Research Article

Composition Operators on Cesàro Function Spaces

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The compact, invertible, Fredholm, and closed range composition operators are characterized. We also make an effort to compute the essential norm of composition operators on the Cesàro function spaces.

1. Introduction and Preliminaries

Let (X, s, μ) be a σ -finite measure space and let $L^0 = L^0(X)$ denote the set of all equivalence classes of complex valued measurable functions defined on X , where $X = [0, 1]$ or $X = [0, \infty)$. Then, for $1 \leq p < \infty$, the Cesàro function space is denoted by $\text{Ces}_p(X)$ and is defined as

$$\text{Ces}_p(X) = \left\{ f \in L^0(X) : \int_X \left(\frac{1}{x} \int_0^x |f(t)| d\mu(t) \right)^p d\mu(x) < \infty \right\}. \quad (1)$$

The Cesàro function space $\text{Ces}_p(X)$ is a Banach space under the norm

$$\|f\| = \left(\int_X \left(\frac{1}{x} \int_0^x |f(t)| d\mu(t) \right)^p d\mu(x) \right)^{1/p}; \quad (2)$$

see [1].

The Cesàro functions spaces $\text{Ces}_p[0, \infty)$ for $1 \leq p \leq \infty$ were considered by Shiue [2], Hassard and Hussein [3], and Sy et al. [4]. The space $\text{Ces}_\infty[0, 1]$ appeared already in 1948 and it is known as the Korenblyum, Krein, and Levin space K (see [5, 6]). Recently, in [7], it is proved that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $\text{Ces}_p(X)$ on both $X = [0, 1]$ and $X = [0, \infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. In [8], Astashkin and Maligranda investigated Rademacher sums in $\text{Ces}_p[0, 1]$ for $1 \leq p \leq \infty$. The description is different for $1 \leq p < \infty$ and $p = \infty$.

Let $T : X \rightarrow X$ be a nonsingular measurable transformation; that is, $\mu T^{-1}(A) = \mu(T^{-1}(A)) = 0$, for each $A \in s$, whenever $\mu(A) = 0$. This condition means that the measure μT^{-1} is absolutely continuous with respect to μ . Let $f_0 = d\mu T^{-1}/d\mu$ be the Radon-Nikodym derivative. In addition, we assume that f_0 is almost everywhere finite valued or equivalently that $(X, T^{-1}(s), \mu)$ is σ -finite. An atom of the measure μ is an element $A \in s$ with $\mu(A) > 0$ such that, for each $F \in s$, if $F \subset A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. Let A be an atom. Since μ is σ -finite, it follows that $\mu(A) < \infty$. Also every s -measurable function f on X is constant almost everywhere on A . It is a well-known fact that every sigma finite measure space (X, s, μ) can be decomposed into two disjoint sets X_1 and X_2 such that μ is atomic over X_1 and X_2 is a countable collection of disjoint atoms (see [9]).

Any nonsingular measurable transformation T induces a linear operator C_T from $\text{Ces}_p(X)$ into the linear space of equivalence classes of s -measurable functions on X defined by $C_T f = f \circ T$, $f \in \text{Ces}_p(X)$. Hence, the nonsingularity of T guarantees that the operator C_T is well defined. If C_T takes $\text{Ces}_p(X)$ into itself, then we call that C_T is a composition operator on $\text{Ces}_p(X)$. By $B(\text{Ces}_p(X))$, we denote the set of all bounded linear operators from $\text{Ces}_p(X)$ into itself.

So far as we know, the earliest appearance of a composition transformation was in 1871 in a paper of Schrljeder [10], where it is asked to find a function f and a number α such that

$$(f \circ T)(z) = \alpha f(z), \quad (3)$$

for every z , in an appropriate domain, whenever the function T is given. If z varies in the open unit disk and T is an analytic function, then a solution is obtained by Koenigs [11]. In 1925, these operators were employed in Littlewood's subordination theory [12]. In the early 1930s, the composition operators were used to study problems in mathematical physics and especially classical mechanics; see Koopman [13]. In those days, these operators were known as substitution operators. The systematic study of composition operators has relatively a very short history. It was started by Nordgren in 1968 in his paper [14]. After this, the study of composition operators has been extended in several directions by several mathematicians. For more details on composition operators, see [15–27] and references therein.

Associated with each σ -finite subalgebra $s_0 \subset s$, there exists an operator $E = E^{s_0}$, which is called conditional expectation operator; on the set of all nonnegative measurable functions f or for each $f \in L^0(X, s, \mu)$, the operator E is uniquely determined by the following conditions:

- (i) $E(f)$ is s_0 -measurable;
- (ii) if A is any s_0 -measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E(f) d\mu$.

The expectation operator E has the following properties:

- (a) $E(f \cdot g \circ T) = E(f) \cdot (g \circ T)$;
- (b) if $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere;
- (c) $E(1) = 1$;
- (d) $E(f)$ has the form $E(f) = g \circ T$ for exactly one σ -measurable function g , in particular, $g = E(f) \circ T^{-1}$ is a well-defined measurable function;
- (e) $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$; this is a Cauchy-Schwartz inequality for conditional expectation;
- (f) for $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.

For deeper study of properties of E , see [28].

Let $\mathcal{B}$ be a Banach space and let $\mathcal{K}$ be the set of all compact operators on $\mathcal{B}$. For $U \in \mathcal{L}(\mathcal{B})$, the Banach algebra of all bounded linear operators on $\mathcal{B}$ into itself, the essential norm of U means the distance from U to $\mathcal{K}$ in the operator norm; namely,

$$\|U\|_e = \inf \{\|U - S\| : S \in \mathcal{K}\}. \quad (4)$$

Clearly, U is compact if and only if $\|U\|_e = 0$. As seen in [29], the essential norm plays an interesting role in the compact problems of concrete operators. Many people have computed the essential norm of various concrete operators. For the study of essential norm of composition operators, see [30–33] and reference therein.

The question of actually calculating the norm and essential norm of composition operators on Cesaro function spaces is not a trivial one. In spite of the difficulties associated with computing the essential norm exactly, it is often possible to find upper and lower bound for the essential norm of

$C_T : \text{Ces}_p(X) \rightarrow \text{Ces}_p(X)$ under certain conditions on p and X .

The main purpose of this paper is to characterize the boundedness, compactness, closed range, and Fredholmness of composition operators on Cesaro function spaces. We also make an effort to compute the essential norm of composition operators in Section 3 of this paper.

2. Composition Operators

In this section of the paper, we will investigate the necessary and sufficient condition for a composition operator to be bounded.

Theorem 1. *Let (X, s, μ) be a σ -finite measure space and let $T : X \rightarrow X$ be nonsingular measurable transformation. Then, T induces a composition operator C_T on $\text{Ces}_p(X)$ if and only if there exists $M > 0$ such that $\mu T^{-1}(E) \leq M\mu(E)$ for every $E \in s$. Moreover,*

$$\|C_T\| = \sup_{0 < \mu(E) < \infty} \left(\frac{\mu(T^{-1}(E))}{\mu(E)} \right)^{1/p}. \quad (5)$$

Proof. Suppose that C_T is a composition operator. If $E \in s$ such that $\mu(E) < \infty$, then $\chi_E \in \text{Ces}_p(X)$ and

$$\mu T^{-1}(E) = \|C_T \chi_E\|^p \leq \|C_T\|^p \|\chi_E\|^p = \|C_T\|^p \mu(E). \quad (6)$$

Let $M = \|C_T\|^p$. Then,

$$\mu T^{-1}(E) \leq M\mu(E). \quad (7)$$

Conversely, suppose that the condition is true. Then, $\mu T^{-1} \ll \mu$ and hence the Radon-Nikodym derivative f_0 of μT^{-1} with respect to μ exists and $f_0 \leq M$ a.e.

Let $f \in \text{Ces}_p(X)$. Then,

$$\begin{aligned} \|C_T f\|^p &= \int_X \left(\frac{1}{x} \int_0^x |(f \circ T)(t)| d\mu(t) \right)^p d\mu(x) \\ &\leq \int_X \left(\frac{1}{x} \int_0^x |f| d\mu T^{-1}(t) \right)^p d\mu(x) \\ &= \int_X \left(\frac{1}{x} \int_0^x |f| f_0 d\mu(t) \right)^p d\mu(x) \\ &\leq M^p \int_X \left(\frac{1}{x} \int_0^x |f(t)| d\mu(t) \right)^p d\mu(x). \end{aligned} \quad (8)$$

Therefore, $\|C_T f\| \leq M \|f\|$.

Now, Let $N = \sup_{0 < \mu(E) < \infty} ((\mu(T^{-1}(E))/\mu(E))^{1/p})$. Then, $((\mu(T^{-1}(E))/\mu(E))^{1/p}) \leq N$ for all $E \in s$ and $\mu(E) \neq 0$. Thus $(\mu T^{-1}(E))^p \leq N^p (\mu(E))^p$ for all $E \in s$. By the first part of the theorem, we have

$$\|C_T f\| \leq N \|f\|, \quad \forall f \in \text{Ces}_p(X). \quad (9)$$

Hence, $\|C_T\| = \sup_{f \neq 0} \|C_T f\|/\|f\| \leq N$. Thus,

$$\|C_T\| \leq \sup_{0 < \mu(E) < \infty} \left(\left(\frac{\mu(T^{-1}(E))}{\mu(E)} \right)^p \right)^{1/p}. \quad (10)$$

On the other hand, Let $M = \|C_T\| = \sup_{f \neq 0} \|C_T f\|/\|f\|$. Thus, $\|C_T f\|/\|f\| \leq M$ for all $f \in \text{Ces}_p(X)$, $f \neq 0$. In particular, for $f = \chi_E$, such that $0 < \mu(E) < \infty$, $E \in \mathcal{S}$, we have $f = \chi_E \in \text{Ces}_p(X)$ and $\|C_T \chi_E\|/\|\chi_E\| = ((\mu(T^{-1}(E))/\mu(E))^p)^{1/p} \leq M$.

Therefore,

$$\sup_{0 < \mu(E) < \infty} \left(\left(\frac{\mu(T^{-1}(E))}{\mu(E)} \right)^p \right)^{1/p} \leq M = \|C_T\|. \quad (11)$$

From (10) and (11), we obtain

$$\|C_T\| = \sup_{0 < \mu(E) < \infty} \left(\left(\frac{\mu(T^{-1}(E))}{\mu(E)} \right)^p \right)^{1/p}. \quad (12)$$

□

Theorem 2. If $C_T : \text{Ces}_p(X) \rightarrow \text{Ces}_p(X)$ is a linear transformation, then C_T is continuous.

Proof. Let $\{f_n\}$ and $\{C_T f_n\}$ be sequences in $\text{Ces}_p(X)$ such that

$$f_n \rightarrow f, \quad C_T f_n \rightarrow g \quad \text{for some } f, g \in \text{Ces}_p(X). \quad (13)$$

Then, we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\|f_{n_k} - f\|(t) \rightarrow 0 \quad \text{for } \mu\text{-almost all } t \in X. \quad (14)$$

From the nonsingularity of T ,

$$(\|f_{n_k} - f\| \circ T)(t) \rightarrow 0 \quad \text{for } \mu\text{-almost all } t \in X. \quad (15)$$

Then, from (13) and (15), we conclude that $C_T f = g$. This proves that graph of C_T is closed and hence, by closed graph theorem, C_T is continuous. □

3. Compactness and Essential Norm of Composition Operators

This section is devoted to the study of compact composition operators on Cesàro function spaces. A necessary and sufficient condition for a composition operator to be compact is reported in this section. The main aim of this section is to compute the essential norm of the composition operators.

Theorem 3. Let $C_T \in B(\text{Ces}_p(X))$. Then, C_T is compact if and only if $\text{Ces}_p(X_\epsilon, \mu T^{-1})$ is finite dimensional, for each $\epsilon > 0$, where

$$X_\epsilon = \left\{ x \in X : \frac{d\mu T^{-1}}{d\mu}(x) \geq \epsilon \right\}. \quad (16)$$

Proof. For $f \in \text{Ces}_p(X)$, we have

$$\begin{aligned} \|C_T f\|_\mu &= \left(\int_X \left(\frac{1}{x} \int_0^x |f \circ T|(t) d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x |f| d\mu T^{-1}(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x |f| d\mu T^{-1}(t) \right)^p d\mu T^{-1}(x) \right)^{1/p} \\ &= \|f\|_{\mu T^{-1}} \\ &= \|If\|_{\mu T^{-1}}. \end{aligned} \quad (17)$$

Then, C_T is compact if and only if $I : \text{Ces}_p(X_\epsilon, \mu T^{-1}) \rightarrow \text{Ces}_p(X_\epsilon, \mu T^{-1})$ is a compact operator if and only if $\text{Ces}_p(X_\epsilon, \mu T^{-1})$ is a finite dimensional, where I is the identity operator. □

Corollary 4. If (X, s, μ) is a nonatomic measure space. Then no nonzero composition operator on $\text{Ces}_p(X)$ is compact.

Let $X = X_1 \cup X_2$ be the decomposition of X into nonatomic and atomic parts, respectively. If $X_2 = \emptyset$ or $\mu(X) = +\infty$ and X_2 consists of finitely many atoms, then, by Theorem 3, $\text{Ces}_p(X)$ does not admit a nonzero compact composition operator. Thus, in this case, $\mathcal{K} = \{0\}$ and hence $\|C_T\|_e = \|C_T\|$.

Now, we present the main result of this section.

Theorem 5. Let X_2 consists of finitely many atoms. Suppose that $\text{Ces}_p(X_\epsilon, \mu T^{-1})$ is a finite dimensional; that is, $X_\epsilon = \{x \in X : (d\mu T^{-1}/d\mu)(x) \geq \epsilon\}$ is a finite dimensional. Let $\alpha = \inf\{\epsilon > 0 : X_\epsilon \text{ is a finite dimensional}\}$. Then,

- (i) $\|C_T\|_e = 0$ iff $\alpha = 0$;
- (ii) $\|C_T\|_e \geq \alpha$ if $0 < \alpha \leq 1$;
- (iii) $\|C_T\|_e \leq \alpha$ if $\alpha > 1$.

Proof. (i) Theorem 3 implies that C_T is compact if and only if $\alpha = 0$. So (i) is the direct consequence of Theorem 3.

(ii) Suppose that $0 < \alpha \leq 1$. Take $0 < \epsilon < 2\alpha$ arbitrary. The definition of α implies that $F = \chi_{\alpha-(\epsilon/2)}$ either contains a nonatomic subset or has infinitely many atoms. If F contains a nonatomic subset, then there are measurable sets $E_n, n \in \mathbb{N}$ such that $E_{n+1} \subseteq E_n \subseteq F, 0 < \mu(E_n) < 1/n$. Define $f_n = 1/(\mu(E_n))^{1/p} \chi_{E_n}$. Then, $\|f_n\| = 1$, for all $n \in \mathbb{N}$. We claim that $f_n \rightarrow 0$ weakly. For this, we show that $\int_X f_n g d\mu \rightarrow 0$, for all $g \in (\text{Ces}_q(X))$. Let $A \subseteq F$ with $\mu(A) < \infty$ and $g = \chi_A$. Then, we have

$$\begin{aligned} \left| \int_X f_n \chi_A d\mu \right| &= \frac{1}{(\mu(E_n))^{1/p}} \mu(A \cap E_n) \\ &\leq \left(\frac{1}{n} \right)^{1-(1/p)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (18)$$

Since simple functions are dense in $\text{Ces}_q(X)$, thus f_n is proved to converge to zero weakly. Now, assume that F consists of

infinitely many atoms. Let $\{E_n\}_{n=0}^\infty$ be disjoint atoms in F . Again, put f_n as above. It is easy to see that, for $A \subseteq F$ with $0 < \mu(A) < \infty$, we have $\mu(A \cap E_n) = 0$ for sufficiently large n . So, in both cases, $\int_X f_n g d\mu \rightarrow 0$. Now, we claim that $\|C_T f_n\| \geq \alpha - (\epsilon/2)$. Since $0 < \alpha - (\epsilon/2) < 1$, we see that

$$\begin{aligned} \|C_T f_n\| &= \left(\int_X \left(\frac{1}{x} \int_0^x |C_T f_n| d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x |f_n| f_0(t) d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &\geq \left(\int_X \left(\frac{1}{x} \int_0^x \left(\alpha - \frac{\epsilon}{2} \right) |f_n| d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\alpha - \frac{\epsilon}{2} \right) \left(\int_X \left(\frac{1}{x} \int_0^x |f_n| d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \alpha - \frac{\epsilon}{2}. \end{aligned} \quad (19)$$

Finally, take a compact operator S on $\text{Ces}_p(X)$ such that $\|C_T - S\| \leq \|C_T\|_e + (\epsilon/2)$. Then, we have

$$\begin{aligned} \|C_T\|_e &> \|C_T - S\| - \frac{\epsilon}{2} \geq \|C_T f_n - S f_n\| - \frac{\epsilon}{2} \\ &\geq \|C_T f_n\| - \|S f_n\| - \frac{\epsilon}{2} \\ &\geq \left(\alpha - \frac{\epsilon}{2} \right) - \|S f_n\| - \frac{\epsilon}{2}, \end{aligned} \quad (20)$$

for all $n \in \mathbb{N}$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|S f_n\| \rightarrow 0$. Hence, $\|C_T\|_e \geq \alpha - \epsilon$. Since ϵ is arbitrary, we obtain $\|C_T\|_e \geq \alpha$.

(iii) Let $\alpha > 1$ and take $\epsilon > 0$ arbitrary. Put $K = X_{\alpha+\epsilon}$. The definition of α implies that K consists of finitely many atoms. So, we can write $K = (E_1, E_2, \dots, E_m)$, where $E_1, E_2, \dots, E_m$ are distinct. Since $(L_{\chi_K} C_T f)X = \sum_{i=1}^m \chi_K(E_i) f(T(E_i))$, for all $f \in \text{Ces}_p(X)$, hence $L_{\chi_K} C_T$ has finite rank. Now, let $F \subseteq X \setminus K$ such that $0 < \mu(F) < \infty$. Then, we have

$$\mu T^{-1}(F) \leq (\alpha + \epsilon) \mu(F). \quad (21)$$

It follows that $\|\chi_F \circ T\| \leq (\alpha + \epsilon) \|\chi_F\|$. Since simple functions are dense in $\text{Ces}_p(X)$. We obtain

$$\sup_{\|f\| \leq 1} \|\chi_{X \setminus K} f \circ T\| \leq \sup_{\|f\| \leq 1} \|\chi_{X \setminus K} f\| \leq \alpha + \epsilon. \quad (22)$$

Finally, Since $L_{\chi_K} C_T$ is a compact operator, we get

$$\begin{aligned} \|C_T - L_{\chi_K} C_T\| &= \sup_{\|f\| \leq 1} \|(1 - \chi_K) C_T f\| \\ &= \sup_{\|f\| \leq 1} \|\chi_{X \setminus K} C_T f\| \\ &\leq \alpha + \epsilon. \end{aligned} \quad (23)$$

It follows that $\|C_T\|_e \leq \alpha + \epsilon$ and, consequently, $\|C_T\|_e \leq \alpha$. $\square$

Example 6. Let $X = (-\infty, 0] \cup \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Let μ be the Lebesgue measure on $(-\infty, 0]$ and $\mu(\{n\}) = 1/2^n$ if $n \in \mathbb{N}$. Define $T : \mathbb{N} \rightarrow \mathbb{N}$ as $T(1) = T(2) = T(3) = 1, T(4) = 2, T(5) = T(6) = 3$, and $T(2n+1) = 5$, for $n \geq 3$.

Consider $T(2n) = 2n - 2$, for $n \geq 4$, and $T(x) = 5x$, for all $x \in (-\infty, 0]$. Then, we can easily calculate $\|C_T\|_e = 3^{-1/p}$ on $\text{Ces}_p(X)$ for $1 < p < \infty$.

4. Fredholm and Isometric Composition Operators

In this section, we first establish a condition for the composition operators to have closed range and then we make the use of it to characterize the Fredholm composition operators. We also make an attempt to compute the adjoint of the composition operators.

Holder's inequality for Cesàro measurable function spaces is that, if $f \in \text{Ces}_p(X)$ and $g \in \text{Ces}_q(X)$ such that $1/p + 1/q = 1$, then

$$\int fg d\mu \leq \|f\|_p \|g\|_q. \quad (24)$$

We find that every $g \in \text{Ces}_q(X)$ gives rise to a bounded linear functional $Fg \in (\text{Ces}_p(X))^*$ which is defined as

$$Fg(f) = \int fg d\mu, \quad \text{for every } f \in \text{Ces}_p(X). \quad (25)$$

For each $f \in \text{Ces}_p(X)$, there exists a unique $T^{-1}(s)$ measurable function $E(f)$ such that $\int gf d\mu = \int gE(f) d\mu$ for $T^{-1}(s)$ measurable function g for which the left integral exists. The function $E(f)$ is called conditional expectation of f with respect to the σ -algebra $T^{-1}(s)$. The operator $P_T : \text{Ces}_p(X) \rightarrow \text{Ces}_p(X)$ defined by $P_T f = f_0 E(f) \circ T^{-1}$ is called the Frobenius Perron operator where $E(f) \circ T^{-1} = g$ if and only if $E(f) = g \circ T$.

Theorem 7. Let $C_T \in B(\text{Ces}_p(X))$. Then C_T has closed range if and only if there exists $\delta > 0$ such that $f_0(x) \geq \delta$ for μ -almost all $x \in \text{supp } f_0 = S$.

Proof. If $f_0(x) \geq \delta$ for μ -almost all $x \in S$, then, for $\eta = \min(\delta, 1/\delta) \leq 1$,

$$\begin{aligned} 1 &\geq \left(\int_X \left(\frac{1}{x} \int_0^x \frac{|C_T f|}{\|C_T f\|} d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x f_0 \frac{|f|}{\|C_T f\|} d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &\geq \left(\int_X \left(\frac{1}{x} \int_0^x \frac{\eta |f|}{\|C_T f\|} d\mu(t) \right)^p d\mu(x) \right)^{1/p}. \end{aligned} \quad (26)$$

Hence, $\|C_T f\| \geq \eta \|f\|$, for all $f \in \text{Ces}_p(S)$, so that C_T has closed range.

Conversely, suppose that C_T has closed range. Then there exists $\delta \geq 0$ such that

$$\|C_T f\| \geq \delta \|f\|, \quad (27)$$

for every $f \in \text{Ces}_p(S)$. Choose a positive integer n such that $1/n < \delta$. If the set $E = \{x \in X : f_0(x) < 1/n\}$ has positive measure, then, for a given measurable subset $F \subset \text{supp } f_0$ such that $0 < \mu(F) < \infty$, we have

$$\mu T^{-1}(E) < \frac{1}{n} \mu(E) \quad (28)$$

or equivalently

$$\|C_T \chi_E\| \leq \frac{1}{n} \|\chi_E\|. \quad (29)$$

This contradicts inequality (27). Hence, f_0 is bounded away from zero on $\text{supp } f_0$. $\square$

Theorem 8. Let $C_T \in B(\text{Ces}_p(X))$. Then, $\ker C_T^*$ is either zero-dimensional or infinite dimensional.

Proof. Suppose $0 \neq g \in \ker C_T^*$. Then $E = \text{supp } g$ is a set of nonzero measure. Now we can partition E into a sequence $\{E_n\}$ of measurable sets, $0 < \mu(E_n) < \infty$. We show that $g\chi_{E_n \circ T} \in \ker C_T^*$. Consider

$$\begin{aligned} C_T^*(g\chi_{E_n \circ T})(f) &= \int_E (g\chi_{E_n \circ T})(C_T f) d\mu \\ &= \int_E g C_T(\chi_{E_n} f) d\mu = 0. \end{aligned} \quad (30)$$

Hence, if $\ker C_T^*$ is not zero-dimensional, it is infinite dimensional. $\square$

Corollary 9. Let $C_T \in B(\text{Ces}_p(X))$. Then, C_T is injective if and only if T is surjective.

Theorem 10. Let $C_T \in B(\text{Ces}_p(X))$. Then, C_T has dense range if and only if $T^{-1}(s) = s$.

Proof. Suppose that C_T has dense range. Let $E \in s$ such that $\chi_E \in \text{Ces}_p(X)$. Then, there exists $\{f_n\} \subset \text{Ces}_p(X)$ such that $C_T f_n \rightarrow \chi_E$. Now, we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $C_T f_{n_k} \rightarrow \chi_E$ a.e. Now, each $C_T f_{n_k}$ is measurable with respect to $T^{-1}(s)$. Therefore, χ_E is measurable with respect to $T^{-1}(s)$ so that $\chi_E = \chi_{T^{-1}(F)}$. Hence, $T^{-1}(s) = s$ a.e.

Conversely, suppose $T^{-1}(s) = s$ a.e. If $E \in s$, $0 < \mu(E) < \infty$, then there exists $F \in s$ such that $\mu(T^{-1}(F) \Delta E) = 0$. Since X is σ -finite, we can find an increasing sequence $\{F_n\}$ of sets of finite measure $F_n \uparrow F$ or $T^{-1}(F) \setminus T^{-1}(F_n) \downarrow 0$. Hence,

for given $\epsilon > 0$, there exists a positive integer n_0 such that $\mu(T^{-1}(F \setminus F_n)) < \epsilon$ for every $n \geq n_0$. Hence,

$$\begin{aligned} \|C_T \chi_F - C_T \chi_{F_n}\| &= \|C_T(\chi_{F \setminus F_n})\| \\ &= \|\chi_{T^{-1}(F \setminus F_n)}\| \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x \chi_{T^{-1}(F \setminus F_n)} d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &< \epsilon, \end{aligned} \quad (31)$$

for all $n \geq n_0$. Then $\chi_F = \chi_{T^{-1}(F)} \in \overline{\text{ran } C_T}$. This proves that C_T has dense range. $\square$

Theorem 11. Let $C_T \in B(\text{Ces}_p(X))$. Then, C_T is Fredholm if and only if C_T is invertible.

Proof. Assume that C_T is Fredholm. In view of Theorem 8, $\ker C_T$ and $\ker C_T^*$ are zero-dimensional so that C_T is injective and $T^{-1}(s) = s$ a.e. Therefore, by Theorem 10, C_T has dense range. Since $\text{ran } C_T$ is closed, so C_T is surjective. This proves the invertibility of C_T . The proof of the converse part is obvious. $\square$

Corollary 12. Let $C_T \in B(\text{Ces}_p(X))$. Then, C_T is an isometry if and only if T is measure preserving.

Proof. If T is measure preserving, then $f_0 = 1$ a.e. Therefore,

$$\begin{aligned} \|C_T f\| &= \left(\int_X \left(\frac{1}{x} \int_0^x |f \circ T|(t) d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x f_0 |f| d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x} \int_0^x |f| d\mu(t) \right)^p d\mu(x) \right)^{1/p} \\ &= \|f\|. \end{aligned} \quad (32)$$

Hence, C_T is an isometry. Conversely, if C_T is an isometry, then

$$\|C_T \chi_E\| = \|\chi_E\|. \quad (33)$$

This implies that $\mu(T^{-1}(E)) = \mu(E)$ for $E \in s$. Hence, $f_0 = 1$ a.e. $\square$

Theorem 13. Let $C_T \in B(\text{Ces}_p(X))$. Then, $C_T^* = P_T$.

Proof. Let $A \in s$ such that $0 < \mu(A) < \infty$. For $g \in \text{Ces}_q(X)$,

$$\begin{aligned} (C_T^* Fg)(\chi_A) &= Fg(C_T \chi_A) \\ &= \int C_T \chi_A g \, d\mu \\ &= \int \chi_A \circ Tg \, d\mu \\ &= \int \chi_A E(g) \circ T^{-1} f_0 \, d\mu \\ &= F_{E(g) \circ T^{-1} f_0}(\chi_A). \end{aligned} \quad (34)$$

Hence, $C_T^* Fg = F_{E(g) \circ T^{-1} f_0}$. After identifying $g \in \text{Ces}_q(X)$ with $Fg \in (\text{Ces}_p(X))^*$, we can write $C_T^* g = E(g) \circ T^{-1} f_0 = P_T g$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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