Research Article

Local Characterizations of Besov and Triebel-Lizorkin Spaces with Variable Exponent

Baohua Dong and Jingshi Xu

Department of Mathematics, Hainan Normal University, Haikou 571158, China

Correspondence should be addressed to Jingshi Xu; jingshixu@126.com

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We introduce new Besov and Triebel-Lizorkin spaces with variable integrable exponent, which are different from those introduced by the second author early. Then we characterize these spaces by the boundedness of the local Hardy-Littlewood maximal operator on variable exponent Lebesgue space. Finally the completeness and the lifting property of these spaces are also given.

1. Introduction

Variable exponent function spaces have attracted many attentions because of their applications in some aspects, such as partial differential equations with nonstandard growth [1], electrorheological fluids [2], and image restoration [3–5]. In fact, since the variable Lebesgue and Sobolev spaces were systemically studied by Kováčik and Rákosník in [6], there are many spaces introduced, such as, Bessel potential spaces with variable exponent, Besov and Triebel-Lizorkin spaces with variable exponents, Morrey spaces with variable exponents, and Hardy spaces with variable exponent; see [7–20] and references therein. When the Hardy-Littlewood maximal operator is bounded on the variable Lebesgue spaces, many results in classical harmonic analysis and function theory hold for the variable exponent case; see [21–23].

Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty, \quad (1)$$

with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \quad (2)$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

We will use the following notations: $p_- := \text{ess inf}\{p(x) : x \in \mathbb{R}^n\}$ and $p_+ := \text{ess sup}\{p(x) : x \in \mathbb{R}^n\}$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Moreover, we define $\mathcal{P}^0(\mathbb{R}^n)$ to be the set of measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with the range in $(0, \infty)$ such that $0 < p_- \leq p_+ < \infty$. Given $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbb{R}^n)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0} \in L^{p(\cdot)/p_0}(\mathbb{R}^n)$, where $0 < p_0 < p_-$ and $p(\cdot)/p_0 \in \mathcal{P}(\mathbb{R}^n)$. We also define a quasinorm on this space by $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \| |f|^{p_0} \|_{L^{p(\cdot)/p_0}(\mathbb{R}^n)}^{1/p_0}$.

Let f be a locally integrable function on $\mathbb{R}^n$; the local variant of the Hardy-Littlewood maximal operator is given by

$$\mathcal{M}_{\text{loc}} f(x) := \sup_{Q \ni x, |Q| \leq C} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad \forall x \in \mathbb{R}^n, \quad (3)$$

for some constant C . We denote $\mathcal{B}_{\text{loc}}(\mathbb{R}^n)$ the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\mathcal{M}_{\text{loc}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In 2013 Danelia et al. gave characterizations of $\mathcal{B}_{\text{loc}}(\mathbb{R}^n)$, a vector-estimate for the local Hardy-Littlewood maximal operator if $p(\cdot) \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$, and a Littlewood-Paley square-function characterization of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot)$ belongs to $\mathcal{B}_{\text{loc}}(\mathbb{R}^n)$ in [24].

In 2001 Rychkov used the boundedness of the local Hardy-Littlewood maximal operator to prove a stronger result of the Peetre type for spaces $F_{p,q}^{s,\omega}(\mathbb{R}^n)$ and $B_{p,q}^{s,\omega}(\mathbb{R}^n)$ and gave the lifting property for these spaces in [25].

Motivated by the previous papers, the goal of this paper is to introduce new Besov and Triebel-Lizorkin spaces with variable exponent. To state our result, we need some notations.

Throughout this paper $|S|$ denotes the Lebesgue measure for a measurable set $S \subset \mathbb{R}^n$. $\mathbb{N}_0$ denotes the set of all nonnegative integers. Let $\mathcal{D} := \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\mathcal{D}'$ be the dual space of $\mathcal{D}$. For $s \in \mathbb{R}$, $s_+ := \max\{s, 0\}$ and $[s]$ is the largest integer less than or equal to s .

Given a function φ on $\mathbb{R}^n$, let $L_\varphi \in \mathbb{N}_0$ denote the maximal number such that φ has vanishing moments up to order L_φ . In other words, $\int_{\mathbb{R}^n} x^\alpha \varphi(x) dx = 0$ for all multiindices α with $|\alpha| \leq L_\varphi$. If no moments of φ vanish, then put $L_\varphi = -1$. A pair of functions (φ_0, φ) is called satisfying the M_s condition, if $\int_{\mathbb{R}^n} \varphi_0(x) dx \neq 0$ and $L_\varphi \geq [s]$.

Take a function $\varphi_0 \in \mathcal{D}$ satisfying M_s condition, which is possible for any s . (Indeed, the assumption $L_\varphi \geq [s]$ is void for $s < 0$ and is satisfied automatically for $0 \leq s < 1$. For $s \geq 1$, any $\varphi_0 \in \mathcal{D}$ with Fourier transform $\widehat{\varphi}_0(\xi) = 1 + O(|\xi|^{[s]+1})$ near the origin will do the job.) The notation $\mathcal{S}'_e(\mathbb{R}^n)$ was introduced by Schott in [26]. More precisely, let it be the set of all $f \in \mathcal{D}'$ for which the estimate

$$|\langle f, \gamma \rangle| \leq A \sup \{ |D^\alpha \gamma(x)| \exp(N|x|) : x \in \mathbb{R}^n, |\alpha| \leq N \},$$

all $\gamma \in \mathcal{D}$,

(4)

is valid with some constants $A = A_f$, $N = N_f$. It is evidently that $\mathcal{S}'_e(\mathbb{R}^n)$ includes temperate distributions $\mathcal{S}'(\mathbb{R}^n)$.

Now, we give the definition of Besov spaces and Triebel-Lizorkin spaces with variable exponent.

Definition 1. Let $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < q \leq \infty$, φ_0 as above, $\varphi(x) := \varphi_0(x) - 2^{-n}\varphi_0(x/2)$, and $\varphi_j(x) := 2^{jn}\varphi(2^j x)$ for $j \in \mathbb{N}$.

- (i) The Besov space with variable exponent $B_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ is the set of f with

$$\left\{ f \in \mathcal{S}'_e(\mathbb{R}^n) : \|f\|_{B_{p(\cdot)}^{s,q}(\mathbb{R}^n)} < \infty \right\}, \quad (5)$$

where

$$\|f\|_{B_{p(\cdot)}^{s,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j * f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}. \quad (6)$$

- (ii) For $p_+ < \infty$, the Triebel-Lizorkin space with variable exponent $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ is the set of f with

$$\left\{ f \in \mathcal{S}'_e(\mathbb{R}^n) : \|f\|_{F_{p(\cdot)}^{s,q}(\mathbb{R}^n)} < \infty \right\}, \quad (7)$$

where

$$\|f\|_{F_{p(\cdot)}^{s,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f(\cdot)|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (8)$$

The key point is to prove that different choices of φ_0 in Definition 1 do not really change the spaces, leading to equivalent quasinorms. For $f \in \mathcal{S}'(\mathbb{R}^n)$ that has been proved by the second author in 2008, see [19]. To go on, we recall variant Peetre-type maximal functions which was introduced by Rychkov in [25]. Let

$$\varphi_{j,A,B}^* f(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_j * f(x-y)|}{m_{j,A,B}(y)}, \quad x \in \mathbb{R}^n, j \in \mathbb{N}_0, \quad (9)$$

where

$$m_{j,A,B}(y) := (1 + 2^j |y|)^A 2^{j|y|B}, \quad A, B \geq 0. \quad (10)$$

Now it is the position to state our main result.

Theorem 2. Let $s \in \mathbb{R}$, $0 < q < \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $0 < p_0 < \min\{p_-, q\}$ such that $p(\cdot)/p_0 \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$. Suppose that $\varphi_0, \zeta_0 \in \mathcal{D}$ and the pairs $(\varphi_0, \varphi := \varphi_0 - 2^{-n}\varphi_0(\cdot/2))$, and $(\zeta_0, \zeta := \zeta_0 - 2^{-n}\zeta_0(\cdot/2))$ satisfy the M_s condition. Then there are positive constants $A_0 := A_0(s, q)$, $B_0 := B_0(n)$, and C such that for each $A \geq A_0$, $B > B_0/p_-$, and all $f \in \mathcal{S}'_e(\mathbb{R}^n)$ one has

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\zeta_{j,A,B}^* f(\cdot)|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{F_{p(\cdot)}^{s,q}(\mathbb{R}^n)}, \quad (11)$$

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \|\zeta_{j,A,B}^* f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} \right\| \leq C \|f\|_{B_{p(\cdot)}^{s,q}(\mathbb{R}^n)}. \quad (12)$$

Since $\varphi_{j,A,B}^* \geq |\varphi_j * f|$ for any φ_0 , one immediately gets a consequence of Theorem 2.

Corollary 3. The spaces $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ and $B_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < q < \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $0 < p_0 < \min\{p_-, q\}$ such that $p(\cdot)/p_0 \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$ are independent of the particular choice of the function φ_0 in Definition 1. The quasinorms arising for different φ_0 are equivalent.

The proof of Theorem 2 will be given in Section 2. In Section 3 we study the completeness and the lifting property of these spaces by using Theorem 2. We will use the notation $a \leq b$ if there exists a constant $C > 0$ such that $a \leq Cb$. If $a \leq b$ and $b \leq a$ we will write $a \approx b$. Finally we claim that C is always a positive constant but it may change from line to line. Other notations will be explained when we meet them.

2. Proof of Theorem 2

We will use the idea of [25] by Rychkov to prove Theorem 2. First we need some lemmas.

Lemma 4 (see [25, Theorem 1.6]). Let a function $\varphi_0 \in \mathcal{D}$ have nonzero integral, and let $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(x/2)$. Then for

any $N \geq 0$ there exist two functions $\psi_0, \psi \in \mathcal{D}$, such that ψ has vanishing moments up to order N and

$$f = \sum_{j=0}^{\infty} \psi_j * \varphi_j * f, \quad \forall f \in \mathcal{D}', \quad (13)$$

where $\psi_j(x) := 2^{jn} \psi_j(2^j x)$ and $\varphi_j(x) := 2^{jn} \varphi(2^j x)$ for $j \in \mathbb{N}$.

Before the next lemma we denote a special convolution operator which is given by

$$K_B f(x) := \int_{\mathbb{R}^n} |f(y)| 2^{-B|x-y|} dy \quad (B \geq 0). \quad (14)$$

Lemma 5 (see [25, Lemma 2.10]). Let $0 < r < \infty$, $\varphi_0 \in \mathcal{D}$, $\int_{\mathbb{R}^n} \varphi_0 dx \neq 0$, and $\varphi = \varphi_0 - 2^{-n} \varphi_0(\cdot/2)$ and $A > n/r$, $B \geq 0$. Then there is a constant C depending only on n, r, φ_0, A, B such that for all $f \in \mathcal{S}'_e(\mathbb{R}^n)$ and each $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, one has

$$\begin{aligned} & \varphi_{j,A,B}^* f(x)^r \\ & \leq C \sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)} \\ & \quad \times \left\{ \mathcal{M}_{\text{loc}}(|\varphi_k * f|^r)(x) + K_{B_r}(|\varphi_k * f|^r)(x) \right\}. \end{aligned} \quad (15)$$

Lemma 6 (see [24, Corollary 3.2]). Let $p(\cdot) \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$ and $1 < q < \infty$; then there exists a positive constant C such that for all sequences $\{f_j\}_{j=0}^{\infty}$ of locally integrable functions on $\mathbb{R}^n$

$$\left\| \left(\sum_{j=0}^{\infty} (\mathcal{M}_{\text{loc}} f_j)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j=0}^{\infty} |f_j|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (16)$$

Lemma 7 (see [23, Lemma 2.1.14]). Let X be a real or complex vector space and ϱ be a semimodular on X . Then $\|x\|_{\varrho} \leq 1$ and $\varrho(x) \leq 1$ are equivalent. If ϱ is continuous, then also $\|x\|_{\varrho} < 1$ and $\varrho(x) < 1$ are equivalent, as are $\|x\|_{\varrho} = 1$ and $\varrho(x) = 1$.

Lemma 8. Let $p(\cdot) \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$ and $1 < q < \infty$. Then there exists a positive constant $B_0 = B_0(n) > 0$ such that for $B \geq B_0/p_-$

$$\left\| \left(\sum_{i=0}^{\infty} (K_B f_i)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left(\sum_{i=0}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad (17)$$

where C is a positive constant and $\{f_i\}_{i=0}^{\infty}$ are locally integrable functions on $\mathbb{R}^n$.

Proof. By homogeneity, it suffices to consider the case

$$\left\| \left(\sum_{i=0}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1. \quad (18)$$

Let $\mathbb{R}^n = \bigcup_{Q \in \mathcal{J}_1} Q$, where $\mathcal{J}_1$ is the set of all unit dyadic cubes in $\mathbb{R}^n$. Then it is easy to get

$$\begin{aligned} K_B f(x) &= \int_{\mathbb{R}^n} |f(y)| 2^{-B|x-y|} dy \\ &\leq \sum_{\mathcal{J}_1 \ni Q} 2^{-B \text{dist}(x,Q)} \int_Q |f(y)| dy. \end{aligned} \quad (19)$$

Since $p_- > 1$, by Minkowski's inequality and Hölder's inequality,

$$\begin{aligned} & \left(\sum_{i=0}^{\infty} K_B f_i(x)^q \right)^{1/q} \\ & \leq \left[\sum_{i=0}^{\infty} \left(\sum_{\mathcal{J}_1 \ni Q} 2^{-B \text{dist}(x,Q)} \int_Q |f_i(y)| dy \right)^q \right]^{1/q} \\ & \leq \sum_{\mathcal{J}_1 \ni Q} 2^{-B \text{dist}(x,Q)} \left[\sum_{i=0}^{\infty} \left(\int_Q |f_i(y)| dy \right)^q \right]^{1/q} \\ & \leq \left\{ \sum_{\mathcal{J}_1 \ni Q} 2^{-B \text{dist}(x,Q) p(x)/2} \right. \\ & \quad \times \left. \left[\sum_{i=0}^{\infty} \left(\int_Q |f_i(y)| dy \right)^q \right]^{p(x)/q} \right\}^{1/p(x)} \\ & \quad \times \left(\sum_{\mathcal{J}_1 \ni Q} 2^{-B \text{dist}(x,Q) p'(x)/2} \right)^{1/p'(x)}. \end{aligned} \quad (20)$$

Since $B > 0$ and $p_- > 1$, the latter factor is uniformly bounded in x . We take the $p(x)$ th power of the above inequality and integrate it. We get

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{i=0}^{\infty} K_B f_i(x)^q \right)^{p(x)/q} dx \\ & \leq \sum_{\mathcal{J}_1 \ni Q} \int_{\mathbb{R}^n} 2^{-B \text{dist}(x,Q) p(x)/2} dx \\ & \quad \times \left[\sum_{i=0}^{\infty} \left(\int_Q |f_i(y)| dy \right)^q \right]^{p(x)/q}. \end{aligned} \quad (21)$$

It is easy to know that

$$\int_{\mathbb{R}^n} 2^{-B \text{dist}(x,Q) p(x)/2} dx \leq \int_Q dx \quad (22)$$

for $p_- > 1$ and also that

$$\int_Q |f_i(y)| dy \leq \mathcal{M}_{\text{loc}} f_i(x), \quad \forall x \in Q. \quad (23)$$

By these two observations and (21), we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{i=0}^{\infty} K_B f_i(x)^q \right)^{p(x)/q} dx \\ & \leq \int_{\mathbb{R}^n} \left[\sum_{i=0}^{\infty} (\mathcal{M}_{\text{loc}} f_i(x))^q \right]^{p(x)/q} dx. \end{aligned} \quad (24)$$

Applying Lemmas 6 and 7 we obtain

$$\int_{\mathbb{R}^n} \left(\sum_{i=0}^{\infty} K_B f_i(x)^q \right)^{p(x)/q} dx \leq 1. \quad (25)$$

Using Lemma 7 again we obtain

$$\left\| \left(\sum_{i=0}^{\infty} (K_B f_i)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1. \quad (26)$$

Thus we have

$$\left\| \left(\sum_{i=0}^{\infty} (K_B f_i)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \left\| \left(\sum_{i=0}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (27)$$

This finishes the proof. $\square$

We give a notation of norm in $L^{p(\cdot)}(\ell_q)$ which will be used in the following context:

$$\left\| \{g_j\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell_q)} := \left\| \left(\sum_{j=0}^{\infty} |g_j|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (28)$$

Lemma 9 (see [27, Lemma 2]). *Let $0 < q \leq \infty$, $\delta > 0$. For any sequence $\{g_j\}_0^{\infty}$ of nonnegative numbers denote*

$$G_j = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k. \quad (29)$$

Then

$$\left\| \{G_j\}_0^{\infty} \right\|_{\ell_q} \leq C \left\| \{g_j\}_0^{\infty} \right\|_{\ell_q} \quad (30)$$

holds, where C is constant and only depends on q, δ .

Lemma 10 (see [19, Lemma 3]). *Let $0 < q \leq \infty$, $\delta > 0$, and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. For any sequence $\{g_j\}_0^{\infty}$ of nonnegative measurable functions on $\mathbb{R}^n$, denote*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x), \quad x \in \mathbb{R}^n. \quad (31)$$

Then

$$\begin{aligned} \left\| \{G_j\}_0^{\infty} \right\|_{L^{p(\cdot)}(\ell_q)} & \leq C_1 \left\| \{g_j\}_0^{\infty} \right\|_{L^{p(\cdot)}(\ell_q)}, \\ \left\| \{G_j\}_0^{\infty} \right\|_{\ell_q(L^{p(\cdot)})} & \leq C_2 \left\| \{g_j\}_0^{\infty} \right\|_{\ell_q(L^{p(\cdot)})} \end{aligned} \quad (32)$$

hold with some constants $C_1 = C_1(q, \delta)$ and $C_2 = C_2(p(\cdot), q, \delta)$.

Proof of Theorem 2. By Lemma 4, take $\psi_0, \psi \in \mathcal{D}$, with large enough L_ψ so that (13) is true. It follows that

$$\begin{aligned} & |\zeta_j * f(x - y)| \\ & \leq \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |\zeta_j * \psi_k(z)| \cdot |\varphi_k * f(x - y - z)| dz \\ & \leq \sum_{k=0}^{\infty} I_{jk} \sup_{z \in \mathbb{R}^n} \frac{|\varphi_k * f(x - y - z)|}{m_{k,A,B}(z)}, \end{aligned} \quad (33)$$

where

$$I_{jk} := \int_{\mathbb{R}^n} |\zeta_j * \psi_k(z)| m_{k,A,B}(z) dz. \quad (34)$$

Because of the elementary inequality

$$\begin{aligned} m_{j,A,B}(z) & \leq m_{j,A,B}(y) m_{j,A,B}(z - y), \\ m_{j,A,B}(y) & \geq \begin{cases} m_{k,A,B}(y), & j \geq k, \\ 2^{(j-k)A} m_{k,A,B}(y), & j < k \end{cases} \end{aligned} \quad (35)$$

we have the following fact:

$$\zeta_{j,A,B}^* f(x) \leq \begin{cases} \sum_{k=0}^{\infty} I_{jk} \varphi_{k,A,B}^* f(x), & \text{if } j \geq k \\ \sum_{k=0}^{\infty} I_{jk} 2^{(k-j)A} \varphi_{k,A,B}^* f(x), & \text{if } j < k. \end{cases} \quad (36)$$

To estimate I_{jk} , note that

$$\|\zeta_j * \psi_k\|_{\infty} \leq \begin{cases} 2^{(k-j)(L_\zeta+1)} 2^{kn}, & \text{if } j \geq k, \\ 2^{(j-k)(L_\psi+1)} 2^{jn}, & \text{if } j < k, \end{cases} \quad (37)$$

which follows easily from the moment conditions on ζ and ψ . Furthermore, $\zeta_j * \psi_k(z)$ is supported in the ball $\{|z| \leq \max(2^{-j}, 2^{-k})\}$, in which

$$m_{k,A,B}(z) \leq \begin{cases} 1, & \text{if } j \geq k \\ 2^{(k-j)A}, & \text{if } j < k. \end{cases} \quad (38)$$

By the last two estimates,

$$I_{jk} \leq \begin{cases} 2^{(k-j)(L_\zeta+1)}, & \text{if } j \geq k \\ 2^{(j-k)(L_\psi+1-A)}, & \text{if } j < k. \end{cases} \quad (39)$$

We put this estimate in (36) and see that if we choose $L_\psi > 2A - [s]$ and take into account $L_\zeta \geq [s]$, then we arrive at

$$2^{js} \zeta_{j,A,B}^* f(x) \leq \sum_{k=0}^{\infty} 2^{-\varepsilon|k-j|} 2^{ks} \varphi_{k,A,B}^* f(x) \quad (40)$$

with some $\varepsilon > 0$. It is easy to see that, in the right side of (40), we have essentially the convolution with the sequence $\{2^{-\varepsilon k}\}$,

which is of course a bounded operator on any ℓ_q , $0 < q < \infty$. Now by Lemmas 9 and 10 for ℓ_q and $L^{p(\cdot)}(\mathbb{R}^n)$ we easily obtain

$$\begin{aligned} & \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\zeta_{j,A,B}^* f(\cdot)|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \leq \left\| \left(\sum_{k=0}^{\infty} 2^{ks} \varphi_{k,A,B}^* f(\cdot)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\ & \left(\sum_{j=0}^{\infty} 2^{jsq} \|\zeta_{j,A,B}^* f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q} \\ & \leq \left(\sum_{k=0}^{\infty} 2^{ksq} \|\varphi_{k,A,B}^* f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}. \end{aligned} \quad (41)$$

In other words, we reduce matters to prove (11) and (12) with $\zeta_0 = \varphi_0$, $\zeta = \varphi$. Below we do it only for (11); the argument for (12) is similar.

Let $0 < r < q$ and $A > A_0 := n/r + \max\{-s, 0\}$. By Lemma 5 and a discrete version of the Hardy inequality

$$\sum_{j=0}^{\infty} 2^{j\theta} \left(\sum_{k=j}^{\infty} |b_k| \right)^{\tau} \leq \sum_{j=0}^{\infty} 2^{j\theta} |b_j|^{\tau} \quad (\theta, \tau > 0), \quad (42)$$

which we apply with $\theta := q(A + s - n/r)$ and $\tau = q/r$, we have

$$\begin{aligned} & \left(\sum_{j=0}^{\infty} 2^{jsq} (\varphi_{j,A,B}^* f)^q \right)^{1/q} \\ & \leq \left(\sum_{j=0}^{\infty} 2^{jsq} \{ \mathcal{M}_{\text{loc}}(|\varphi_j * f|^r)(x) \right. \\ & \quad \left. + K_{Br} (|\varphi_j * f|^r)(x) \}^{q/r} \right)^{1/q}. \end{aligned} \quad (43)$$

Note that $Br \cdot p_-/r \geq B_0$ and $1 < q/r < \infty$. Let $r := p_0$; by Lemmas 6 and 8, the operators $f \mapsto \{\mathcal{M}_{\text{loc}}(|f|^r)\}^{1/r}$ and $f \mapsto \{K_{Br}(|f|^r)\}^{1/r}$ are all bounded on $L^{p(\cdot)}(\ell_q)$. Hence the desired estimate (11) with $\zeta_0 = \varphi_0$, $\zeta = \varphi$, follows.

This finishes the proof. $\square$

3. Some Applications

In this section, we will consider the completeness, the lifting property, and the related quasinorms of these spaces introduced in previous section.

Theorem 11. *Let $s \in \mathbb{R}$, $0 < q < \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $0 < p_0 < \min\{p_-, q\}$ such that $p(\cdot)/p_0 \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$. Then the quasinormed spaces $B_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ and $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ are quasi-Banach spaces.*

Proof. We only give the proof for $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ and for $B_{p(\cdot)}^{s,q}(\mathbb{R}^n)$; it can be proved by the similar way. We use the similar argument in [25]. Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$ and $\gamma \in \mathcal{D}$ with $\text{supp } \gamma \subset B(0, 1)$. We set $\zeta_0 = \gamma$ and $j = 0$ in the left side of (11). Analyzing the proof of Theorem 2 shows that only finite numbers of derivatives of the kernels are involved in the estimates, and therefore we know

$$\left\| \sup_{y \in \mathbb{R}^n} \frac{|f * \gamma(\cdot - y)|}{(1 + |y|)^A 2^{|y|B}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{F_{p(\cdot)}^{s,q}(\mathbb{R}^n)} \sup_{|\alpha| \leq L} |D^\alpha \gamma|, \quad (44)$$

where L is a constant and depends on p_- , q , s , n , but not on f and γ .

It is easy to know

$$\begin{aligned} \sup_{y \in \mathbb{R}^n} \frac{|f * \gamma(x - y)|}{(1 + |y|)^A 2^{|y|B}} &= \sup_{u \in \mathbb{R}^n} \frac{|f * \gamma(u)|}{(1 + |x - u|)^A 2^{|x - u|B}} \\ &\geq \sup_{|u - x| \leq 1} \frac{|f * \gamma(u)|}{(1 + |u - x|)^A 2^{|x - u|B}}. \end{aligned} \quad (45)$$

We take $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ on both sides of the last inequality and get

$$\begin{aligned} & \left\| \sup_{y \in \mathbb{R}^n} \frac{|f * \gamma(\cdot - y)|}{(1 + |y|)^A 2^{|y|B}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= \left\| \sup_{u \in \mathbb{R}^n} \frac{|f * \gamma(u)|}{(1 + |\cdot - u|)^A 2^{|\cdot - u|B}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\geq \left\| \sup_{|\cdot - u| \leq 1} \frac{|f * \gamma(u)|}{(1 + |\cdot - u|)^A 2^{|\cdot - u|B}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\geq |f * \gamma(u)| \left\| \sup_{|v| \leq 1} \frac{1}{(1 + |v|)^A 2^{|v|B}} \cdot \chi_{B(u, 1)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\geq |f * \gamma(u)|. \end{aligned} \quad (46)$$

By (44) and (46) we have

$$|f * \gamma(x)| \leq \|f\|_{F_{p(\cdot)}^{s,q}(\mathbb{R}^n)} \left(\sup_{|\alpha| \leq L} |D^\alpha \gamma| \right) \exp(N|x|), \quad (47)$$

where N, L are constants and depend on p_- , q , s , n , but not on f and γ .

Then we know that the following estimate

$$\begin{aligned} & |f * \gamma(x)| \\ & \leq \|f\|_{F_{p(\cdot)}^{s,q}(\mathbb{R}^n)} \sup \{ |D^\alpha \gamma(x)| \exp(N|x|) : x \in \mathbb{R}^n, |\alpha| \leq L \} \end{aligned} \quad (48)$$

is valid for all $f \in \mathcal{S}'_e(\mathbb{R}^n)$ and $\gamma \in \mathcal{D}$ with some constants N, L which may depend on p_- , q , s , n , but not on f and γ . Thus we obtain that $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ is continuously embedded in $\mathcal{S}'_e(\mathbb{R}^n)$.

Now we conclude the proof of the theorem in a normal way. If a sequence of distributions $\{f_j\}$ is Cauchy sequence in

$F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$, then by (48) it converges “pointwise.” By the completeness of $\mathcal{D}'$, the sequence has a limit f in $\mathcal{D}'$. Again by (48), we have $f \in \mathcal{S}'_e(\mathbb{R}^n)$, since Cauchy sequences are bounded. Finally, by Lebesgue’s theorem on dominated convergence it is easily seen that $f_j \rightarrow f$ in $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$.

This finishes the proof. $\square$

In next context, we study the action of the Bessel potential operators in our Besov and Triebel-Lizorkin spaces with variable exponent. More precisely, we consider the following t -dilated version:

$$\mathcal{T}_t^a = (\text{id} - t^2 \Delta)^{-a/2}, \quad a \in \mathbb{R}, \quad t > 0, \quad (49)$$

where id denotes the identity operator.

For $f \in \mathcal{S}(\mathbb{R}^n)$ this operator acts by the rule $\mathcal{T}_t^a f = (K^a)_t * f$, where

$$K^a = \mathcal{F}^{-1} \left\{ (1 + 4\pi^2 |x|^2)^{-a/2} \right\} \quad (50)$$

formally $\int e^{2\pi i \langle x, \cdot \rangle} (1 + 4\pi^2 |x|^2)^{-a/2} dx \in \mathcal{S}'(\mathbb{R}^n).$

It is well known that if $a > 0$, then $K^a \in L^1(\mathbb{R}^n)$ and has the representation

$$K^a(x) \approx \int_0^\infty u^{a-n} e^{-\pi|x|^2/u^2} e^{-u^2/4\pi} \frac{du}{u} \quad (51)$$

(see Stein’s book [28] for these matters), from which it follows rather easily that K^a with $a > 0$ is $\mathcal{C}^\infty$ away from the origin and

$$|D^\alpha K^a(x)| \lesssim e^{-E|x|}, \quad |x| \geq 1 \quad (52)$$

with an absolute constant $E > 0$. By the identity

$$K^a = (\text{id} - \Delta)^N K^{a+2N}, \quad N \in \mathbb{N}, \quad (53)$$

we see that for $a \leq 0$ the distribution K^a agrees in $\mathbb{R}^n \setminus \{0\}$ with a $\mathcal{C}^\infty$ function, which again satisfies (52).

By the same argument in page 170 of [25] we know that the convolution $K_t^a * f$ can be defined as an element of $\mathcal{S}'_e(\mathbb{R}^n)$ for any $f \in B_{p(\cdot)}^{s,q}(\mathbb{R}^n) \cup F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$, provided that $t \leq t_0(n) \cdot p_-$. The next theorem states explicitly where it acts.

Theorem 12 (the lifting property). *Let $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $0 < p_0 < \min\{p_-, q\}$ such that $p(\cdot)/p_0 \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$. Then there is a constant $t_0 = t_0(n) > 0$ so that for all $0 < q < \infty$, $s \in \mathbb{R}$, and every positive $t < t_0 \cdot p_-$ one has*

$$\begin{aligned} \mathcal{T}_t^a : B_{p(\cdot)}^{s,q}(\mathbb{R}^n) &\longrightarrow B_{p(\cdot)}^{s+a,q}(\mathbb{R}^n) \quad \text{isomorphically,} \\ \mathcal{T}_t^a : F_{p(\cdot)}^{s,q}(\mathbb{R}^n) &\longrightarrow F_{p(\cdot)}^{s+a,q}(\mathbb{R}^n) \quad \text{isomorphically.} \end{aligned} \quad (54)$$

Proof. The idea of the proof comes from [25]. We use again (13) with φ and ψ having vanishing moments up to large order L . By an argument similar to that one used above to define $\mathcal{T}_t^a$

on Besov and Triebel-Lizorkin spaces with variable exponent, one can establish the identity

$$\varphi_l * (K_t^a * f) = \sum_{j=0}^{\infty} K_t^a * \varphi_l * \psi_j * (\varphi_j * f), \quad l \in \mathbb{N}_0. \quad (55)$$

From [25] by choosing L sufficiently large, we have

$$2^{l(s+a)} |\varphi_l * (K_t^a * f)(x)| \leq \sum_{j=0}^{\infty} 2^{-\varepsilon|l-j|} 2^{js} \varphi_{j,A,B}^* f(x), \quad (56)$$

$\varepsilon > 0, \quad t < \frac{E}{2B \ln 2}.$

Now by using Theorem 2 with $A = A_0$ and $B = B_0/p_-$, it follows easily that if f belongs to $B_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ or $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$, then $K_t^a * f$ is in $B_{p(\cdot)}^{s+a,q}(\mathbb{R}^n)$ or $F_{p(\cdot)}^{s+a,q}(\mathbb{R}^n)$, respectively. Then the condition on t becomes $t < t_0 \cdot p_-$ with $t_0 = E/2B_0 \ln 2$.

The fact that the maps in (54) are actually onto follows from the identity $\mathcal{T}_t^{-a} \mathcal{T}_t^a = \text{id}$.

This finishes the proof. $\square$

It follows from Theorem 12 that $\|\mathcal{T}_t^{-s} f\|_{F_{p(\cdot)}^{0,q}(\mathbb{R}^n)}$ is an equivalent quasinorm on $F_{p(\cdot)}^{s,q}(\mathbb{R}^n)$ for small $t > 0$ and analogously for $B_{p(\cdot)}^{s,q}(\mathbb{R}^n)$. The next theorem gives a version of this result for $s \in \mathbb{N}$ involving “pure” derivatives.

Theorem 13. *Let $s \in \mathbb{N}$, $0 < q < \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $0 < p_0 < \min\{p_-, q\}$ such that $p(\cdot)/p_0 \in \mathcal{B}_{\text{loc}}(\mathbb{R}^n)$. Then for any $|\alpha| \leq s$*

$$\begin{aligned} \|f\|_{F_{p(\cdot)}^{s,q}} &\approx \sum_{|\alpha| \leq s} \|D^\alpha f\|_{F_{p(\cdot)}^{0,q}}, \quad \forall f \in \mathcal{S}'_e, \\ \|f\|_{B_{p(\cdot)}^{s,q}} &\approx \sum_{|\alpha| \leq s} \|D^\alpha f\|_{B_{p(\cdot)}^{0,q}}, \quad \forall f \in \mathcal{S}'_e. \end{aligned} \quad (57)$$

Proof. For brevity, we only give the outline of the proof for Triebel-Lizorkin space with variable exponent; for Besov space with variable exponent it can be proved by similar way. The “ $\geq$ ” inequality follows immediately from Definition 1 by partial integration and invoking Theorem 2.

The “ $\leq$ ” inequality for s even follows from Theorem 12. To obtain it for s odd, it suffices to consider the case $s = 1$. In view of Theorem 12, it is sufficient to prove the estimate

$$\|K_t^1 * f\|_{F_{p(\cdot)}^{0,q}} \leq \sum_{|\alpha| \leq 1} \|D^\alpha f\|_{F_{p(\cdot)}^{0,q}} \quad \text{for small } t. \quad (58)$$

From [25] again, we have

$$\varphi_j * f = 2^{-j} \sum_{\nu=1}^n (\varphi^\nu)_j * \frac{\partial f}{\partial x_\nu}. \quad (59)$$

All this leads to the following counterpart of (56):

$$\begin{aligned} |\varphi_l * (K_t^a * f)(x)| &\leq 2^{-\varepsilon l} \varphi_{0,A,B}^* f(x) \\ &+ \sum_{j=1}^{\infty} 2^{-\varepsilon|l-j|} \sum_{\nu=1}^n (\varphi^\nu)^*_{j,A,B} \left(\frac{\partial f}{\partial x_\nu} \right)(x), \\ \varepsilon &> 0, \end{aligned} \quad (60)$$

where, for $\nu = 1, \dots, n$, $\varphi^\nu \in \mathcal{D}$ has vanishing moments up to order $L - 1$ and satisfy

$$\varphi = \sum_{\nu=1}^n \left(\frac{\partial}{\partial x_\nu} \right) \varphi^\nu. \quad (61)$$

From (58), it is easily deduced by virtue of Theorem 12.

This finishes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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