Research Article

The Commutators of Fractional Integrals on Generalized Herz Spaces

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Received 7 February 2014; Accepted 14 March 2014; Published 3 April 2014

Academic Editor: Dragan Djordjevic

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Let I_l be the fractional integral, $0 < l < n$ and let $b \in \text{BMO}$. We will obtain the weighted estimates for the commutator $[b, I_l]$ on the generalized Herz spaces.

1. Introduction and Results

Let $0 < l < n$ the fractional integral operator I_l is defined by

$$I_l f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-l}} dy. \quad (1)$$

Consider the commutator associated with the fractional integrals I_l and the locally integrable function b ,

$$[b, I_l] f(x) = b(x) I_l f(x) - I_l (bf)(x). \quad (2)$$

The area of the commutators of fractional integrals has been under intensive research. When $b \in \text{BMO}(\mathbb{R}^n)$, $1/p_2 = 1/p_1 - l/n$ and $1 < p_1 < n/l$, and Chanillo proved in [1] that $[b, I_l]$ is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$. Segovia and Torrea in [2] proved that $[b, I_l]$ is also bounded from $L^{p_1}(\omega^{p_1}, \mathbb{R}^n)$ to $L^{p_2}(\omega^{p_2}, \mathbb{R}^n)$ if $\omega \in A_{p_1, p_2}(\mathbb{R}^n)$. Lu and Yang in [3] studied $[b, I_l]$ Herz spaces; they proved that if $1/p_2 = 1/p_1 - l/n$, $1 < p_1 < n/l$, $-n/p_1 + l < \alpha < n(1 - 1/p_1)$ and $0 < q_1 \leq q_2 < \infty$, then $[b, I_l]$ maps $\dot{K}_{p_1}^{\alpha, q_1}(\mathbb{R}^n)$ continuously into $\dot{K}_{p_2}^{\alpha, q_2}(\mathbb{R}^n)$.

In [4], Komori and Matsuoka considered the boundedness of singular integral operators and fractional integral operators on weighted Herz spaces; they introduced generalized Herz spaces, and, as corollaries of their general theory, they obtained the boundedness of these operators on weighted Herz spaces.

For a sequence $\varphi = \{\varphi(k)\}_{k=-\infty}^{\infty}$, $\varphi(k) > 0$, we suppose that φ satisfies doubling condition of order (s, t) and write $\varphi \in D(s, t)$ if there exists $C \geq 1$ such that

$$C^{-1} 2^{s(k-j)} \leq \frac{\varphi(k)}{\varphi(j)} \leq C 2^{t(k-j)}, \quad (3)$$

for $k > j$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$ and let $\chi_k = \chi_{C_k}$ be the characteristic function of the set C_k for any $k \in \mathbb{Z}$. Suppose that ω is a weight function on $\mathbb{R}^n$. For $1 < p < \infty$, $0 < q < \infty$, the generalized Herz space is defined by

$$K_p^q(\varphi, \omega)(\mathbb{R}^n) = \left\{ f : f \text{ is a measurable function on } \mathbb{R}^n, \right. \quad (4)$$

$$\left. \|f\|_{K_p^q(\varphi, \omega)} < \infty \right\},$$

where

$$\|f\|_{K_p^q(\varphi, \omega)} = \left(\sum_{k=-\infty}^{+\infty} \varphi(k)^q \|f \chi_k\|_{L^p(\omega)}^q \right)^{1/q}, \quad (5)$$

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx \right)^{1/q}. \quad (6)$$

Let $\varphi(k) = \omega_1(B_k)^{\alpha/n}$, $\omega = \omega_2$, and ω_1, ω_2 be weight functions in (5); then, $K_p^q(\varphi, \omega)(\mathbb{R}^n)$ is the weighted Herz spaces $\dot{K}_p^\alpha(\omega_1, \omega_2)(\mathbb{R}^n)$ (see [5]).

Let $\delta > 0$. We say $\omega \in RD(\delta)$ (centered reverse doubling) if there is a positive number C such that

$$\frac{\omega(B_k)}{\omega(B_j)} \geq C2^{\delta(k-j)} \quad \text{for } k > j. \quad (7)$$

In 2000, Lu et al. in [6] proved boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators on weighted Herz spaces. In [7], Wang proved the boundedness properties of intrinsic square function on weighted Herz spaces. Tomita in [8] studied the Strang-Fix theory for approximation order in weighted Herz spaces with general Muckenhoupt weights. In [9], Hu and Wang consider the boundedness properties of commutator operators generated by $BMO(\mathbb{R}^n)$ function and intrinsic square function on weighted Herz spaces.

In this paper, we investigate the boundedness of commutators generated by fractional integrals and $BMO(\mathbb{R}^n)$ on generalized Herz spaces $K_p^q(\varphi, \omega)(\mathbb{R}^n)$ with general Muckenhoupt weights. Our main results in this paper are formulated as follows.

Theorem 1. Let $b \in BMO(\mathbb{R}^n)$, $1 < p_1 < n/l$, $0 < q < \infty$, $\delta > 0$ and let $1/p_2 = 1/p_1 - l/n$. One assumes that

- (1) $\varphi \in D(a, b)$, where $-(\delta/p_2) < a \leq b < n(1 - (1/p_1))$;
- (2) $\omega^{p_2} \in A_{r, \cdot}$, where $r = \min(1 + (p_2/p_1'), 1 + (p_2/p_1') - (bp_2/n))$;
- (3) $\omega^{p_2} \in RD(\delta)$.

Then, $[b, I_l]$ is bounded from $K_{p_1}^q(\varphi, \omega^{p_1})(\mathbb{R}^n)$ to $K_{p_2}^q(\varphi, \omega^{p_2})(\mathbb{R}^n)$.

Remark 2. When $b \leq 0$, condition (2) is equivalent to the condition that $\omega \in A_{p_1, p_2}(\mathbb{R}^n)$, but when $b > 0$, (2) is stronger than A_{p_1, p_2} -condition. Komori and Matsuoka in [4] showed that the condition (2) is the best possible by a counterexample.

Let $\varphi \in D(0, \alpha)$ in Theorem 1; we have

Corollary 3. Let $1 < p_1 < n/l$, $0 < q < \infty$, $0 \leq \alpha < n(1 - 1/p_1)$ and let $1/p_2 = 1/p_1 - l/n$. If $b \in BMO(\mathbb{R}^n)$, $\omega^{p_2} \in A_{1+p_2/p_1' - \alpha p_2/n}(\mathbb{R}^n)$, then $[b, I_l]$ is bounded from $\dot{K}_{p_1}^{\alpha, q}(1, \omega^{p_1})(\mathbb{R}^n)$ to $\dot{K}_{p_2}^{\alpha, q}(1, \omega^{p_2})(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we give some basic notation and definitions and recall some preliminaries. In Section 3, we will prove Theorem 1.

2. Definitions and Preliminaries

We begin with some properties of A_p weights which play a great role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0

and radius r_B . For any ball B and $\lambda > 0$, λB denotes the ball concentric with B whose radius is λ times as long. For a given weight function ω and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set the weighted measure $\omega(E) = \int_E \omega(x) dx$.

A weight ω is said to belong to $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-p'} dx \right)^{p-1} \leq C, \quad (8)$$

where s' is the dual of s such that $1/s + 1/s' = 1$. The class $A_1(\mathbb{R}^n)$ is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \cdot \operatorname{ess\,inf}_{x \in B} \omega(x) \quad (9)$$

for every ball $B \subset \mathbb{R}^n$.

A weight ω is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers C and δ so that

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta, \quad (10)$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n). \quad (11)$$

The classical $A_p(\mathbb{R}^n)$ weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal function in [10].

Lemma 4 (see [10, 12]). Suppose $\omega \in A_p(\mathbb{R}^n)$ and the following statements hold.

- (i) For any $1 \leq p < \infty$, there are positive numbers C and δ such that

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C2^{np(k-j)} \quad \text{for } k > j, \quad (12)$$

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C2^{np\delta(k-j)} \quad \text{for } k < j;$$

- (ii) $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$ for any $1 \leq p_1 < p_2 \leq \infty$;

- (iii) For any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.

We also need another weight class $A_{p, q}(\mathbb{R}^n)$ introduced by Muckenhoupt and Wheeden in [11] to study weighted boundedness of fractional integral operators.

Given $1 \leq p \leq q < \infty$, we say that $\omega \in A_{p, q}(\mathbb{R}^n)$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$, the inequality

$$\left(\frac{1}{|B|} \int_B \omega(y)^{-p'} dy \right)^{1/p'} \left(\frac{1}{|B|} \int_B \omega(y)^q dy \right)^{1/q} \leq C \quad (13)$$

holds when $1 < p < \infty$, and the inequality

$$\left(\frac{1}{|B|} \int_B \omega(y)^q dy \right)^{1/q} \leq C \cdot \operatorname{ess\,inf}_{x \in B} \omega(x) \quad (14)$$

for every ball $B \subset \mathbb{R}^n$

holds when $p = 1$.

By (13), we have

$$\left(\int_B \omega(y)^{-p'} dy \right)^{1/p'} \left(\int_B \omega(y)^q dy \right)^{1/q} \leq C |B|^{1/p' + 1/q}. \quad (15)$$

We summarize some properties about weights $A_{p,q}(\mathbb{R}^n)$ (see [11, 12]).

Lemma 5. *Given $1 \leq p \leq q < \infty$,*

- (i) $\omega \in A_{p,q}(\mathbb{R}^n)$ if and only if $\omega^q \in A_{1+q/p'}(\mathbb{R}^n)$;
- (ii) $\omega \in A_{p,q}(\mathbb{R}^n)$ if and only if $\omega^{-p'} \in A_{1+p'/q}(\mathbb{R}^n)$;
- (iii) $\omega \in A_{p,p}(\mathbb{R}^n)$ if and only if $\omega^p \in A_p(\mathbb{R}^n)$;
- (iv) If $p_1 < p_2$ and $q_2 > q_1$, then $A_{p_1,q_1}(\mathbb{R}^n) \subset A_{p_2,q_2}(\mathbb{R}^n)$.

A locally integrable function b is said to be in $\operatorname{BMO}(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| dx = \|b\|_* < \infty, \quad (16)$$

where $b_B = |B|^{-1} \int_B b(y) dy$.

Lemma 6 (John-Nirenberg inequality; see [13]). *Let $b \in \operatorname{BMO}(\mathbb{R}^n)$. Then, for any ball $B \subset \mathbb{R}^n$, there exist positive constants C_1 and C_2 such that, for all $\lambda > 0$,*

$$|\{x \in B : |b(x) - b_B| > \lambda\}| \leq C_1 |B| \exp\left(-\frac{C_2 \lambda}{\|b\|_*}\right). \quad (17)$$

Lemma 7 (see [14]). *Let $\omega \in A_\infty(\mathbb{R}^n)$. Then, the norm of $\operatorname{BMO}(\omega, \mathbb{R}^n)$ is equivalent to the norm of $\operatorname{BMO}(\mathbb{R}^n)$, where*

$$\begin{aligned} \operatorname{BMO}(\omega, \mathbb{R}^n) &= \left\{ b : \|b\|_{*,\omega} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\omega(B)} \right. \\ &\quad \times \left. \int_B |b(x) - b_{B,\omega}| \omega(x) dx < \infty \right\}, \quad (18) \\ b_{B,\omega} &= \frac{1}{\omega(B)} \int_B b(z) \omega(z) dz. \end{aligned}$$

Lemma 8. *Suppose $\omega \in A_\infty(\mathbb{R}^n)$ and $b \in \operatorname{BMO}(\mathbb{R}^n)$. Then, for any $p \geq 1$, one has*

$$\left(\frac{1}{\omega(B)} \int_B |b(x) - b_{B,\omega}|^p \omega(x) dx \right)^{1/p} \leq C \|b\|_*. \quad (19)$$

Proof. Since $\omega(x) \in A_\infty(\mathbb{R}^n)$, (10) and Lemma 6 imply that

$$\omega(\{x \in B : |b(x) - b_B| > \lambda\}) \leq C \omega(B) \exp\left(-\frac{C_2 \delta \lambda}{\|b\|_*}\right). \quad (20)$$

Therefore,

$$\begin{aligned} &\int_B |b(x) - b_B|^p \omega(x) dx \\ &\leq p \int_0^\infty \lambda^{p-1} \omega(\{x \in B : |b(x) - b_B| > \lambda\}) d\lambda \\ &\leq C \omega(B) \int_0^\infty \lambda^{p-1} \exp\left(-\frac{C_2 \delta \lambda}{\|b\|_*}\right) d\lambda \\ &\leq C \omega(B) \|b\|_*^p. \end{aligned} \quad (21)$$

Hence,

$$\begin{aligned} &\frac{1}{\omega(B)} \int_B |b(x) - b_{B,\omega}|^p \omega(x) dx \\ &\leq \frac{C}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x) dx \\ &\leq C \|b\|_*^p. \end{aligned} \quad (22)$$

□

3. Proof of Theorem 1

Let $f \in K_{p_1}^q(\varphi, \omega^{p_1})(\mathbb{R}^n)$, $b \in \operatorname{BMO}(\mathbb{R}^n)$. Then,

$$\begin{aligned} &\|[b, I_l] f\|_{K_{p_2}^q(\varphi, \omega^{p_2})}^q \\ &\leq C \sum_{j=-\infty}^\infty \varphi(j)^q \left(\sum_{k=-\infty}^{j-2} \|[b, I_l](f\chi_k)\chi_j(x)\|_{L^{p_2}(\omega^{p_2})} \right)^q \\ &\quad + C \sum_{j=-\infty}^\infty \varphi(j)^q \left(\sum_{k=j-1}^{j+1} \|[b, I_l](f\chi_k)\chi_j(x)\|_{L^{p_2}(\omega^{p_2})} \right)^q \\ &\quad + C \sum_{j=-\infty}^\infty \varphi(j)^q \left(\sum_{k=j+2}^\infty \|[b, I_l](f\chi_k)\chi_j(x)\|_{L^{p_2}(\omega^{p_2})} \right)^q \\ &= L_1 + L_2 + L_3. \end{aligned} \quad (23)$$

Since $\omega^{p_2} \in A_r$, where $r = \min\{1 + (p_2/p'_1), 1 + (p_2/p'_1) - (bp_2/n)\}$, then, by (i) of Lemma 5, we know that $\omega \in A_{p_1, p_2}(\mathbb{R}^n)$. Using the fact that $[b, I_l]$ is a bounded operator from $L^{p_1}(\omega^{p_1}, \mathbb{R}^n)$ to $L^{p_2}(\omega^{p_2}, \mathbb{R}^n)$ (see [2]), we obtain

$$\begin{aligned} L_2 &= C \sum_{j=-\infty}^\infty \varphi(j)^q \left(\sum_{k=j-1}^{j+1} \|[b, I_l](f\chi_k)\chi_j(x)\|_{L^{p_2}(\omega^{p_2})} \right)^q \\ &\leq C \|b\|_*^q \sum_{j=-\infty}^\infty \varphi(j)^q \sum_{k=j-1}^{j+1} \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\ &\leq C \|b\|_*^q \sum_{j=-\infty}^\infty \varphi(j)^q \|f\chi_j\|_{L^{p_1}(\omega^{p_1})}^q \\ &\leq C \|b\|_*^q \|f\|_{K_{p_1}^q(\varphi, \omega^{p_1})}^q. \end{aligned} \quad (24)$$

We now estimate L_1 . Obviously,

$$\begin{aligned} & \left\| [b, I_l] (f\chi_k) \chi_j(x) \right\|_{L^{p_2}(\omega^{p_2})} \\ & \leq C \left(\int_{C_j} \left| (b(x) - b_{B_k}) \int_{C_k} \frac{1}{|x-y|^{n-l}} \right. \right. \\ & \quad \times (f\chi_k)(y) dy \left. \right|^{p_2} \omega(x)^{p_2} dx \Big)^{1/p_2} \\ & + C \left(\int_{C_j} \left| \int_{C_k} \frac{1}{|x-y|^{n-l}} (b(y) - b_{B_k}) \right. \right. \\ & \quad \times (f\chi_k)(y) dy \left. \right|^{p_2} \omega(x)^{p_2} dx \Big)^{1/p_2} \\ & = M_1 + M_2. \end{aligned} \quad (25)$$

First, we estimate M_1 . From $j \geq k+2$, one can see then that

$$\begin{aligned} M_1 &= \left(\int_{C_j} \left| (b(x) - b_{B_k}) \int_{C_k} \frac{1}{|x-y|^{n-l}} \right. \right. \\ & \quad \times (f\chi_k)(y) dy \left. \right|^{p_2} \omega(x)^{p_2} dx \Big)^{1/p_2} \\ & \leq C 2^{-j(n-l)} \|f\chi_k\|_{L^1} \\ & \quad \times \left(\left(\int_{B_j} |b(x) - b_{B_j, \omega^{p_2}}|^{p_2} \omega(x)^{p_2} dx \right)^{1/p_2} \right. \\ & \quad + (|b_{B_j} - b_{B_j, \omega^{p_2}}| + |b_{B_j} - b_{B_k}|) \\ & \quad \times \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2} \Big). \end{aligned} \quad (26)$$

Since $\omega^{p_2} \in A_r$, then, by Lemma 8,

$$\begin{aligned} & \left(\int_{B_j} |b(x) - b_{B_j, \omega^{p_2}}|^{p_2} \omega(x)^{p_2} dx \right)^{1/p_2} \\ & \leq C \|b\|_* \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2}. \end{aligned} \quad (27)$$

Similar to the proof of Lemma 8, we can deduce that

$$|b_{B_j} - b_{B_j, \omega^{p_2}}| \leq C \|b\|_*. \quad (28)$$

From the definition of $\text{BMO}(\mathbb{R}^n)$, it is easy to see that

$$|b_{B_j} - b_{B_k}| \leq C(j-k) \|b\|_*. \quad (29)$$

By Hölder's inequality, we get

$$\|f\chi_k\|_{L^1} \leq C \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \left(\int_{B_k} \omega^{-p'_1} dx \right)^{1/p'_1}. \quad (30)$$

Then,

$$\begin{aligned} M_1 & \leq C(j-k) 2^{-j(n-l)} \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \\ & \quad \times \left(\int_{B_k} \omega^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2}. \end{aligned} \quad (31)$$

We now turn to estimate M_2 . Since $\omega \in A_{p_1, p_2}(\mathbb{R}^n)$, then, by (ii) of Lemma 5, we know $\omega^{-p'_1} \in A_\infty(\mathbb{R}^n)$. Similar to the proof of Lemma 8, we have

$$\begin{aligned} & \left(\int_{B_k} |b(x) - b_{B_k}|^{p'_1} \omega^{-p'_1} dx \right)^{1/p_2} \\ & \leq C \|b\|_* \left(\int_{B_k} \omega(x)^{-p'_1} dx \right)^{1/p'_1}. \end{aligned} \quad (32)$$

Using Hölder's inequality and (32),

$$\begin{aligned} M_2 & \leq C 2^{-j(n-l)} \int_{B_k} |b(y) - b_{B_k}| |f\chi_k(y)| dy \\ & \quad \times \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2} \\ & \leq C 2^{-j(n-l)} \left(\int_{B_k} |b(y) - b_{B_k}|^{p'_1} \omega(y)^{-p'_1} dy \right)^{1/p'_1} \\ & \quad \times \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2} \\ & \leq C 2^{-j(n-l)} \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \\ & \quad \times \left(\int_{B_k} \omega(x)^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2}. \end{aligned} \quad (33)$$

Since $\omega(x)^{p_2} \in A_r$, by (iii) of Lemma 4, we have $\omega(x)^{p_2} \in A_{\bar{r}}$ for some $\bar{r} < r$. Then, from (15) and (i) of Lemma 4,

$$\begin{aligned} & \left(\int_{B_k} \omega(x)^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_j} \omega^{p_2} dx \right)^{1/p_2} \\ & = \left(\int_{B_k} \omega^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_k} \omega(x)^{p_2} dx \right)^{1/p_2} \\ & \quad \times \left(\frac{\omega^{p_2}(B_j)}{\omega^{p_2}(B_k)} \right)^{1/p_2} \\ & \leq C 2^{kn(1/p'_1 + 1/p_2) + (j-k)n\bar{r}/p_2}. \end{aligned} \quad (34)$$

Hence,

$$\begin{aligned} & \left\| [b, I_l] (f\chi_k) \chi_j(x) \right\|_{L^{p_2}(\omega^{p_2})} \\ & \leq C \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} (j-k) 2^{-j(n-l)} \\ & \quad \times \left(\int_{B_k} \omega^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2} \\ & \leq C \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} (j-k) 2^{-j(n-l)} \\ & \leq C \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \\ & \quad \times (j-k) 2^{j(-n+l+n\bar{r}/p_2)} 2^{nk(1/p'_1 + 1/p_2 - \bar{r}/p_2)} \\ & = C \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} (j-k) 2^{(j-k)(-n+l+n\bar{r}/p_2)}. \end{aligned} \quad (35)$$

Using the condition that $\varphi \in D(a, b)$, we obtain

$$\begin{aligned}
 L_1 &= C \sum_{j=-\infty}^{\infty} \varphi(j)^q \left(\sum_{k=-\infty}^{j-2} \left\| [b, I_l] (f\chi_k) \chi_j(x) \right\|_{L^{p_2}(\omega^{p_2})} \right)^q \\
 &\leq C \|b\|_*^q \sum_{j=-\infty}^{\infty} \varphi(j)^q \\
 &\quad \times \left(\sum_{k=-\infty}^{j-2} \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} (j-k) 2^{(j-k)(-n+l+n\bar{r}/p_2)} \right)^q \\
 &\leq C \|b\|_*^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j-2} \varphi(k) \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \right. \\
 &\quad \left. \times (j-k) 2^{(j-k)(-n+l+n\bar{r}/p_2+b)} \right)^q. \tag{36}
 \end{aligned}$$

Since $r \leq 1 + p_2/p'_1 - bp_2/n$, we have $-n+l+n\bar{r}/p_2+b \leq 0$, and, then, $-n+l+n\bar{r}/p_2+b < 0$. When $0 < q \leq 1$, we have

$$\begin{aligned}
 L_1 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\quad \times \sum_{j=k+2}^{\infty} (j-k)^q 2^{(j-k)q(-n+l+n\bar{r}/p_2+b)} \\
 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\leq C \|b\|_*^q \|f\|_{K_p^q(\mathbb{R}^n)}^q. \tag{37}
 \end{aligned}$$

When $q > 1$, by Hölder's inequality, we have

$$\begin{aligned}
 L_1 &\leq C \|b\|_*^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j-2} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \right. \\
 &\quad \left. \times (j-k) 2^{(j-k)(-n+l+n\bar{r}/p_2+b)} \right) \\
 &\quad \times \left(\sum_{k=-\infty}^{j-2} (j-k) 2^{(j-k)(-n+l+n\bar{r}/p_2+b)} \right)^{q/q'} \\
 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\leq C \|b\|_*^q \|f\|_{K_p^q(\mathbb{R}^n)}^q. \tag{38}
 \end{aligned}$$

Now, we estimate L_3 . Since $k \geq j+2$, similar to the estimation of L_1 , we have

$$\begin{aligned}
 &\left\| [b, I_l] (f\chi_k) \chi_j(x) \right\|_{L^{p_2}(\omega^{p_2})} \\
 &\leq C (k-j) 2^{-k(n-l)} \|b\|_* \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \\
 &\quad \times \left(\int_{B_k} \omega(x)^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2}. \tag{39}
 \end{aligned}$$

From the fact $\omega \in A_{p_1, p_2}(\mathbb{R}^n)$ and (15), we get

$$\begin{aligned}
 &\left(\int_{B_k} \omega^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_k} \omega(x)^{p_2} dx \right)^{1/p_2} \\
 &\leq C 2^{kn(1/p'_1+1/p_2)}. \tag{40}
 \end{aligned}$$

By $\omega^{p_2} \in RD(\delta)$, we have

$$\left(\frac{\omega^{p_2}(B_j)}{\omega^{p_2}(B_k)} \right)^{1/p_2} \leq C 2^{(j-k)\delta/p_2}. \tag{41}$$

Then,

$$\begin{aligned}
 &\left(\int_{B_k} \omega(x)^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_j} \omega(x)^{p_2} dx \right)^{1/p_2} \\
 &= \left(\int_{B_k} \omega^{-p'_1} dx \right)^{1/p'_1} \left(\int_{B_k} \omega(x)^{p_2} dx \right)^{1/p_2} \\
 &\quad \times \left(\frac{\omega^{p_2}(B_j)}{\omega^{p_2}(B_k)} \right)^{1/p_2} \\
 &\leq C 2^{kn(1/p'_1+1/p_2)+(j-k)\delta/p_2}. \tag{42}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 &\left\| [b, I_l] (f\chi_k) \chi_j(x) \right\|_{L^{p_2}(\omega^{p_2})} \\
 &\leq C \|b\|_* (k-j) 2^{kn(1/p'_1+1/p_2)+(j-k)\delta/p_2-k(n-l)} \\
 &= C \|b\|_* (k-j) 2^{(j-k)\delta/p_2} \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}. \tag{43}
 \end{aligned}$$

Using the condition that $\varphi \in D(a, b)$,

$$\begin{aligned}
 L_3 &= C \sum_{j=-\infty}^{\infty} \varphi(j)^q \left(\sum_{k=j+2}^{\infty} \left\| [b, I_l] (f\chi_k) \chi_j(x) \right\|_{L^{p_2}(\omega^{p_2})} \right)^q \\
 &\leq C \|b\|_*^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=j+2}^{\infty} \varphi(k) \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \right. \\
 &\quad \left. \times (k-j) 2^{(j-k)(\delta/p_2+a)} \right)^q. \tag{44}
 \end{aligned}$$

When $0 < q \leq 1$, we have

$$\begin{aligned}
 L_3 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\quad \times \sum_{j=-\infty}^{k-2} (k-j)^q 2^{(j-k)(\delta/p_2+a)q} \\
 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\leq C \|b\|_*^q \|f\|_{K_p^q(\varphi, \omega^{p_1})}^q. \tag{45}
 \end{aligned}$$

When $q > 1$, by the Hölder inequality, we get

$$\begin{aligned}
 L_3 &\leq C \|b\|_*^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=j+2}^{\infty} \varphi(k) \|f\chi_k\|_{L^{p_1}(\omega^{p_1})} \right. \\
 &\quad \left. \times (k-j) 2^{(j-k)(\delta/p_2+a)} \right)^q \\
 &\leq C \|b\|_*^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=j+2}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \right. \\
 &\quad \left. \times (k-j) 2^{(j-k)(\delta/p_2+a)} \right) \\
 &\quad \times \left(\sum_{k=j+2}^{\infty} (k-j) 2^{(j-k)(\delta/p_2+a)} \right)^{q/q'} \quad (46) \\
 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\quad \times \left(\sum_{j=-\infty}^{k-2} (k-j)^q 2^{(j-k)(\delta/p_2+a)q} \right) \\
 &\leq C \|b\|_*^q \sum_{k=-\infty}^{\infty} \varphi(k)^q \|f\chi_k\|_{L^{p_1}(\omega^{p_1})}^q \\
 &\leq C \|b\|_*^q \|f\|_{K_p^q(\varphi, \omega^{p_1})}^q.
 \end{aligned}$$

Combining the above estimates for L_1 , L_2 , and L_3 , the proof of Theorem 1 is completed.

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgment

The authors wish to thank the referee for his/her useful suggestions for the improvement of the paper.

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