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Research Article

Generalized Steffensen Type Inequalities Involving Convex Functions

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In this paper generalized Steffensen type inequalities related to the class of functions that are "convex at point c" are derived and as a consequence inequalities involving the class of convex functions are obtained. Moreover, linear functionals from the difference of the right- and left-hand side of the obtained generalized inequalities are constructed and new families of exponentially convex functions related to constructed functionals are derived.

1. Introduction

The well-known Steffensen inequality [1] reads as follows.

Theorem 1. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Then one has*

$$\int_{b-\lambda}^{b} f(t)\,dt \le \int_a^b f(t)\,g(t)\,dt \le \int_a^{a+\lambda} f(t)\,dt. \tag{1}$$

The inequalities are reversed for f is nondecreasing.

Since its appearance in 1918 numerous papers have been devoted to generalizations and refinements of Steffensen's inequality. In [2] Pečarić proved the following generalization.

Theorem 2. *Let h be a positive integrable function on $[a,b]$ and f an integrable function such that f/h is nondecreasing on $[a,b]$. If g is a real-valued integrable function on $[a,b]$ such that $0 \le g \le 1$, then*

$$\int_a^b f(t)\,g(t)\,dt \ge \int_a^{a+\lambda} f(t)\,dt \tag{2}$$

holds, where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)\,dt = \int_a^b h(t)\,g(t)\,dt. \tag{3}$$

If f/h is a nonincreasing function, then the reverse inequality in (2) holds.

By substitutions $g(x) \to 1 - g(x)$ and $\lambda \to b - a - \lambda$, Theorem 2 becomes as follows.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then*

$$\int_a^b f(t)\,g(t)\,dt \le \int_{b-\lambda}^{b} f(t)\,dt \tag{4}$$

holds, where λ is the solution of the equation

$$\int_{b-\lambda}^{b} h(t)\,dt = \int_a^b h(t)\,g(t)\,dt. \tag{5}$$

If f/h is a nonincreasing function, then the reverse inequality in (4) holds.

In 2000 Mercer [3] gave a generalization that contains various already known generalizations, one of which is the aforementioned generalization given by Pečarić. In 2007 Wu and Srivastava [4] noted that Mercer's result is incorrect and they have corrected it and gave a refinement of Steffensen's inequality. Liu [5] also noted that Mercer's result is incorrect as stated.

Motivated by refinement of Steffensen's inequality given in [4], Pečarić et al. [6] obtained the following refined version of results given in Theorems 2 and 3.

Corollary 4. *Let h be a positive integrable function on $[a,b]$ and f, g integrable functions on $[a,b]$ such that f/h is nonincreasing and $0 \le g \le 1$. Then*

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dt &\le \int_a^{a+\lambda}\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(a+\lambda)}{h(a+\lambda)}\right]h(t)\left[1-g(t)\right]\right)dt\\ &\le \int_a^{a+\lambda} f(t)\,dt,\end{aligned} \tag{6}$$

where λ is given by (3).

If f/h is a nondecreasing function, then the reverse inequality in (6) *holds.*

Corollary 5. *Let h be a positive integrable function on $[a,b]$ and f,g integrable functions on $[a,b]$ such that f/h is nonincreasing and $0 \le g \le 1$. Then*

$$\begin{aligned}\int_{b-\lambda}^b f(t)\,dt &\le \int_{b-\lambda}^b\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(b-\lambda)}{h(b-\lambda)}\right]h(t)\left[1-g(t)\right]\right)dt\\ &\le \int_a^b f(t)\,g(t)\,dt,\end{aligned} \tag{7}$$

where λ is given by (5).

If f/h is a nondecreasing function, then the reverse inequality in (7) *holds.*

In this paper we obtain generalized Steffensen type inequalities, related to the aforementioned generalizations and refinements of Steffensen's inequality, for the class of functions that are convex at point c. Moreover, we construct linear functionals from the difference of the right- and left-hand side of obtained generalized inequalities and derive new families of exponentially convex functions related to constructed functionals.

2. Main Results

Let us begin by introducing a class of functions that extends the class of convex functions.

Definition 6. Let $h : [a,b] \to \mathbb{R}$ be a positive function, $f : [a,b] \to \mathbb{R}$ a function, and $c \in (a,b)$. We say that f/h belongs to the class $\mathcal{M}_1^c[a,b]$ ($\mathcal{M}_2^c[a,b]$) if there exists a constant A such that the function $F(x)/h(x) = (f(x)/h(x)) - Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.

As noted in [7] we can describe the property from Definition 6 as "convexity at point c." In [7] Pečarić and Smoljak also proved that there is a connection between the class of functions $\mathcal{M}_1^c[a,b]$ and the class of convex functions. This connection is given in the following theorem.

Theorem 7. *The function f/h is convex (concave) on $[a,b]$ if and only if $f/h \in \mathcal{M}_1^c[a,b]$ ($f/h \in \mathcal{M}_2^c[a,b]$) for every $c \in (a,b)$.*

Applying the generalizations of Steffensen's inequality given in the Introduction to functions that are convex at point c we obtain the following results.

Theorem 8. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$. Let λ_1 be the solution of the equation*

$$\int_a^{a+\lambda_1} h(t)\,dt = \int_a^c h(t)\,g(t)\,dt \tag{8}$$

and let λ_2 be the solution of the equation

$$\int_{b-\lambda_2}^b h(t)\,dt = \int_c^b h(t)\,g(t)\,dt. \tag{9}$$

If $f/h \in \mathcal{M}_1^c[a,b]$ and

$$\int_a^b th(t)\,g(t)\,dt = \int_a^{a+\lambda_1} th(t)\,dt + \int_{b-\lambda_2}^b th(t)\,dt, \tag{10}$$

then

$$\int_a^b f(t)\,g(t)\,dt \le \int_a^{a+\lambda_1} f(t)\,dt + \int_{b-\lambda_2}^b f(t)\,dt. \tag{11}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (10) *holds, the inequality in* (11) *is reversed.*

Proof. Let $f/h \in \mathcal{M}_1^c[a,b]$ and let $F(x) = f(x) - Axh(x)$, where A is the constant from Definition 6. Since $F/h : [a,c] \to \mathbb{R}$ is nonincreasing, from Theorem 2 we obtain

$$\begin{aligned}0 &\le \int_a^{a+\lambda_1} F(t)\,dt - \int_a^c F(t)\,g(t)\,dt\\ &= \int_a^{a+\lambda_1} f(t)\,dt - \int_a^c f(t)\,g(t)\,dt\\ &\quad - A\left(\int_a^{a+\lambda_1} th(t)\,dt - \int_a^c th(t)\,g(t)\,dt\right).\end{aligned} \tag{12}$$

Since $F/h : [c,b] \to \mathbb{R}$ is nondecreasing, from Theorem 3 we obtain

$$\begin{aligned}0 &\ge \int_c^b F(t)\,g(t)\,dt - \int_{b-\lambda_2}^b F(t)\,dt\\ &= \int_c^b f(t)\,g(t)\,dt - \int_{b-\lambda_2}^b f(t)\,dt\\ &\quad - A\left(\int_c^b th(t)\,g(t)\,dt - \int_{b-\lambda_2}^b th(t)\,dt\right).\end{aligned} \tag{13}$$

Now from (12) and (13) we obtain

$$\int_a^{a+\lambda_1} f(t)\,dt + \int_{b-\lambda_2}^b f(t)\,dt - \int_a^b f(t)\,g(t)\,dt \geq A\left(\int_a^{a+\lambda_1} th(t)\,dt + \int_{b-\lambda_2}^b th(t)\,dt - \int_a^b th(t)\,g(t)\,dt\right). \tag{14}$$

Hence, if (10) is satisfied, then (11) holds.

It is similar for $f/h \in \mathcal{M}_2^c[a,b]$. □

Remark 9. From the proof we deduce that condition (10) can be weakened. So, for $f/h \in \mathcal{M}_1^c[a,b]$ inequality (11) still holds if (10) is replaced by the weaker condition

$$A\left(\int_a^{a+\lambda_1} th(t)\,dt + \int_{b-\lambda_2}^b th(t)\,dt - \int_a^b th(t)\,g(t)\,dt\right) \geq 0, \tag{15}$$

where A is the constant from Definition 6. For $f/h \in \mathcal{M}_2^c[a,b]$ the reverse inequality in (11) holds if (10) is replaced by (15) with the reverse inequality.

Moreover, condition (15) can be further weakened if the function f/h is monotonic.

First, let us show that for $f/h \in \mathcal{M}_1^c[a,b]$ we have

$$\left(\frac{f}{h}\right)'_-(c) \leq A \leq \left(\frac{f}{h}\right)'_+(c). \tag{16}$$

Since F/h is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$ for all distinct points $x_1, x_2 \in [a,c]$ and $y_1, y_2 \in [c,b]$ we have

$$\begin{aligned}\left[x_1, x_2; \frac{F}{h}\right] &= \left[x_1, x_2; \frac{f}{h}\right] - A \leq 0 \leq \left[y_1, y_2; \frac{f}{h}\right] - A \\ &= \left[y_1, y_2; \frac{F}{h}\right].\end{aligned} \tag{17}$$

Therefore, if $(f/h)'_-(c)$ and $(f/h)'_+(c)$ exist, letting $x_i \nearrow c$ and $y_i \searrow c$, $i = 1,2$ we get (16). Similarly, for $f/h \in \mathcal{M}_2^c[a,b]$ we have (16) with the reverse inequality.

Hence, if the function $f/h \in \mathcal{M}_1^c[a,b]$ is nondecreasing or $f/h \in \mathcal{M}_2^c[a,b]$ is nonincreasing, from (15) we obtain that (10) can be weakened to

$$\int_a^b th(t)\,g(t)\,dt \leq \int_a^{a+\lambda_1} th(t)\,dt + \int_{b-\lambda_2}^b th(t)\,dt. \tag{18}$$

Further, if $f/h \in \mathcal{M}_1^c[a,b]$ is nonincreasing or $f/h \in \mathcal{M}_2^c[a,b]$ is nondecreasing, (10) can be weakened to (18) with the reverse inequality.

Theorem 10. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of the equation*

$$\int_{c-\lambda_1}^c h(t)\,dt = \int_a^c h(t)\,g(t)\,dt \tag{19}$$

and let λ_2 be the solution of the equation

$$\int_c^{c+\lambda_2} h(t)\,dt = \int_c^b h(t)\,g(t)\,dt. \tag{20}$$

If $f/h \in \mathcal{M}_1^c[a,b]$ and

$$\int_a^b th(t)\,g(t)\,dt = \int_{c-\lambda_1}^{c+\lambda_2} th(t)\,dt, \tag{21}$$

then

$$\int_a^b f(t)\,g(t)\,dt \geq \int_{c-\lambda_1}^{c+\lambda_2} f(t)\,dt. \tag{22}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (21) holds, the inequality in (22) is reversed.

Proof. Let $f/h \in \mathcal{M}_1^c[a,b]$ and let $F(x) = f(x) - Axh(x)$, where A is the constant from Definition 6. $F/h : [a,c] \to \mathbb{R}$ is nonincreasing, so from Theorem 3 we obtain

$$\begin{aligned}0 &\leq \int_a^c f(t)\,g(t)\,dt - \int_{c-\lambda_1}^c f(t)\,dt \\ &\quad - A\left(\int_a^c th(t)\,g(t)\,dt - \int_{c-\lambda_1}^c th(t)\,dt\right).\end{aligned} \tag{23}$$

$F/h : [c,b] \to \mathbb{R}$ is nondecreasing, so from Theorem 2 we obtain

$$\begin{aligned}0 &\geq \int_c^{c+\lambda_2} f(t)\,dt - \int_c^b f(t)\,g(t)\,dt \\ &\quad - A\left(\int_c^{c+\lambda_2} th(t)\,dt - \int_c^b th(t)\,g(t)\,dt\right).\end{aligned} \tag{24}$$

Hence, from (23) and (24) we conclude

$$\begin{aligned}&\int_a^b f(t)\,g(t)\,dt - \int_{c-\lambda_1}^{c+\lambda_2} f(t)\,dt \\ &\quad \geq A\left(\int_a^b th(t)\,g(t)\,dt - \int_{c-\lambda_1}^{c+\lambda_2} th(t)\,dt\right).\end{aligned} \tag{25}$$

Hence, if $\int_a^b th(t)g(t)dt = \int_{c-\lambda_1}^{c+\lambda_2} th(t)dt$, then (22) holds.

It is similar for $f/h \in \mathcal{M}_2^c[a,b]$. □

Remark 11. For $f/h \in \mathcal{M}_1^c[a,b]$ inequality (22) still holds if condition (21) is replaced by the weaker condition

$$A\left(\int_a^b th(t)\,g(t)\,dt - \int_{c-\lambda_1}^{c+\lambda_2} th(t)\,dt\right) \geq 0, \tag{26}$$

where A is the constant from Definition 6. Also, for $f/h \in \mathcal{M}_2^c[a,b]$ the reverse inequality in (22) holds if (21) is replaced by (26) with the reverse inequality.

Additionally, condition (26) can be further weakened if the function f/h is monotonic. Similarly, as in Remark 9, if the function $f/h \in \mathcal{M}_1^c[a,b]$ is nondecreasing or $f/h \in \mathcal{M}_2^c[a,b]$ is nonincreasing, from (26) we obtain that (21) can be weakened to

$$\int_a^b th(t)\,g(t)\,dt \geq \int_{c-\lambda_1}^{c+\lambda_2} th(t)\,dt. \tag{27}$$

Further, if $f/h \in \mathcal{M}_1^c[a,b]$ is nonincreasing or $f/h \in \mathcal{M}_2^c[a,b]$ is nondecreasing, (21) can be weakened to (27) with the reverse inequality.

As a consequence of Theorems 8 and 10 we obtain generalized Steffensen type inequalities that involve convex functions.

Corollary 12. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of (8) and λ_2 the solution of (9). If $f/h : [a,b] \to \mathbb{R}$ is convex and (10) holds, then the inequality in (11) holds.*

If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (11) is reversed.

Proof. Since f/h is convex, from Theorem 7, we have that $f/h \in \mathcal{M}_1^c[a,b]$ for every $c \in (a,b)$. Hence, we can apply Theorem 8. □

Corollary 13. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of (19) and λ_2 the solution of (20). If $f/h : [a,b] \to \mathbb{R}$ is convex and (21) holds, then the inequality in (22) holds.*

If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (22) is reversed.

Proof. The proof is similar to that of Corollary 12 applying Theorem 10. □

Remark 14. Similarly, as in Remarks 9 and 11, we obtain that conditions (10) and (21) in Corollaries 12 and 13 can be weakened if, additionally, the function f/h is monotonic.

Remark 15. For $h \equiv 1$ in Theorems 8 and 10 and Corollaries 12 and 13 we obtain the results given in [7].

In [6, Theorems 2.4 and 2.5] Pečarić et al. gave a corrected version of Mercer's result [3, Theorem 2] which follows from Theorems 2 and 3. In the following theorems we obtain generalizations of these results for functions from the class $\mathcal{M}_1^c[a,b]$.

Theorem 16. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq h$. Let λ_1 be the solution of the equation $\int_a^{a+\lambda_1} h(t)dt = \int_a^c g(t)dt$ and λ_2 the solution of the equation $\int_{b-\lambda_2}^b h(t)dt = \int_c^b g(t)dt$. If $f \in \mathcal{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)\,dt = \int_a^{a+\lambda_1} th(t)\,dt + \int_{b-\lambda_2}^b th(t)\,dt, \tag{28}$$

then

$$\int_a^b f(t)\,g(t)\,dt \leq \int_a^{a+\lambda_1} f(t)\,h(t)\,dt + \int_{b-\lambda_2}^b f(t)\,h(t)\,dt. \tag{29}$$

If $f \in \mathcal{M}_2^c[a,b]$ and (28) holds, the inequality in (29) is reversed.

Proof. Take $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 8. □

Theorem 17. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq h$. Let λ_1 be the solution of the equation $\int_{c-\lambda_1}^c h(t)dt = \int_a^c g(t)dt$ and λ_2 the solution of the equation $\int_c^{c+\lambda_2} h(t)dt = \int_c^b g(t)dt$. If $f \in \mathcal{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)\,dt = \int_{c-\lambda_1}^{c+\lambda_2} th(t)\,dt, \tag{30}$$

then

$$\int_a^b f(t)\,g(t)\,dt \geq \int_{c-\lambda_1}^{c+\lambda_2} f(t)\,h(t)\,dt. \tag{31}$$

If $f \in \mathcal{M}_2^c[a,b]$ and (30) holds, the inequality in (31) is reversed.

Proof. Take $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 10. □

In [6, Theorem 2.6] Pečarić et al. showed that Mercer's generalization [3, Theorem 3] is equivalent to Theorem 2. Further, in [6, Theorem 2.7] they obtained analogue theorem equivalent to Theorem 3. Motivated by the mentioned generalizations in the following theorems we obtain generalizations for functions from class $\mathcal{M}_1^c[a,b]$.

Theorem 18. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g,k : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \leq g \leq k$. Let λ_1 be the solution of the equation*

$$\int_a^{a+\lambda_1} k(t)\,h(t)\,dt = \int_a^c h(t)\,g(t)\,dt \tag{32}$$

and let λ_2 be the solution of the equation

$$\int_{b-\lambda_2}^b k(t)\,h(t)\,dt = \int_c^b h(t)\,g(t)\,dt. \tag{33}$$

If $f/h \in \mathcal{M}_1^c[a,b]$ and

$$\int_a^b th(t)\,g(t)\,dt = \int_a^{a+\lambda_1} tk(t)\,h(t)\,dt + \int_{b-\lambda_2}^b tk(t)\,h(t)\,dt, \tag{34}$$

then

$$\int_a^b f(t)\,g(t)\,dt \le \int_a^{a+\lambda_1} f(t)\,k(t)\,dt + \int_{b-\lambda_2}^b f(t)\,k(t)\,dt. \tag{35}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (34) holds, the inequality in (35) is reversed.

Proof. Take $h \mapsto kh$, $g \mapsto g/k$ and $f \mapsto fk$ in Theorem 8. □

Theorem 19. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g,k : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \le g \le k$. Let λ_1 be the solution of the equation*

$$\int_{c-\lambda_1}^c k(t)\,h(t)\,dt = \int_a^c h(t)\,g(t)\,dt \tag{36}$$

and let λ_2 be the solution of the equation

$$\int_c^{c+\lambda_2} k(t)\,h(t)\,dt = \int_c^b h(t)\,g(t)\,dt. \tag{37}$$

If $f/h \in \mathcal{M}_1^c[a,b]$ and

$$\int_a^b th(t)\,g(t)\,dt = \int_{c-\lambda_1}^{c+\lambda_2} tk(t)\,h(t)\,dt, \tag{38}$$

then

$$\int_a^b f(t)\,g(t)\,dt \ge \int_{c-\lambda_1}^{c+\lambda_2} f(t)\,k(t)\,dt. \tag{39}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (38) holds, the inequality in (39) is reversed.

Proof. Take $h \mapsto kh$, $g \mapsto g/k$ and $f \mapsto fk$ in Theorem 10. □

Remark 20. Taking $k \equiv 1$ in Theorems 18 and 19 we obtain Theorems 8 and 10, respectively.

In the following theorems we obtain refined version of results given in Theorems 8 and 10.

Theorem 21. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$. Let λ_1 be the solution of (8) and λ_2 the solution of (9). If $f/h \in \mathcal{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)\,g(t)\,dt &= \int_a^{a+\lambda_1} \left(th(t) - [t-a-\lambda_1]\,h(t)\,[1-g(t)]\right)dt \\ &\quad + \int_{b-\lambda_2}^b \left(th(t) - [t-b+\lambda_2]\,h(t)\,[1-g(t)]\right)dt,\end{aligned} \tag{40}$$

then

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dt &\le \int_a^{a+\lambda_1} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right] \times h(t)\,[1-g(t)]\right)dt \\ &\quad + \int_{b-\lambda_2}^b \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right] \times h(t)\,[1-g(t)]\right)dt.\end{aligned} \tag{41}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (40) holds, the inequality in (41) is reversed.

Proof. The proof is similar to that of Theorem 8 applying Corollary 4 for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Corollary 5 for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Theorem 22. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$. Let λ_1 be the solution of (19) and λ_2 the solution of (20). If $f/h \in \mathcal{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)\,g(t)\,dt &= \int_{c-\lambda_1}^c \left(th(t) - [t-c+\lambda_1]\,h(t)\,[1-g(t)]\right)dt \\ &\quad + \int_c^{c+\lambda_2} \left(th(t) - [t-c-\lambda_2]\,h(t)\,[1-g(t)]\right)dt,\end{aligned} \tag{42}$$

then

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dt &\ge \int_{c-\lambda_1}^c \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right] \times h(t)\,[1-g(t)]\right)dt\end{aligned}$$

$$+\int_{c}^{c+\lambda_2}\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right]\times h(t)\left[1-g(t)\right]\right)dt. \tag{43}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (42) holds, the inequality in (43) is reversed.

Proof. The proof is similar to that of Theorem 10 applying Corollary 5 for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Corollary 4 for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Motivated by sharpened and generalized versions of Theorems 2 and 3 obtained by Pečarić at al. in [6, Corollaries 2.4 and 2.5] we obtain the following results.

Theorem 23. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g,\psi : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \le \psi \le g \le 1-\psi$. Let λ_1 be the solution of (8) and λ_2 the solution of (9). If $f/h \in \mathcal{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)\,g(t)\,dt &= \int_a^{a+\lambda_1} th(t)\,dt-\int_a^c \left|t-a-\lambda_1\right| h(t)\,\psi(t)\,dt \\ &\quad+\int_{b-\lambda_2}^b th(t)\,dt+\int_c^b \left|t-b+\lambda_2\right| h(t)\,\psi(t)\,dt,\end{aligned} \tag{44}$$

then

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dt &\le \int_a^{a+\lambda_1} f(t)\,dt-\int_a^c\left|\frac{f(t)}{h(t)}-\frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right| h(t)\,\psi(t)\,dt \\ &\quad+\int_{b-\lambda_2}^b f(t)\,dt+\int_c^b\left|\frac{f(t)}{h(t)}-\frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right| h(t)\,\psi(t)\,dt.\end{aligned} \tag{45}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (44) holds, the inequality in (45) is reversed.

Proof. The proof is similar to that of Theorem 8 applying [6, Corollary 2.3] for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and [6, Corollary 2.4] for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Theorem 24. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, let $f : [a,b] \to \mathbb{R}$ be an integrable function, and let $c \in (a,b)$. Let $g,\psi : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \le \psi \le g \le 1-\psi$. Let λ_1 be the solution of (19) and λ_2 the solution of (20). If $f/h \in \mathcal{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)\,g(t)\,dt &= \int_{c-\lambda_1}^{c+\lambda_2} th(t)\,dt-\int_a^c \left|t-c+\lambda_1\right| h(t)\,\psi(t)\,dt \\ &\quad+\int_c^b \left|t-c-\lambda_2\right| h(t)\,\psi(t)\,dt,\end{aligned} \tag{46}$$

then

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dt &\ge \int_{c-\lambda_1}^{c+\lambda_2} f(t)\,dt+\int_a^c\left|\frac{f(t)}{h(t)}-\frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right| h(t)\,\psi(t)\,dt \\ &\quad-\int_c^b\left|\frac{f(t)}{h(t)}-\frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right| h(t)\,\psi(t)\,dt.\end{aligned} \tag{47}$$

If $f/h \in \mathcal{M}_2^c[a,b]$ and (46) holds, the inequality in (47) is reversed.

Proof. The proof is similar to that of Theorem 10 applying [6, Corollary 2.4] for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and [6, Corollary 2.3] for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Remark 25. Generalized Steffensen type inequalities obtained in Theorems 18–24 also hold if the function f/h is convex (concave). This follows from Theorem 7; that is, if f/h is a convex function, then $f/h \in \mathcal{M}_1^c[a,b]$ for every $c \in (a,b)$.

Remark 26. Similarly, as in Remarks 9 and 11, we obtain that conditions (28), (30), (34), (38), (40), (42), (44), and (46) can be weakened, but here we omit the details.

3. Mean Value Theorems

Generalizations of Steffensen type inequalities given by (11), (22), (35), (39), (41), and (43) are linear in f. Hence, we can define the following linear functionals:

$$L_1(f)=\int_a^{a+\lambda_1} f(t)\,dt+\int_{b-\lambda_2}^b f(t)\,dt-\int_a^b f(t)\,g(t)\,dt, \tag{48}$$

$$L_2(f)=\int_a^b f(t)\,g(t)\,dt-\int_{c-\lambda_1}^{c+\lambda_2} f(t)\,dt, \tag{49}$$

$$\begin{aligned}L_3(f) &= \int_a^{a+\lambda_1} f(t)\,k(t)\,dt+\int_{b-\lambda_2}^b f(t)\,k(t)\,dt \\ &\quad-\int_a^b f(t)\,g(t)\,dt,\end{aligned} \tag{50}$$

$$L_4(f)=\int_a^b f(t)\,g(t)\,dt-\int_{c-\lambda_1}^{c+\lambda_2} f(t)\,k(t)\,dt, \tag{51}$$

$$L_5(f) = \int_a^{a+\lambda_1} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)} \right] \times h(t)[1-g(t)] \right) dt + \int_{b-\lambda_2}^{b} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)} \right] \times h(t)[1-g(t)] \right) dt - \int_a^b f(t)g(t)\,dt, \tag{52}$$

$$L_6(f) = \int_a^b f(t)g(t)\,dt - \int_{c-\lambda_1}^{c} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)} \right] \times h(t)[1-g(t)] \right) dt - \int_c^{c+\lambda_2} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)} \right] \times h(t)[1-g(t)] \right) dt. \tag{53}$$

Under the assumptions of Theorems 8, 10, 18, 19, 21, and 22 we have that $L_i(f) \geq 0$, $i = 1, \ldots, 6$, for $f/h \in \mathcal{M}_1^c[a,b]$. Further, Corollaries 12 and 13 and Remark 25 assure that $L_i(f) \geq 0$, $i = 1, \ldots, 6$, for any convex function f/h.

Let us begin by showing a Lagrange type mean value theorem for the functional L_1.

Theorem 27. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of* (8) *and λ_2 the solution of* (9). *Let $f : [a,b] \to \mathbb{R}$ be such that $f/h \in C^2[a,b]$. If* (10) *holds, then there exists $\xi \in [a,b]$ such that*

$$L_1(f) = \frac{1}{2}\left(\frac{f(\xi)}{h(\xi)}\right)'' \times \left[\int_a^b t^2 h(t) g(t)\,dt - \int_a^{a+\lambda_1} t^2 h(t)\,dt - \int_{b-\lambda_2}^{b} t^2 h(t)\,dt \right], \tag{54}$$

where L_1 is defined by (48).

Proof. Since $f/h \in C^2[a,b]$, there exist

$$m = \min_{x\in[a,b]} \left(\frac{f(x)}{h(x)}\right)'', \qquad M = \max_{x\in[a,b]} \left(\frac{f(x)}{h(x)}\right)''. \tag{55}$$

Let

$$\Phi_1(x) = f(x) - \frac{m}{2}x^2 h(x), \qquad \Phi_2(x) = \frac{M}{2}x^2 h(x) - f(x). \tag{56}$$

The functions Φ_1/h and Φ_2/h are convex since $\Phi_i/h \geq 0$, $i = 1, 2$. Hence, $L_1(\Phi_i) \geq 0$, $i = 1, 2$, and we obtain

$$\frac{m}{2} L_1(\tilde{f}) \leq L_1(f) \leq \frac{M}{2} L_1(\tilde{f}), \tag{57}$$

where $\tilde{f}(x) = x^2 h(x)$. Since $\tilde{f}/h$ is convex we have $L_1(\tilde{f}) \geq 0$.

If $L_1(\tilde{f}) = 0$, then (57) implies $L_1(f) = 0$ and (54) holds for every $\xi \in [a,b]$. Otherwise, multiplying (57) by $2/L_1(\tilde{f})$ we obtain

$$m \leq \frac{2L_1(f)}{L_1(\tilde{f})} \leq M, \tag{58}$$

so continuinity of $(f/h)''$ ensures the existence of $\xi \in [a,b]$ satisfying (54). □

We continue with a Cauchy type mean value theorem for the functional L_1.

Theorem 28. *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of* (8) *and λ_2 the solution of* (9). *Let the functions f and F be such that $f/h, F/h \in C^2[a,b]$. If* (10) *holds and $L_1(F) \neq 0$, then there exists $\xi \in [a,b]$ such that*

$$\frac{L_1(f)}{L_1(F)} = \frac{(f/h)''(\xi)}{(F/h)''(\xi)}, \tag{59}$$

where L_1 is defined by (48).

Proof. Define $\Psi(x) = L_1(F)f(x) - L_1(f)F(x)$. Due to linearity of L_1 we have $L_1(\Psi) = 0$. Now by Theorem 27 there exist $\xi, \xi_1 \in [a,b]$ such that

$$0 = L_1(\Psi) = \frac{1}{2}\left(\frac{\Psi(\xi)}{h(\xi)}\right)'' L_1(\tilde{f}), \qquad 0 \neq L_1(F) = \frac{1}{2}\left(\frac{F(\xi_1)}{h(\xi_1)}\right)'' L_1(\tilde{f}), \tag{60}$$

where $\tilde{f}(x) = x^2 h(x)$. Therefore, $L_1(\tilde{f}) \neq 0$ and

$$0 = \left(\frac{\Psi(\xi)}{h(\xi)}\right)'' = L_1(F)\left(\frac{f(\xi)}{h(\xi)}\right)'' - L_1(f)\left(\frac{F(\xi)}{h(\xi)}\right)'', \tag{61}$$

which gives the claim of the theorem. □

Remark 29. As in Theorem 27 we can obtain Lagrange type mean value theorems for the functionals L_i, $i = 2, \ldots, 6$. Similarly, Cauchy type mean value theorems can be obtained for the functionals L_i, $i = 2, \ldots, 6$. Hence, we can obtain that there exist $\xi_i \in [a,b]$, $i = 2, \ldots, 6$, such that

$$\frac{L_i(f)}{L_i(F)} = \frac{(f/h)''(\xi_i)}{(F/h)''(\xi_i)}, \quad i = 2, \ldots, 6. \tag{62}$$

4. Exponential Convexity

We begin by recalling some definitions and results on exponential convexity; see [8, 9].

Definition 30. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0 \tag{63}$$

holds for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$.

A function $\psi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 31. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are nonnegative functions.

Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for all $k \leq n$ and $k \in \mathbb{N}$.

Definition 32. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense on I for every $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex on I if it is exponentially convex in the Jensen sense and continuous on I.

Remark 33. A function $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense, that is,

$$\psi\left(\frac{x+y}{2}\right)^2 \leq \psi(x)\psi(y), \quad \forall x, y \in I, \tag{64}$$

if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0 \tag{65}$$

holds for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$, that is, if and only if ψ is 2-exponentially convex in the Jensen sense. By induction from (64) we have

$$\psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) \leq \psi(x)^{1/2^k} \psi(y)^{1-(1/2^k)}. \tag{66}$$

Therefore, if ψ is continuous and $\psi(x) = 0$ for some $x \in I$, then from the last inequality and nonnegativity of ψ (see Remark 31) we get

$$\psi(y) = \lim_{k\to\infty} \psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) = 0 \quad \forall y \in I. \tag{67}$$

Hence, either 2-exponentially convex function is identically equal to zero or it is strictly positive and log-convex.

We also use the following well known results for convex functions.

Lemma 34. *A function $\psi : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\psi(x_1) + (x_1 - x_3)\psi(x_2) + (x_2 - x_1)\psi(x_3) \geq 0 \tag{68}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Proposition 35. *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality holds:*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{69}$$

If the function f is concave, the inequality is reversed.

Definition 36. Let f be a function defined on $[a, b]$. The nth order divided difference of f at distinct points $x_0, x_1, \ldots, x_n$ in $[a, b]$ is defined recursively by

$$[x_j; f] = f(x_j), \quad j = 0, \ldots, n,$$

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{70}$$

Remark 37. The value $[x_0, x_1, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$. Previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \cdots \leq x_k$ and letting

$$\left[\underbrace{x, \ldots, x}_{j+1 \text{ times}}; f\right] = \frac{f^{(j)}(x)}{j!}, \tag{71}$$

provided that $f^{(j)}(x)$ exists.

Next, we construct exponentially convex functions using the previously defined functionals L_i, $i = 1, \ldots, 6$. In the sequel the notation log denotes the natural logarithm function and I, J denote intervals in $\mathbb{R}$.

Theorem 38. *Let $\Omega = \{f_p/h : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for all mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p/h]$ is n-exponentially convex in the Jensen sense on J. Let L_i, $i = 1, \ldots, 6$, be linear functionals defined by* (48)–(53). *Then the mapping $p \mapsto L_i(f_p)$ is n-exponentially convex in the Jensen sense on J.*

If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is n-exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$ and $p_j \in J$, $j = 1, \ldots, n$, we define the function

$$\Phi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{(p_j+p_k)/2}(x). \tag{72}$$

Since the mapping $p \mapsto [x_0, x_1, x_2; f_p/h]$ is n-exponentially convex in the Jensen sense we have

$$\left[x_0, x_1, x_2; \frac{\Phi}{h}\right] = \sum_{j,k=1}^{n} \xi_j \xi_k \left[x_0, x_1, x_2; \frac{f_{(p_j+p_k)/2}}{h}\right] \geq 0. \tag{73}$$

So Φ/h is a convex function and

$$0 \le L_i(\Phi) = \sum_{j,k=1}^{n} \xi_j \xi_k L_i\left(f_{(p_j+p_k)/2}\right), \quad i = 1, \ldots, 6. \tag{74}$$

Therefore, the mapping $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the mapping $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition. □

If the assumptions of Theorem 38 hold for all $n \in \mathbb{N}$, then we have the following corollary.

Corollary 39. *Let $\Omega = \{f_p/h : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for all mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p/h]$ is exponentially convex in the Jensen sense on J. Let L_i, $i = 1, \ldots, 6$, be linear functionals defined by* (48)–(53). *Then the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense on J.*

If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.

Corollary 40. *Let $\Omega = \{f_p/h : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for all mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p/h]$ is 2-exponentially convex in the Jensen sense on J. Let L_i, $i = 1, \ldots, 6$, be linear functionals defined by* (48)–(53). *Then the following statements hold.*

(i) *If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then, for $r, s, t \in J$, such that $r < s < t$, we have*

$$\left[L_i(f_s)\right]^{t-r} \le \left[L_i(f_r)\right]^{t-s} \left[L_i(f_t)\right]^{s-r}, \quad i = 1, \ldots, 6. \tag{75}$$

(ii) *If the mapping $p \mapsto L_i(f_p)$ is positive and differentiable on J, then for all $p, q, u, v \in J$ such that $p \le u$ and $q \le v$ we have*

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega), \tag{76}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\dfrac{L_i(f_p)}{L_i(f_q)}\right)^{1/(p-q)}, & p \ne q, \\ \exp\left(\dfrac{(d/dp) L_i(f_p)}{L_i(f_p)}\right), & p = q. \end{cases} \tag{77}$$

Proof. (i) By Theorem 38 the mapping $p \mapsto L_i(f_p)$ is 2-exponentially convex. Hence, by Remark 33, either this mapping is identically equal to zero, in which case inequality (75) holds trivially with zeros on both sides, or it is strictly positive and log-convex. Therefore, for $r, s, t \in J$ such that $r < s < t$ Lemma 34 gives

$$(t-s)\log L_i(f_r) + (r-t)\log L_i(f_s) + (s-r)\log L_i(f_t) \ge 0, \tag{78}$$

which is equivalent to inequality (75).

(ii) By (i) we have that the mapping $p \mapsto L_i(f_p)$ is log-convex on J; that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 35 with $p \le u$, $q \le v$, $p \ne q$, $u \ne v$, we obtain

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p-q} \le \frac{\log L_i(f_u) - \log L_i(f_v)}{u-v}; \tag{79}$$

that is

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega). \tag{80}$$

The limit cases $p = q$ and $u = v$ are obtained by taking the limits $p \to q$ and $u \to v$. □

Remark 41. The results stated in Theorem 38 and Corollaries 39 and 40 still hold when some or all of the points $x_0, x_1, x_2 \in I$ coincide. The proofs are obtained by recalling Remark 37 and a suitable characterization of convexity.

We continue with an example of a family of functions which satisfies the previous conditions.

Let h be a positive integrable function and let

$$\Upsilon = \left\{\frac{f_p}{h} : \mathbb{R} \longrightarrow (0, \infty) \mid p \in \mathbb{R}\right\} \tag{81}$$

be a family of functions where f_p is defined by

$$f_p(x) = \begin{cases} \dfrac{1}{p^2} e^{px} h(x), & p \ne 0, \\ \dfrac{1}{2} x^2 h(x), & p = 0. \end{cases} \tag{82}$$

We have $(d^2/dx^2)(f_p(x)/h(x)) = e^{px} > 0$, so f_p/h is convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto (d^2/dx^2)(f_p(x)/h(x))$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 38 we have that $p \mapsto [t_0, t_1, t_2; f_p/h]$ is exponentially convex (and so exponentially convex in the Jensen sense). We see that the family Υ satisfies the assumptions of Corollary 39, so mappings $p \mapsto L_i(f_p)$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex.

For this family of functions $\mu_{p,q}(L_i, \Upsilon)$, $i = 1, \ldots, 6$, from (77) becomes

$$\mu_{p,q}(L_i, \Upsilon) = \begin{cases} \left(\dfrac{L_i(f_p)}{L_i(f_q)}\right)^{1/(p-q)}, & p \ne q, \\ \exp\left(\dfrac{L_i(\mathrm{id} \cdot f_p)}{L_i(f_p)} - \dfrac{2}{p}\right), & p = q \ne 0, \\ \exp\left(\dfrac{L_i(\mathrm{id} \cdot f_0)}{3L_i(f_0)}\right), & p = q = 0. \end{cases} \tag{83}$$

Explicitly for $\mu_{p,q}(L_1, \Upsilon)$ we have the following:

(i) for $p \neq q$,

$$\begin{aligned}\mu_{p,q}(L_1, \Upsilon) &= \left(\frac{q^2}{p^2}\left(\int_a^b e^{pt}h(t)\,g(t)\,dt - \int_a^{a+\lambda_1} e^{pt}h(t)\,dt - \int_{b-\lambda_2}^b e^{pt}h(t)\,dt\right)\right.\\ &\quad\times\left.\left(\int_a^b e^{qt}h(t)\,g(t)\,dt - \int_a^{a+\lambda_1} e^{qt}h(t)\,dt - \int_{b-\lambda_2}^b e^{qt}h(t)\,dt\right)^{-1}\right)^{1/(p-q)};\end{aligned} \tag{84}$$

(ii) for $p = q \neq 0$,

$$\begin{aligned}\mu_{p,p}(L_1, \Upsilon) &= \exp\left(\left(\int_a^b te^{pt}h(t)\,g(t)\,dt - \int_a^{a+\lambda_1} te^{pt}h(t)\,dt - \int_{b-\lambda_2}^b te^{pt}h(t)\,dt\right)\right.\\ &\quad\times\left.\left(\int_a^b e^{pt}h(t)\,g(t)\,dt - \int_a^{a+\lambda_1} e^{pt}h(t)\,dt - \int_{b-\lambda_2}^b e^{pt}h(t)\,dt\right)^{-1} - \frac{2}{p}\right);\end{aligned} \tag{85}$$

(iii) for $p = q = 0$,

$$\begin{aligned}\mu_{0,0}(L_1, \Upsilon) &= \exp\left(\frac{1}{3}\left(\int_a^b t^3h(t)\,g(t)\,dt - \int_a^{a+\lambda_1} t^3h(t)\,dt - \int_{b-\lambda_2}^b t^3h(t)\,dt\right)\right.\\ &\quad\times\left.\left(\int_a^b t^2h(t)\,g(t)\,dt - \int_a^{a+\lambda_1} t^2h(t)\,dt - \int_{a-\lambda_2}^b t^2h(t)\,dt\right)^{-1}\right).\end{aligned} \tag{86}$$

Theorem 28 applied on functions $f_p/h, f_q/h \in \Upsilon$ implies that

$$M_{p,q}(L_1, \Upsilon) = \log \mu_{p,q}(L_1, \Upsilon) \tag{87}$$

satisfies $a \leq M_{p,q}(L_1, \Upsilon) \leq b$. Hence $M_{p,q}$ is a monotonic mean by (76).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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