamma^s(U)}. \quad (30)$$

(2) *If $s > \sigma$, then $H_\gamma^s(G) \hookrightarrow H_\gamma^\sigma(U)$. Moreover, the following inequality holds:*

$$\|f\|_{H_\gamma^\sigma(U)} \leq \|f\|_{H_\gamma^s(U)}. \quad (31)$$

(3) *If $1/(1 + \gamma(\cdot)^2) \in L^s(G^\wedge)$, then $H_\gamma^s(U) \hookrightarrow C(U)$. Moreover, there exists a constant $C(\gamma, s)$ such that for each $f \in H_\gamma^s(U)$, the following inequality holds:*

$$\|f\|_{C(U)} \leq C(\gamma, s) \|f\|_{H_\gamma^s(U)}. \quad (32)$$

(4) *If $\alpha > s$ and $1/(1 + \gamma(\cdot)^2) \in L^\alpha(G^\wedge)$, then $H_\gamma^s(U) \hookrightarrow L^{\alpha^*}(U)$, where $\alpha^* = 2\alpha/(\alpha - s)$. Moreover, there exists a constant $D(\gamma, s)$ such that for each $f \in H_\gamma^s(U)$, the following inequality holds:*

$$\|f\|_{L^{\alpha^*}(U)} \leq D(\gamma, s) \|f\|_{H_\gamma^s(U)}. \quad (33)$$

We now prove the following compactness theorem in detail.

Theorem 14. *Let G be an LCA group and let U be a subset of G of finite measure. Assume that $1/(1 + \gamma^2(\cdot)) \in L^\alpha(G^\wedge)$ for some $\alpha > s$ and that*

$$\frac{|\xi(h) - 1|}{(1 + \gamma^2(\xi))^s} \xrightarrow{h \rightarrow e} 0 \quad \text{uniformly with respect to } \xi \in G^\wedge. \quad (34)$$

Then, for all $p < \alpha^$, one has the compact embedding*

$$H_\gamma^s(U) \hookrightarrow L^p(U). \quad (35)$$

The convergence concept used in (34) is explained in our previous paper [1]. Let us mention that a condition similar to (34) appears in the characterization of precompact sets in $L^2(G)$ via Fourier transform; see [30].

Proof. We will need two lemmas which we proved in [1].

Lemma 15. *Let $F \in H_\gamma^s(G)$ and assume that*

$$\frac{|\xi(h) - 1|}{(1 + \gamma^2(\xi))^s} \xrightarrow{h \rightarrow e} 0 \quad \text{uniformly with respect to } \xi \in G^\wedge. \quad (36)$$

Then, for every $h \in G$,

$$\int_G |F(xh) - F(x)|^2 dx \leq C(h) \|F\|_{H_\gamma^s(G)}^2, \quad (37)$$

where $C(h) \xrightarrow{h \rightarrow e} 0$.

Lemma 16. Let us denote by $\mathcal{F}$ the set of all symmetric unit neighbourhoods of group G partially ordered by inclusion. Let $(\phi_V)_{V \in \mathcal{F}}$ be a Dirac net and let $F \in H_V^s(G)$. Then,

$$\int_G |F * \phi_V(x) - F(x)|^2 dx \leq \|F\|_{H_V^s(G)} \sup_{y \in V} C(y). \quad (38)$$

Now we can continue with the proof of Theorem 14. Let f_n be a bounded sequence in $H_V^s(U)$; that is, $\|f_n\|_{H_V^s(U)} \leq C$. From the very definition, there exist functions F_n such that $F_n = f_n$ in U , $F_n \in H_V^s(G)$ and $\|F_n\|_{H_V^s(G)} \leq \|f_n\|_{H_V^s(U)} + 1$ for $n = 1, 2, \dots$. Now, by Theorem 13, we get that the sequence f_n is bounded in $L^{\alpha^*}(U)$. Hence, there exists a subsequence f_{n_k} such that $f_{n_k} \rightharpoonup f$ in $L^{\alpha^*}(U)$ and there also exists a subsequence F_{n_k} such that $F_{n_k} \rightharpoonup F$ in $L^{\alpha^*}(G)$. For simplicity, we simply write f_n and F_n for these subsequences. From the basic properties of weak convergence, we have that $F = f$ in U . We will show that for $q < \alpha^*$, we have

$$f_n \longrightarrow f \text{ in } L^q(U). \quad (39)$$

Let us set $F_{(V)} = F * \phi_V$. From the triangle inequality, we get

$$\begin{aligned} \|f_n - f\|_{L^2(U)} &\leq \|f_n - F_{n(V)}\|_{L^2(U)} + \|F_{n(V)} - f\|_{L^2(U)} \\ &\leq \|F_n - F_{n(V)}\|_{L^2(G)} + \|F_{n(V)} - F_{(V)}\|_{L^2(U)} \\ &\quad + \|F_{(V)} - f\|_{L^2(G)}. \end{aligned} \quad (40)$$

From Lemma 15, we get

$$\begin{aligned} \sup_n \int_G |F_{n(V)}(x) - F_n(x)|^2 dx &\leq \sup_n \|F_n\|_{H_V^s(G)}^2 \sup_{y \in V} C(y) \\ &\leq (C+1)^2 \sup_{y \in V} C(y). \end{aligned} \quad (41)$$

Furthermore it can be easily shown that for each $\epsilon > 0$, there exists V_ϵ such that for all $V \in \mathcal{F}$ with $V \subset V_\epsilon$, we have

$$\|F_{(V)} - f\|_{L^2(G)} \leq \frac{\epsilon}{3}. \quad (42)$$

Hence, we can choose V_ϵ such that

$$\|f_n - f\|_{L^2(U)} \leq \frac{2}{3}\epsilon + \|F_{n(V_\epsilon)} - F_{(V_\epsilon)}\|_{L^2(U)}. \quad (43)$$

We need to estimate $\|F_{n(V_\epsilon)} - F_{(V_\epsilon)}\|_{L^2(U)}$. Since $F_n \rightharpoonup F$ in $L^{\alpha^*}(G)$, we get that $F_{n(V_\epsilon)}(x) \xrightarrow{n \rightarrow \infty} F_{(V_\epsilon)}(x)$. Moreover, we have

$$\begin{aligned} &|F_{n(V_\epsilon)}(x) - F_{(V_\epsilon)}(x)|^2 \\ &= \left| \int_G (F_n(y) - F(y)) \phi_{V_\epsilon}(y^{-1}x) dy \right|^2 \end{aligned} \quad (44)$$

Hence, since U has a finite measure, we get from Lebesgue convergence theorem that

$$\|F_{n(V_\epsilon)} - F_{(V_\epsilon)}\|_{L^2(U)} \leq \frac{\epsilon}{3}. \quad (45)$$

We conclude that $f_n \rightarrow f$ in $L^2(U)$. In order to finish the proof, it is enough to use the Vitali convergence theorem. We conclude that $f_n \rightarrow f$ in $L^q(U)$ for $q < \alpha^*$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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