Research Article

Marcinkiewicz Integral Operators and Commutators on Herz Spaces with Variable Exponents

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Our aim in this paper is to give the boundedness of the Marcinkiewicz integral μ_Ω on Herz spaces $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$, where the two main indices are variable. Meanwhile, we consider the boundedness of the higher order commutator $\mu_{\Omega,b}^m$ generated by μ_Ω and a function b in $\text{BMO}(\mathbb{R}^n)$ on these spaces.

1. Introduction

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma(x')$. Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and has mean zero on $\mathbb{S}^{n-1}$, that is,

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0. \quad (1)$$

Then the Marcinkiewicz integral μ_Ω in higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \quad (2)$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy. \quad (3)$$

Denote by $\mathbb{N}$ the set of all positive integer numbers. Let $m \in \mathbb{N}$ and $b \in \text{BMO}(\mathbb{R}^n)$; the higher order commutator $\mu_{\Omega,b}^m$ is defined by

$$\mu_{\Omega,b}^m(f)(x) = \left(\int_0^\infty |F_{\Omega,b,t}^m(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \quad (4)$$

where

$$F_{\Omega,b,t}^m(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x) - b(y))^m f(y) dy. \quad (5)$$

Stein [1] defined the operator μ_Ω and proved that if $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1})$ ($0 < \gamma \leq 1$), then μ_Ω is of type (p, p) ($1 < p \leq 2$) and of weak type $(1, 1)$. Benedek et al. [2] showed that μ_Ω is of type (p, p) ($1 < p < \infty$) with $\Omega \in C^1(\mathbb{S}^{n-1})$. Ding et al. [3] improved the previous results to the case of $\Omega \in H^1(\mathbb{S}^{n-1})$, where $H^1(\mathbb{S}^{n-1})$ denotes the Hardy space on $\mathbb{S}^{n-1}$. Obviously, $\mu_{\Omega,b}^1 = [b, \mu_\Omega]$, which was defined by Torchinsky and Wang in [4]; moreover, they proved that if $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1})$ ($0 < \gamma \leq 1$), then $[b, \mu_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$). Ding et al. [5] weakened the smoothness of the kernel to a rough kernel and showed that if $\Omega \in L^q(\mathbb{S}^{n-1})$ ($1 < q \leq \infty$), then $[b, \mu_\Omega]$ is of type (p, p) ($1 < p < \infty$). Ding et al. [6] established the weighted weak $L\log L$ type estimates for $\mu_{\Omega,b}^m$ when $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1})$ ($0 < \gamma \leq 1$). Recently, Zhang [7] improved the previous result and proved that $\mu_{\Omega,b}^m$ enjoys the same weighted weak $L\log L$ type estimates when the kernel Ω satisfies a kind of Dini's conditions. For further details on recent developments on this field, we refer the readers to [8, 9] and references therein.

Function spaces with variable exponents were intensively studied during the past 20 years, due to their applications to PDE with nonstandard growth conditions and so on; we mention [10, 11], for instance. Since the fundamental paper [12] by Kováčik and Rákosník appeared in 1991, the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ have attracted a great attention and many interesting results have been obtained;

see [13–15]. Izuki [16, 17] defined the Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and $q \in (0, \infty]$. Wang et al. [18, 19] obtained the boundedness of μ_Ω and $[b, \mu_\Omega]$ on $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$. Almeida and Drihem [20] established the boundedness of a wide class of sublinear operators, which includes maximal, potential, and Calderón-Zygmund operators, on Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$, where the two main exponents α and p are both variable. In this paper we will give boundedness results for μ_Ω and $\mu_{\Omega,b}^m$ on Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$.

For brevity, $|E|$ denotes the Lebesgue measure for a measurable set $E \subset \mathbb{R}^n$. f_E denotes the integral average of f on E , that is, $f_E = |E|^{-1} \int_E f(x) dx$. $p'(\cdot)$ stands for the conjugate exponent $1/p(\cdot) + 1/p'(\cdot) = 1$. $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. C denotes a positive constant, which may have different values even in the same line. $f \lesssim g$ means that $f \leq Cg$, and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Preliminaries and Main Results

Let $E \subset \mathbb{R}^n$ with $|E| > 0$, and let $p(\cdot) : E \rightarrow [1, \infty)$ be a measurable function. Let us first recall some definitions and notations.

Definition 1. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some constant } \lambda > 0 \right\}. \quad (6)$$

This is a Banach space with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \quad (7)$$

Let $f \in L_{\text{loc}}^1(E)$; the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r) \cap E} |f(y)| dy. \quad (8)$$

Denote

$$p_- = \text{ess inf } \{p(x) : x \in E\},$$

$$p_+ = \text{ess sup } \{p(x) : x \in E\},$$

$$\mathcal{P}(E) = \{p(\cdot) : p_- > 1, p_+ < \infty\},$$

$$\mathcal{B}(E) = \{p(\cdot) \in \mathcal{P}(E) : M \text{ is bounded on } L^{p(\cdot)}(E)\}. \quad (9)$$

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $R_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$. For $m \in \mathbb{N}$, one denotes $\widetilde{\chi}_m = \chi_{R_m}$ if $m \geq 1$, and $\widetilde{\chi}_0 = \chi_{B_0}$. By ℓ^q ($0 < q \leq \infty$), we denote the discrete Lebesgue space equipped by the usual quasinorm.

Definition 2. Let $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(1) The homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined by

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} < \infty \right\}, \quad (10)$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} = \left\| \left\{ 2^{\alpha(\cdot)k} f \chi_k \right\}_{k=-\infty}^{\infty} \right\|_{\ell^q(\mathbb{Z})}. \quad (11)$$

(2) The inhomogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ is defined by

$$K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n) : \|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} < \infty \right\}, \quad (12)$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} = \left\| \left\{ 2^{\alpha(\cdot)m} f \widetilde{\chi}_m \right\}_{m=0}^{\infty} \right\|_{\ell^q(\mathbb{N})}, \quad (13)$$

with the usual modification when $q = \infty$.

Remark 3. It is obvious that if $0 < q_1 \leq q_2 \leq \infty$, then $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q_1}(\mathbb{R}^n) \subset \dot{K}_{p(\cdot)}^{\alpha(\cdot),q_2}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha(\cdot),q_1}(\mathbb{R}^n) \subset K_{p(\cdot)}^{\alpha(\cdot),q_2}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constants, then $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) = \dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n) = K_p^{\alpha,q}(\mathbb{R}^n)$ are classical Herz spaces; see [21, 22].

Definition 4. A function $\alpha(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is called log-Hölder continuous at the origin, if there exists a constant $C_{\log} > 0$ such that

$$|\alpha(x) - \alpha(0)| \leq \frac{C_{\log}}{\log(e + 1/|x|)}, \quad (14)$$

for all $x \in \mathbb{R}^n$. If, for some $\alpha_\infty \in \mathbb{R}$ and $C_{\log} > 0$, there holds

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_{\log}}{\log(e + |x|)} \quad (15)$$

for all $x \in \mathbb{R}^n$, then $\alpha(\cdot)$ is called log-Hölder continuous at infinity.

Let one denote

$$\|\{h_k\}\|_{\ell_{\geq}^q(L^{p(\cdot)})} = \left(\sum_{k \geq 0} \|h_k\|_{L^{p(\cdot)}}^q \right)^{1/q}, \quad (16)$$

$$\|\{h_k\}\|_{\ell_{<}^q(L^{p(\cdot)})} = \left(\sum_{k < 0} \|h_k\|_{L^{p(\cdot)}}^q \right)^{1/q}$$

for sequences $\{h_k\}_{k \in \mathbb{Z}}$ of measurable functions (with the usual modification when $q = \infty$).

Proposition 5 (see [20]). *Let $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\begin{aligned} \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} &\approx \left\| \left\{ 2^{\alpha(0)k} f \chi_k \right\} \right\|_{\ell^q_{\mathbb{Z}}(L^{p(\cdot)})} \\ &+ \left\| \left\{ 2^{\alpha_\infty k} f \chi_k \right\} \right\|_{\ell^q_{\mathbb{Z}}(L^{p(\cdot)})}. \end{aligned} \quad (17)$$

Before stating the main results of this paper, we introduce some key lemmas that will be used later.

Lemma 6 (generalized Hölder's inequality [12]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$; if $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x) g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \quad (18)$$

where $r_p = 1 + 1/p_- - 1/p_+$.

We remark that the following Lemmas 7–9 were shown in Izuki [17, 23], and Lemma 10 was considered by Wang et al. in [18].

Lemma 7. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; then one has, for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1. \quad (19)$$

Lemma 8. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; then one has, for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq \left(\frac{|S|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq \left(\frac{|S|}{|B|} \right)^{\delta_2}, \quad (20)$$

where δ_1 and δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Lemma 9. *Let $m \in \mathbb{N}$, $b \in BMO(\mathbb{R}^n)$, and $k > i$ ($k, i \in \mathbb{N}$); then one has*

$$\begin{aligned} \sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\approx \|b\|_{BMO}^m, \\ \|(b - b_{B_i})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq (k - i)^m \|b\|_{BMO}^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (21)$$

Lemma 10. *Let $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1})$ ($0 < \gamma \leq 1$), $b \in BMO(\mathbb{R}^n)$, and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; then one has*

$$\begin{aligned} \|\mu_\Omega(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\ \|\mu_{\Omega,b}^m(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \|b\|_{BMO}^m \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (22)$$

Our results in this paper can be stated as follows.

Theorem 11. *Let $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1})$ ($0 < \gamma \leq 1$), $0 < q \leq \infty$, and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. And let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder*

continuous both at the origin and at infinity, such that $-\alpha(0) < \alpha_\infty < \alpha_\infty$, where $0 < \delta_1, \delta_2 < 1$ are the constants appearing in Lemma 8; then the operator μ_Ω is bounded on $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$.

Theorem 12. *Let $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1})$ ($0 < \gamma \leq 1$), $b \in BMO(\mathbb{R}^n)$, $0 < q \leq \infty$, and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. And let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity, such that $-\alpha(0) < \alpha_\infty < \alpha_\infty$, where $0 < \delta_1, \delta_2 < 1$ are the constants appearing in Lemma 8; then the higher order commutator $\mu_{\Omega,b}^m$ is bounded on $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$.*

Remark 13. If $\alpha(\cdot) \equiv \alpha$ is constant, then the statements corresponding to Theorems 11 and 12 can be found in [19, 24]. We consider only $0 < q < \infty$ in Section 3. The arguments are similar in the case $q = \infty$.

3. Proofs of the Theorems

In this section, we prove the boundedness of μ_Ω and $\mu_{\Omega,b}^m$ on $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ (the same arguments can be used in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$); some of our decomposition techniques are similar to those used by Dong and Xu in [25].

Proof of Theorem 11. In view of Proposition 5, we have

$$\begin{aligned} \|\mu_\Omega(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} &\approx \left\| \left\{ 2^{\alpha(0)k} \mu_\Omega(f) \chi_k \right\} \right\|_{\ell^q_{\mathbb{Z}}(L^{p(\cdot)})} \\ &+ \left\| \left\{ 2^{\alpha_\infty k} \mu_\Omega(f) \chi_k \right\} \right\|_{\ell^q_{\mathbb{Z}}(L^{p(\cdot)})} \\ &= I_< + I_>. \end{aligned} \quad (23)$$

Let $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$; write

$$f(x) = \sum_{i=-\infty}^{\infty} f(x) \chi_i(x) = \sum_{i=-\infty}^{\infty} f_i(x). \quad (24)$$

Minkowski's inequality implies that

$$\begin{aligned} I_< &= \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \|\mu_\Omega(f) \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\ &\leq \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{i=-\infty}^{k-2} \|\mu_\Omega(f_i) \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &+ \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{i=k-1}^{k+1} \|\mu_\Omega(f_i) \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &+ \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{i=k+2}^{\infty} \|\mu_\Omega(f_i) \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &= E_< + F_< + G_<. \end{aligned} \quad (25)$$

Similarly we obtain

$$\begin{aligned}
 I_{>} &= \left\{ \sum_{k=0}^{\infty} 2^{\alpha_{\infty} k q} \left\| \mu_{\Omega}(f) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\
 &\lesssim \left\{ \sum_{k=0}^{\infty} 2^{\alpha_{\infty} k q} \left(\sum_{i=-\infty}^{k-2} \left\| \mu_{\Omega}(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
 &\quad + \left\{ \sum_{k=0}^{\infty} 2^{\alpha_{\infty} k q} \left(\sum_{i=k-1}^{k+1} \left\| \mu_{\Omega}(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \quad (26) \\
 &\quad + \left\{ \sum_{k=0}^{\infty} 2^{\alpha_{\infty} k q} \left(\sum_{i=k+2}^{\infty} \left\| \mu_{\Omega}(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
 &= E_{>} + F_{>} + G_{>}.
 \end{aligned}$$

Thus we get

$$\left\| \mu_{\Omega}(f) \right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(\mathbb{R}^n)} \leq E + F + G, \quad (27)$$

where $E = E_{<} + E_{>}$, $F = F_{<} + F_{>}$, and $G = G_{<} + G_{>}$.

For F , Lemma 10 yields

$$\begin{aligned}
 F &= F_{<} + F_{>} \\
 &\lesssim \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0) k q} \left\| f_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\
 &\quad + \left\{ \sum_{k=0}^{\infty} 2^{\alpha_{\infty} k q} \left\| f_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \quad (28) \\
 &\lesssim \left\| \{ 2^{\alpha(0) k} f \chi_k \} \right\|_{\ell_{<}^q(L^{p(\cdot)})} + \left\| \{ 2^{\alpha_{\infty} k} f \chi_k \} \right\|_{\ell_{>}^q(L^{p(\cdot)})} \\
 &\approx \left\| f \right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(\mathbb{R}^n)}.
 \end{aligned}$$

Now we turn to estimate E . Observe that if $x \in R_k$, $y \in R_i$, and $i \leq k-2$, then $|x-y| \approx |x| \approx 2^k$ and

$$\left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right| \lesssim \frac{|y|}{|x-y|^3}. \quad (29)$$

Since $\Omega \in \text{Lip}_\gamma(\mathbb{S}^{n-1}) \subset L^\infty(\mathbb{S}^{n-1})$, by Minkowski's inequality and Lemma 6, we have

$$\begin{aligned}
 |\mu_{\Omega}(f_i)(x)| &\leq \left(\int_0^{|x|} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_i(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 &\quad + \left(\int_{|x|}^{\infty} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_i(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
 &\lesssim \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \left(\int_{|x-y| \leq t, |x| \geq t} \frac{dt}{t^3} \right)^{1/2} dy \\
 &\quad + \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \left(\int_{|x|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \\
 &\lesssim \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \cdot \frac{|y|^{1/2}}{|x-y|^{3/2}} dy + \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \cdot \frac{1}{|x|} dy \\
 &\lesssim 2^{(i-k)/2} 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
 &\quad + 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
 &\lesssim 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \quad (30)
 \end{aligned}$$

Lemmas 7 and 8 lead to

$$\begin{aligned}
 &\left\| \mu_{\Omega}(f_i)(x) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
 &\lesssim 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
 &\lesssim \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \quad (31) \\
 &\lesssim 2^{(i-k)n\delta_2} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
 \end{aligned}$$

Thus we get

$$\begin{aligned}
 E_{<} &\lesssim \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left(\sum_{i=-\infty}^{k-2} 2^{(i-k)n\delta_2} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
 &\approx \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha(0)i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(i-k)(n\delta_2 - \alpha(0))} \right)^q \right\}^{1/q}. \quad (32)
 \end{aligned}$$

If $1 < q < \infty$, since $n\delta_2 - \alpha(0) > 0$, Hölder's inequality implies that

$$\begin{aligned}
 E_{<} &\lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha(0)i q} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)(n\delta_2 - \alpha(0))q/2} \right) \right. \\
 &\quad \times \left. \left(\sum_{i=-\infty}^{k-2} 2^{(i-k)(n\delta_2 - \alpha(0))q'/2} \right)^{q/q'} \right\}^{1/q} \quad (33) \\
 &\lesssim \left\{ \sum_{i=-\infty}^{-3} 2^{\alpha(0)i q} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{-1} 2^{(i-k)(n\delta_2 - \alpha(0))q/2} \right\}^{1/q} \\
 &\lesssim \left\| \{ 2^{\alpha(0)k} f \chi_k \} \right\|_{\ell_{<}^q(L^{p(\cdot)})} \\
 &\lesssim \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(\mathbb{R}^n)}.
 \end{aligned}$$

If $0 < q \leq 1$, by the well-known inequality

$$\left(\sum_{i=1}^{\infty} a_i \right)^q \leq \sum_{i=1}^{\infty} a_i^q \quad (a_i > 0, i = 1, 2, \dots), \quad (34)$$

we obtain

$$\begin{aligned} E_{<} &\leq \left\{ \sum_{k=-\infty}^{-1} \sum_{i=-\infty}^{k-2} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)(n\delta_2 - \alpha(0))q} \right\}^{1/q} \\ &\approx \left\{ \sum_{i=-\infty}^{-3} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{-1} 2^{(i-k)(n\delta_2 - \alpha(0))q} \right\}^{1/q} \\ &\leq \left\| 2^{\alpha(0)k} f\chi_k \right\|_{\ell_{<}^q(L^{p(\cdot)})} \\ &\leq \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \end{aligned} \quad (35)$$

Similarly we have

$$\begin{aligned} E_{>} &\leq \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_{\infty}q} \left(\sum_{i=-\infty}^{k-2} 2^{(i-k)n\delta_2} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &\approx \left\{ \sum_{k=0}^{\infty} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha_{\infty}i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(i-k)(n\delta_2 - \alpha_{\infty})} \right)^q \right\}^{1/q}. \end{aligned} \quad (36)$$

If $1 < q < \infty$, since $\alpha_{\infty} + n\delta_2 > 2\alpha_{\infty} > 2\alpha(0)$, then we get

$$\begin{aligned} E_{>} &\leq \left\{ \sum_{k=0}^{\infty} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)(n\delta_2 - \alpha_{\infty})q/2} \right) \right. \\ &\quad \times \left. \left(\sum_{i=-\infty}^{k-2} 2^{(i-k)(n\delta_2 - \alpha_{\infty})q'/2} \right)^{q/q'} \right\}^{1/q} \\ &\leq \left\{ \sum_{k=0}^{\infty} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)(n\delta_2 - \alpha_{\infty})q/2} \right) \right\}^{1/q} \\ &\approx \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=0}^{\infty} 2^{(i-k)(n\delta_2 - \alpha_{\infty})q/2} \right. \\ &\quad \left. + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{\infty} 2^{(i-k)(n\delta_2 - \alpha_{\infty})q/2} \right\}^{1/q} \\ &\leq \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha(0)iq} 2^{(\alpha_{\infty} + n\delta_2 - 2\alpha(0))iq/2} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right. \\ &\quad \left. + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\ &\leq \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \end{aligned}$$

$$\begin{aligned} &\leq \left\| \{2^{\alpha(0)k} f\chi_k\} \right\|_{\ell_{<}^q(L^{p(\cdot)})} + \left\| \{2^{\alpha_{\infty}k} f\chi_k\} \right\|_{\ell_{>}^q(L^{p(\cdot)})} \\ &\approx \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \end{aligned}$$

(37)

If $0 < q \leq 1$, since $\alpha(0) \leq \alpha_{\infty}$, we obtain

$$\begin{aligned} E_{>} &\leq \left\{ \sum_{k=0}^{\infty} \sum_{i=-\infty}^{k-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)(n\delta_2 - \alpha_{\infty})q} \right\}^{1/q} \\ &\approx \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=0}^{\infty} 2^{(i-k)(n\delta_2 - \alpha_{\infty})q} \right. \\ &\quad \left. + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{\infty} 2^{(i-k)(n\delta_2 - \alpha_{\infty})q} \right\}^{1/q} \\ &\leq \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\ &\leq \left\| \{2^{\alpha(0)k} f\chi_k\} \right\|_{\ell_{<}^q(L^{p(\cdot)})} + \left\| \{2^{\alpha_{\infty}k} f\chi_k\} \right\|_{\ell_{>}^q(L^{p(\cdot)})} \\ &\approx \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \end{aligned} \quad (38)$$

Thus, we arrive at

$$E = E_{<} + E_{>} \leq \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \quad (39)$$

For G , observe that if $x \in R_k$, $y \in R_i$, and $i \geq k+2$, then $|x - y| \approx |y| \approx 2^i$ and

$$\left| \frac{1}{|x - y|^2} - \frac{1}{|y|^2} \right| \leq \frac{|x|}{|x - y|^3}. \quad (40)$$

From Minkowski's inequality and Lemma 6, it follows that

$$\begin{aligned} &|\mu_{\Omega}(f_i)(x)| \\ &\leq \left(\int_0^{|y|} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_i(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\quad + \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_i(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\leq t, |y|\geq t} \frac{dt}{t^3} \right)^{1/2} dy \\ &\quad + \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \left(\int_{|y|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \\ &\leq \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \cdot \frac{|x|^{1/2}}{|x-y|^{3/2}} dy + \int_{R_i} \frac{|f_i(y)|}{|x-y|^{n-1}} \cdot \frac{1}{|y|} dy \end{aligned}$$

$$\begin{aligned}
&\leq 2^{(k-i)/2} 2^{-in} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad + C 2^{-in} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq 2^{-in} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{41}$$

By Lemmas 7 and 8, we have

$$\begin{aligned}
&\|\mu_\Omega(f_i)(x) \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq 2^{-in} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\
&\leq 2^{(k-i)n\delta_1} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{42}$$

Thus we get

$$\begin{aligned}
G_{<} &\leq \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left(\sum_{i=k+2}^{\infty} 2^{(k-i)n\delta_1} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\approx \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{i=k+2}^{\infty} 2^{\alpha(0)i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-i)(n\delta_1+\alpha(0))} \right)^q \right\}^{1/q}, \\
G_{>} &\leq \left\{ \sum_{k=0}^{\infty} \left(\sum_{i=k+2}^{\infty} 2^{\alpha_\infty i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-i)(n\delta_1+\alpha_\infty)} \right)^q \right\}^{1/q}.
\end{aligned} \tag{43}$$

Using the same arguments as that for $E_{<}$ and $E_{>}$, we get

$$G = G_{<} + G_{>} \leq \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \tag{44}$$

Hence the proof of Theorem 11 is completed. $\square$

Proof of Theorem 12. We apply Proposition 5 again and get

$$\begin{aligned}
\|\mu_{\Omega,b}^m(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} &\approx \left\| \{2^{\alpha(0)k} \mu_{\Omega,b}^m(f) \chi_k\} \right\|_{\ell_{<}^q(L^{p(\cdot)})} \\
&\quad + \left\| \{2^{\alpha_\infty k} \mu_{\Omega,b}^m(f) \chi_k\} \right\|_{\ell_{>}^q(L^{p(\cdot)})} \\
&= J_{<} + J_{>}.
\end{aligned} \tag{45}$$

Let $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$, and write

$$f(x) = \sum_{i=-\infty}^{\infty} f(x) \chi_i(x) = \sum_{i=-\infty}^{\infty} f_i(x). \tag{46}$$

By Minkowski's inequality, we have

$$\begin{aligned}
J_{<} &= \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left\| \mu_{\Omega,b}^m(f) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\
&\leq \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{i=k-1}^{k-2} \left\| \mu_{\Omega,b}^m(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{i=k-1}^{k+1} \left\| \mu_{\Omega,b}^m(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{i=k+2}^{\infty} \left\| \mu_{\Omega,b}^m(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&= U_{<} + V_{<} + W_{<}.
\end{aligned} \tag{47}$$

By the same way, we obtain

$$\begin{aligned}
J_{>} &= \left\{ \sum_{k=0}^{\infty} 2^{\alpha_\infty kq} \left\| \mu_{\Omega,b}^m(f) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\
&\leq \left\{ \sum_{k=0}^{\infty} 2^{\alpha_\infty kq} \left(\sum_{i=-\infty}^{k-2} \left\| \mu_{\Omega,b}^m(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{\alpha_\infty kq} \left(\sum_{i=k-1}^{k+1} \left\| \mu_{\Omega,b}^m(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{\alpha_\infty kq} \left(\sum_{i=k+2}^{\infty} \left\| \mu_{\Omega,b}^m(f_i) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&= U_{>} + V_{>} + W_{>}.
\end{aligned} \tag{48}$$

Thus, we have

$$\|\mu_{\Omega,b}^m(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \leq U + V + W, \tag{49}$$

where $U = U_{<} + U_{>}$, $V = V_{<} + V_{>}$, and $W = W_{<} + W_{>}$.

For V , by Lemma 10, we have

$$\begin{aligned}
V &= V_{<} + V_{>} \\
&\leq \left\{ \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \|f_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{\alpha_\infty kq} \|f_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\
&\leq \left\| \{2^{\alpha(0)k} f \chi_k\} \right\|_{\ell_{<}^q(L^{p(\cdot)})} + \left\| \{2^{\alpha_\infty k} f \chi_k\} \right\|_{\ell_{>}^q(L^{p(\cdot)})} \\
&\approx \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}.
\end{aligned} \tag{50}$$

For U , observe that if $x \in R_k$, $y \in R_i$, and $i \leq k-2$, then

$$\begin{aligned}
& |\mu_{\Omega,b}^m(f_i)(x)| \\
& \leq \left(\int_0^{|x|} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)]^m f_i(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& \quad + \left(\int_{|x|}^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)]^m f_i(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& \leq \int_{R_i} \frac{|b(x)-b(y)|^m |f_i(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\leq t, |x|\geq t} \frac{dt}{t^3} \right)^{1/2} dy \\
& \quad + \int_{R_i} \frac{|b(x)-b(y)|^m |f_i(y)|}{|x-y|^{n-1}} \left(\int_{|x|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \\
& \leq \int_{R_i} \frac{|b(x)-b(y)|^m |f_i(y)|}{|x-y|^{n-1}} \cdot \frac{|y|^{1/2}}{|x-y|^{3/2}} dy \\
& \quad + \int_{R_i} \frac{|b(x)-b(y)|^m |f_i(y)|}{|x-y|^{n-1}} \cdot \frac{1}{|x|} dy \\
& \leq 2^{(i-k)/2} 2^{-kn} \int_{R_i} |b(x)-b(y)|^m |f_i(y)| dy \\
& \quad + 2^{-kn} \int_{R_i} |b(x)-b(y)|^m |f_i(y)| dy \\
& \leq 2^{-kn} \int_{R_i} |b(x)-b(y)|^m |f_i(y)| dy \\
& \leq 2^{-kn} \sum_{j=0}^m C_m^j |b(x)-b_{B_i}|^{m-j} \int_{R_i} |b_{B_i}-b(y)|^j |f_i(y)| dy \\
& \leq 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{j=0}^m C_m^j |b(x)-b_{B_i}|^{m-j} \|(b_{B_i}-b)^j \chi_i\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{51}
\end{aligned}$$

An application of Lemmas 7, 8, and 10 gives

$$\begin{aligned}
& \|\mu_{\Omega,b}^m(f_i) \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leq 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \quad \times \sum_{j=0}^m C_m^j \|(b(x)-b_{B_i})^{m-j} \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|(b_{B_i}-b)^j \chi_i\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
& \leq 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \quad \times \sum_{j=0}^m C_m^j (k-i)^{m-j} \|b\|_{\text{BMO}}^{m-j} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|b\|_{\text{BMO}}^j \|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
& \leq (k-i+1)^m 2^{-kn} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
& \leq (k-i+1)^m \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_i}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
& \leq (k-i+1)^m 2^{(i-k)n\delta_2} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{52}
\end{aligned}$$

For convenience below we put $\sigma = n\delta_2 - \alpha(0)$; if $1 < q < \infty$, then we use Hölder's inequality and obtain

$$\begin{aligned}
U_{<} & \leq \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha(0)i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} (k-i+1)^m 2^{(i-k)\sigma} \right)^q \right\}^{1/q} \\
& \leq \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{i=-\infty}^{k-2} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)\sigma q/2} \right) \right. \\
& \quad \times \left. \left(\sum_{i=-\infty}^{k-2} (k-i+1)^{mq'} 2^{(i-k)\sigma q'/2} \right)^{q/q'} \right\}^{1/q} \\
& \leq \left\{ \sum_{i=-\infty}^{-3} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{-1} 2^{(i-k)\sigma q/2} \right\}^{1/q} \\
& \leq \left\| \{2^{\alpha(0)k} f \chi_k\} \right\|_{\ell^q_{<}(L^{p(\cdot)})} \\
& \leq \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \tag{53}
\end{aligned}$$

If $0 < q \leq 1$, then we get

$$\begin{aligned}
U_{<} & \leq \left\{ \sum_{k=-\infty}^{-1} \sum_{i=-\infty}^{k-2} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q (k-i+1)^{mq} 2^{(i-k)\sigma q} \right\}^{1/q} \\
& \approx \left\{ \sum_{i=-\infty}^{-3} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{-1} (k-i+1)^{mq} 2^{(i-k)\sigma q} \right\}^{1/q} \\
& \leq \left\| \{2^{\alpha(0)k} f \chi_k\} \right\|_{\ell^q_{<}(L^{p(\cdot)})} \\
& \leq \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \tag{54}
\end{aligned}$$

Similarly, we put $\eta = n\delta_2 - \alpha_\infty$; if $1 < q < \infty$, by Hölder's inequality, we obtain

$$\begin{aligned}
U_{>} & \leq \left\{ \sum_{k=0}^\infty \left(\sum_{i=-\infty}^{k-2} 2^{\alpha_\infty i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} (k-i+1)^m 2^{(i-k)\eta} \right)^q \right\}^{1/q} \\
& \leq \left\{ \sum_{k=0}^\infty \left(\sum_{i=-\infty}^{k-2} 2^{\alpha_\infty iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)\eta q/2} \right) \right. \\
& \quad \times \left. \left(\sum_{i=-\infty}^{k-2} (k-i+1)^{mq'} 2^{(i-k)\eta q'/2} \right)^{q/q'} \right\}^{1/q} \\
& \leq \left\{ \sum_{k=0}^\infty \left(\sum_{i=-\infty}^{k-2} 2^{\alpha_\infty iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{(i-k)\eta q/2} \right) \right\}^{1/q}. \tag{55}
\end{aligned}$$

By the same arguments as $E_>$, we get

$$\begin{aligned} U_> &\leq \left\| \{2^{\alpha(0)k} f\chi_k\} \right\|_{\ell^q_{\zeta}(L^{p(\cdot)}(\mathbb{R}^n))} + \left\| \{2^{\alpha_{\infty}k} f\chi_k\} \right\|_{\ell^q_{\zeta}(L^{p(\cdot)}(\mathbb{R}^n))} \\ &\approx \|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (56)$$

If $0 < q \leq 1$, we obtain

$$\begin{aligned} U_> &\leq \left\{ \sum_{k=0}^{\infty} \sum_{i=-\infty}^{k-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q (k-i+1)^{mq} 2^{(i-k)\eta q} \right\}^{1/q} \\ &\approx \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=0}^{\infty} (k-i+1)^{mq} 2^{(i-k)\eta q} \right. \\ &\quad \left. + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{k=i+2}^{\infty} (k-i+1)^{mq} 2^{(i-k)\eta q} \right\}^{1/q} \\ &\leq \left\{ \sum_{i=-\infty}^{-2} 2^{\alpha(0)iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q + \sum_{i=-1}^{\infty} 2^{\alpha_{\infty}iq} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q} \\ &\leq \left\| \{2^{\alpha(0)k} f\chi_k\} \right\|_{\ell^q_{\zeta}(L^{p(\cdot)}(\mathbb{R}^n))} + \left\| \{2^{\alpha_{\infty}k} f\chi_k\} \right\|_{\ell^q_{\zeta}(L^{p(\cdot)}(\mathbb{R}^n))} \\ &\approx \|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (57)$$

Thus, we have

$$U = U_< + U_> \lesssim \|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}. \quad (58)$$

For W , if $x \in R_k$, $y \in R_i$, and $i \geq k+2$, as in the arguments for G and U , we obtain

$$\left\| \mu_{\Omega,b}^m(f_j)\chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq (i-k+1)^m 2^{(k-i)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (59)$$

Thus we get

$$\begin{aligned} W_< &\lesssim \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left(\sum_{i=k+2}^{\infty} (i-k+1)^m 2^{(k-i)n\delta_1} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\ &\approx \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{i=k+2}^{\infty} 2^{\alpha(0)i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right. \right. \\ &\quad \left. \left. \times (i-k+1)^m 2^{(k-i)(n\delta_1+\alpha(0))} \right)^q \right\}^{1/q}, \end{aligned}$$

$W_>$

$$\lesssim \left\{ \sum_{k=0}^{\infty} \left(\sum_{i=k+2}^{\infty} 2^{\alpha_{\infty}i} \|f_i\|_{L^{p(\cdot)}(\mathbb{R}^n)} (i-k+1)^m 2^{(k-i)(n\delta_1+\alpha_{\infty})} \right)^q \right\}^{1/q}. \quad (60)$$

Similar to the estimates of $U_<$ and $U_>$, we get

$$W = W_< + W_> \lesssim \|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}. \quad (61)$$

Hence the proof of Theorem 12 is completed. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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