Anisotropic Two-Microlocal Spaces and Regularity

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We define $D\cdot u$-anisotropic two-microlocal spaces by decay conditions on anisotropic wavelet coefficients on any $D\cdot u$-anisotropic wavelet basis of $L^2(\mathbb{R}^d)$. We prove that these spaces allow the characterizing of pointwise anisotropic Hölder regularity. We also prove an anisotropic wavelet criterion for anisotropic uniform regularity. We finally prove that both this criterion and anisotropic $D\cdot u$-two-microlocal spaces are independent of the chosen anisotropic $D\cdot u$-orthonormal wavelet basis.

1. Introduction

Two-microlocal spaces were introduced by Bony [1] for the study of the propagation of singularities of solutions of hyperbolic PDEs. These spaces became much simpler when Jaffard [2] characterized them by decay conditions on isotropic wavelet coefficients. These spaces yield very accurate information on the local oscillations of a function $f$ near a point $x_0$ and the regularity of its fractional derivatives and primitives at $x_0$. The properties of the two-microlocal domain were investigated by Seuret and Véhel [3].

However, many natural mathematical objects, as well as many multidimensional signals and images from real physical problems, present strong anisotropies (see [4, 5] and references therein). For instance, this is the case in the textures of medical images (mammographies, osteoporosis, muscular tissues, etc.), hydrology, fracture surfaces analysis, and so forth (see [4, 6–12]). Anisotropic pointwise Hölder regularity and anisotropic Besov spaces have been introduced (see [13]). Anisotropic Besov spaces have played a central role in the mathematical modeling of anisotropic textures. They also have been used to study some PDEs; see [14] and for the study of semielliptic pseudodifferential operators whose symbols have different degrees of smoothness along different directions see [15]. Two-microlocal spaces have to be changed in order to fit anisotropic behaviors. Let $u = (u_1, \ldots, u_d)$ be such that

$$0 < u_1 \leq u_2 \leq \cdots \leq u_d, \quad \sum_{i=1}^d u_i = d. \quad (1)$$

The vector $u$ is called anisotropy. For $t > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we call anisotropic dilation the map

$$t^u x = (t^{u_1} x_1, \ldots, t^{u_d} x_d). \quad (2)$$

In [16] (resp., [17]) Calderón and Torchinsky (resp., Folland and Stein) have developed a theory of anisotropic $\mathcal{B}^p (\mathbb{R}^d)$ spaces by replacing the Euclidean norm by a homogeneous quasinorm $\rho_u$; recall that $\rho_u$ is defined on $\mathbb{R}^d$ by $\rho_u(0) = 0$ and, for all $x \neq 0$, $\rho_u(x)$ is the unique $r > 0$ for which $|r^{-u} x| = 1$, where $| \cdot |$ is the Euclidean norm on $\mathbb{R}^d$.

The function $\rho_u$ is continuous and homogeneous in the sense that

$$\rho_u (r^u x) = r \rho_u (x), \quad \forall r > 0 \text{ and all } x \in \mathbb{R}^d. \quad (3)$$

The corresponding $u$-ball $B_u (x, r) := \{ y \in \mathbb{R}^d; \rho_u (x - y) < r \}$ of $\rho_u$-radius $r$, centered on $x$, is an ellipse of axis of lengths $2r^{u_1}, \ldots, 2r^{u_d}$, centered on $x$. 

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In the isotropic case ($u_i = 1$ for all $1 \leq i \leq d$), the homogeneous quasinorm $\rho_u$ coincides with the Euclidean norm.

If we set
\[
|x|_u = \left( \sum_{i=1}^d |x_i|^{u_i} \right)^{1/u_u}
\] (4)
then $| \cdot |_u$ is also a homogeneous quasinorm in the sense that $|r^u x|_u = r |x|_u$ and there exists a constant $C > 0$ such that, for all $x, y \in \mathbb{R}^d$,
\[
|x + y|_u \leq C (|x|_u + |y|_u).
\] (5)
The homogeneous quasinorm $| \cdot |_u$ is equivalent to $\rho_u$ because
\[
\frac{1}{d} |x|_u \leq \rho_u(x) \leq d^{1/u_u} |x|_u.
\] (6)

In [18–20], we adapted the notion of pointwise regularity to the anisotropy. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$. For $I = (i_1, \ldots, i_d) \in \mathbb{N}^d$, we set $d(I) = \sum_{i=1}^d u_i/i_i$ and $d_u(I) = u_i d(I) = \sum_{i=1}^d u_i i_i/I_i$. Thus $d(I)$ is the degree of homogeneity of the differential operator $\partial^I$, or, as we will say, its homogeneous degree. If $P = \sum_i a_i x_i^{i_i}, a_i \in \mathbb{R}$, or $C$ is a polynomial, we define its homogeneous degree to be $d(P) := \max|d(I) : a_i \neq 0|$. We also define its $u$-homogeneous degree to be
\[
d_u(P) := u_i d(P) = \max |d_u(I) : a_i \neq 0|.
\] (7)

**Definition 1.** Let $s > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$ or let $C$ be a function.

(1) Let $x_0 \in \mathbb{R}^d$. We say that $f$ belongs to $C_u^s(x_0)$ (resp., $C_{u, \text{log}}(x_0)$) if $f \in L^\infty(\mathbb{R}^d)$ and if there exists a constant $C > 0$, $0 < \varepsilon < 1$, and a polynomial $P$ of $u$-homogeneous degree less than $s$ such that
\[
|f(x) - f(x - x_0)| \leq C |x - x_0|^{s'}
\] (8)

respectively,
\[
|f(x) - f(x - x_0)| \leq C |x - x_0|^{s'} \log \left( \frac{1}{|x - x_0|} \right).
\] (9)

(2) We say that $f$ belongs to $C_u^{\infty}(\mathbb{R}^d)$ (resp., $C_{u, \text{log}}^{\infty}(\mathbb{R}^d)$) if $f \in L^\infty(\mathbb{R}^d)$ and if (8) (resp., (9)) holds for any $x_0$ in $\mathbb{R}^d$ with uniform constant $C$.

For $j \in \mathbb{Z}$, let $I_j$ be a finite set with cardinality bounded independently of $j$, and for $\alpha \in I_j$ let
\[
D^\alpha_a : x = (x_1, \ldots, x_d) \mapsto (a_{i_1} x_1, \ldots, a_{i_d} x_d)
\] (10)
such that there exists $0 < C \leq 1 \leq C'$ such that for all $j \in \mathbb{Z}$, for all $i \in \{1, \ldots, d\}$, and for all $\alpha \in I_j$,
\[
C \leq a_{i_j} \leq C'.
\] (11)

Remark that
\[
C^{1/u_u}|x|_u \leq |D^\beta_{a} x|_u \leq \frac{d}{d_u} |x|_u
\] (12)

So we write
\[
|D^\beta_{a} x|_u = |x|_u.
\] (13)

For $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$, put $|\beta| = \beta_1 + \cdots + \beta_d$ and $\partial^\beta = \partial^{\beta_1} \cdots \partial^{\beta_d}$. In [13], the following definition was given.

**Definition 2.** Let $D = (D^\beta_{a})_{\beta \in \mathbb{Z}, \alpha \in I_j}$ be as above. Let $N > 0$. A homogeneous $D\cdot u$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order $N$ is a family \(\{f_{j,k}^{\alpha} : j \in \mathbb{Z}, \alpha \in I_j, k \in \mathbb{Z}^d\}\), satisfying the following conditions.

(1) $f_{j,k}^{\alpha} \in \mathcal{C}^N(\mathbb{R}^d)$ (regularity condition).

(2) For all $M > 0$, there exists $C > 0$ such that, for all $j \in \mathbb{Z}, k \in \mathbb{Z}^d$, $\alpha \in I_j$, and $x \in \mathbb{R}^d|\partial^\beta f_{j,k}^{\alpha}(x)| \leq C \frac{2^{d/2 + d_u(\beta)}}{1 + [2^{m}d^\alpha x - k]_u^{-M}}$ if $|\beta| \leq N$ (localization condition).

(3) $\int_{\mathbb{R}^d} x^\beta f_{j,k}^{\alpha}(x) dx = 0$ if $|\beta| < N$ (vanishing moment condition).

(4) $\{f_{j,k}^{\alpha} : j \in \mathbb{Z}, \alpha \in I_j, k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

For the inhomogeneous version we need the following definition.

**Definition 3.** Let $D = (D^\beta_{a})_{\beta \in \mathbb{Z}, \alpha \in I_j}$ be as above. Let $N > 0$. An inhomogeneous $D\cdot u$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order $N$ is a collection of union of two families \(\{f_{j,k}^{\alpha} : j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\}\) and \(\{g_{k} : k \in \mathbb{Z}^d\}\), satisfying the following conditions.

(1) $f_{j,k}^{\alpha} \in \mathcal{C}^N(\mathbb{R}^d)$ and $g_{k} \in \mathcal{C}^N(\mathbb{R}^d)$ (regularity condition).

(2) For all $M > 0$, there exists $C > 0$ such that, for all $j \in \mathbb{Z}, k \in \mathbb{Z}^d, \alpha \in I_j$, and $x \in \mathbb{R}^d|\partial^\beta f_{j,k}^{\alpha}(x)| \leq C \frac{2^{d/2 + d_u(\beta)}}{1 + [2^{m}d^\alpha x - k]_u^{-M}}$ and $|\partial^\beta g_{k}(x)| \leq C(1 + |x - k|_u)^{-M}$ if $|\beta| \leq N$ (localization condition).

(3) $\int_{\mathbb{R}^d} x^\beta f_{j,k}^{\alpha}(x) dx = 0$ if $|\beta| < N$ (vanishing moment condition).

(4) $\int_{\mathbb{R}^d} g_{k}(x) dx = 1$.

(5) $\{f_{j,k}^{\alpha} : j \in \mathbb{N}, \alpha \in I_j, k \in \mathbb{Z}^d\} \cup \{g_{k} : k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

The inhomogeneous (resp., homogeneous version of) Triebel family of anisotropic wavelets [21, 22] are examples of inhomogeneous (resp., homogeneous) $D\cdot u$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$. Let $(\psi_j)_{j \in \mathbb{Z}}$ be a one-dimensional multiresolution analysis in $L^2(\mathbb{R})$ with a scaling
function (called father wavelet) $\psi_F$ and an associated wavelet (called mother wavelet) $\psi_M$. Let $I_{j,u}$, $j \in \mathbb{Z}$, be the set of all elements $(G, l)$ with $G = (G_1, \ldots, G_d) \in [F, M]^d$ such that at least one component $G_i$ is an $M$ and $l = (l_1, \ldots, l_d) \in \mathbb{Z}^d$ such that

$$l_i = \lfloor j u_i \rfloor, \quad \text{if } G_i = F,$$

$$\lceil j u_i \rceil \leq l_i < \lfloor (j + 1) u_i \rfloor, \quad \text{if } G_i = M, \quad \lfloor (j + 1) u_i \rfloor \leq \lceil j u_i \rceil,$$

$$l_i = \lceil j u_i \rceil, \quad \text{if } G_i = M, \quad \lfloor (j + 1) u_i \rfloor = \lceil j u_i \rceil.$$  \hspace{1cm} (14)

Remark that the cardinality of $I_{j,u}$ is bounded, independently of $j$, by $(2^d - 1) \prod_{i=1}^{d} (u_i + 2)$.

We easily get the following result.

**Proposition 4.** Let $D^{(G)}_{F,j,k,u} = \mathbb{R}^{j_{-} - j_{+} - 1} \times \cdots \times \mathbb{R}^{j_{-} - j_{+} - 1} \times \mathbb{R}^{d}$ and

$$\Psi^{(G)}_{j,k,u}(x) = 2^{j/2} \prod_{i=1}^{d} \Psi_{G_i}\left(2^{j_{-} - j_{+} - 1} \times x_{j_{+} - j_{+} - 1} - k_{i}\right).$$  \hspace{1cm} (15)

Let

$$\Phi_k(x) = 2^{d/2} \prod_{i=1}^{d} \Psi_{G_i}\left(2^{j_{-} - j_{+} - 1} \times x_{j_{+} - j_{+} - 1} - k_{i}\right).$$  \hspace{1cm} (16)

Suppose that $\psi_F$ and $\psi_M$ are Daubechies [23] (resp., Lemarié-Rieusset and Meyer [24]) wavelets and let $N$ be the common regularity of $\psi_F$ and $\psi_M$ (resp., $N = \infty$). Then the family of $\Psi_{j,k,u}^{(G)}$, $j \in \mathbb{Z}$, $(G, l) \in I_{j,u}$, $k \in \mathbb{Z}^d$ (resp., the collection of the union of $\{\Phi_k, k \in \mathbb{Z}^d\}$ and $\Psi_{j,k,u}^{(G)}$, $j \in \mathbb{N}$, $(G, l) \in I_{j,u}$, $k \in \mathbb{Z}^d$) is a homogeneous (resp., inhomogeneous) $D$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$, of order $N$. We call them the Triebel anisotropic orthonormal bases.

**Remark 5.** Let $f^{G}_{j,k,u}, j \in \mathbb{N}$, $\alpha \in I_j$, $k \in \mathbb{Z}^d$ be an inhomogeneous $D$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$ of order $N$. For $F \in L^2(\mathbb{R}^d)$, let $(f^{G}_{j,k,u}) = (|F(x)| f^{G}_{j,k,u}(x)dx)$ and $(\Phi_k) = (|F(x)| \Phi_k(x)dx)$ denote the $D$-anisotropic wavelet coefficients of $F$. Then

$$F(x) = \sum_{k \in \mathbb{Z}^d} \sum_{\alpha \in I_j} c^G_{j,k,u} \Phi_k(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} c^G_{j,k,u} f^{G}_{j,k,u}(x).$$  \hspace{1cm} (17)

We now define $D$-anisotropic two-microlocal spaces.

**Definition 6.** Let $s > 0$ and $-s \leq s' \leq 0$. Let $x_0 \in \mathbb{R}^d$. Define $D$-anisotropic two-microlocal space $C^s_{D,u}(x_0)$ as the space of functions $F \in L^2(\mathbb{R}^d)$ such that there exists $C > 0$ satisfying

$$|c^G_{j,k,u}| \leq C 2^{j/2} \prod_{i=1}^{d} |x_{j_{+} - j_{+} - 1} - k_{i}|^{s'},$$  \hspace{1cm} (18)

In the next section, we will recall the Mean Value Theorem and Taylor’s theorem with remainder for the homogeneous quasinorm $\rho_a$.

In the third section, we will prove the following two theorems which characterize uniform anisotropic regularity (resp., pointwise anisotropic regularity) by decay condition on $D$-anisotropic wavelet coefficients (resp., by $D$-anisotropic two-microlocal spaces); we denote by $\Delta$ the additive subsemigroup of $\mathbb{R}$ generated by $0, 1, u_2/u_1, \ldots$ and $u_d/u_1$. In other words, $\Delta$ is the set of all numbers $d(I)$ as $I$ ranges over $\mathbb{N}^d$.

**Theorem 7.** Let $s > 0$. Let $f^{G}_{j,k,u}, j \in \mathbb{N}$, $\alpha \in I_j$, $k \in \mathbb{Z}^d$ be any inhomogeneous $D$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$, of order $N$, and $[N] > s/u_1$. Let $F \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

1. If $F \in C^s_{D}(\mathbb{R}^d)$, then there exists a constant $C > 0$ such that

$$|c^G_{j,k,u}| \leq C 2^{j/2} 2^{s/2} 2^{s'/2}, \quad \forall j, k, \alpha.$$  \hspace{1cm} (19)

2. Conversely, if (19) holds, then $F \in C^s_{D}(\mathbb{R}^d)$ if $u_1s \notin \Delta$ and $F \in C^s_{D,u}(\mathbb{R}^d)$ if $u_1s \in \Delta$.

**Theorem 8.** Let $s > 0$. Let $f^{G}_{j,k,u}, j \in \mathbb{N}$, $\alpha \in I_j$, $k \in \mathbb{Z}^d$ be any inhomogeneous $D$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$, of order $N$, and $[N] > s/u_1$. Let $F \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

1. If $F \in C^s_{D}(x_0)$, then $F \in C^{s,s}(x_0)$.

2. Conversely, if $F \in C^{s,s}(x_0)$ and if $F \in C^s_{D}(\mathbb{R}^d)$ for $0 < \beta < s$, then $F \in C^{s,s}(x_0)$.

3. If there exist $-s \leq s' \leq 0$ such that $F \in C^{s,s}(x_0)$, then $F \in C^{s,s}(x_0)$ if $u_1s \notin \Delta$ and $F \in C^{s,s}(x_0)$ if $u_1s \in \Delta$.

The spaces $C^s_{D,u}(x_0)$ are defined by conditions on the wavelet coefficients; therefore we should check that this definition is independent of the wavelet basis chosen. This will be done in the fourth section. We will also show that Theorem 7 does not depend on the chosen $D$-anisotropic orthonormal wavelet basis of $L^2(\mathbb{R}^d)$.

2. **Mean Value Theorem and Taylor’s Theorem**

The homogeneous quasinorm $\rho_a$ satisfies the following properties:

$$\forall i \in \{1, \ldots, d\}, \quad \|x\|^{1/u_i} \leq \rho_a(x),$$

there exists $\tilde{r} \in \{1, \ldots, d\}$ such that $x_{\tilde{r}}^2 \geq \frac{1}{d} \rho_a(x)^{2u_i}$,

$$\rho_a(x) \leq 1 \iff \|x\| \leq 1,$$  \hspace{1cm} (20)

$$\|x\|^{1/u_i} \leq \rho_a(x) \leq \|x\|^{1/u_i}, \quad \text{if } \rho_a(x) \leq 1,$$

$$\|x\|^{1/u_i} \leq \rho_a(x) \leq \|x\|^{1/u_i}, \quad \text{if } \rho_a(x) \geq 1.$$
In [16, 17], there are versions of Mean Value Theorem and Taylor's theorem with remainder for the homogeneous quasinorm $\rho_u$. Using the fact that $\rho_u$ and $|\cdot|_u$ are equivalent we deduce the following results.

**The $u$-Mean Value Theorem.** There exist two positive constants $C$ and $v$ such that, for all functions $f$ of class $C^k$ on $\mathbb{R}^d$ and all $x, y \in \mathbb{R}^d$,

$$|f(y) - f(x)| \leq C \sum_{i=1}^{d} \left| y - x \right|_{\rho_u}^i \sup_{|h|_u \leq 1} \left| \partial_i f(x + h) \right|. \quad (21)$$

**The $u$-Taylor Inequality.** Suppose $\delta \in \Delta (\delta > 0)$, and $k = [\delta]$. There are two constants $C_\delta > 0$ and $v > 0$ such that, for all functions $f$ of class $C^{(k+1)}$ on $\mathbb{R}^d$ and all $x, y \in \mathbb{R}^d$,

$$|f(y) - P(y-x)| \leq C_\delta \sum_{|l| \leq d(\delta)} |y-x|_{\rho_u}^{|l|} \sup_{|h|_u \leq 1} \left| \partial^l f(x + h) \right|, \quad (22)$$

where $P$ is the Taylor polynomial of $f$ at $x$ of homogeneous degree $\delta$ as

$$P(y-x) = \sum_{I \leq d(\delta)} \frac{\partial^I f(x)}{I!} (y-x)^I. \quad (23)$$

### 3. Proofs of Theorems 7 and 8

Let us first prove Theorem 7.

**Proof.** (1) Let $F \in C_u^a(\mathbb{R}^d)$. We will prove the decay decreasing (19) for the anisotropic wavelet coefficients.

There exists a polynomial $P = \sum j_i x_i^{d_i}$ of $u$-homogeneous degree $d_u(P) = u_d(P) = u_l \max[d(I) : a_I \neq 0]$ less then $s$ such that (8) holds for any $x$ and $x_0 \in \mathbb{R}^d$ with uniform constant $C$. Since $|I| \leq d(I)$, then the vanishing moment condition for $f_{\alpha, j, k, u}$ for $|N| > s/\alpha_t$ implies that

$$|c_{\alpha, j, k, u}| = \left| \int_{\mathbb{R}^d} F(x) f_{\alpha, j, k, u}^\alpha(x) dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \left( F(x) - P(x - 2^{-j\alpha}(D^\alpha_j)^{-1}k) \right) f_{\alpha, j, k, u}^\alpha(x) dx \right|$$

$$\leq C \int \left| x - 2^{-j\alpha}(D^\alpha_j)^{-1}k \right|_u^i \left| f_{\alpha, j, k, u}^\alpha(x) \right| dx. \quad (24)$$

Thanks to the localization condition, we get

$$|c_{\alpha, j, k, u}| \leq C 2^{d_j/2} \int \left| x - 2^{-j\alpha}(D^\alpha_j)^{-1}k \right|_u^i \left( 1 + 2^{j\alpha} D^\alpha_j x - k \right)_u^{-M} dx. \quad (25)$$

But $|x - 2^{-j\alpha}(D^\alpha_j)^{-1}k|_u = |2^{-j\alpha}(D^\alpha_j)^{-1}(2^{j\alpha} D^\alpha_j x - k)|_u = 2^{j\alpha} D^\alpha_j x - k$ in the sense given in (13). It follows that

$$|c_{\alpha, j, k, u}| \leq C 2^{d_j/2} 2^{-j\alpha}. \quad (26)$$

(2) Conversely, assume the decay decreasing (19). We will prove that $F \in C_u^a(\mathbb{R}^d)$ if $u_j s \notin \Delta$ and $F \in C_u^a(\log(\mathbb{R}^d))$ if $u_j s \in \Delta$.

We know that $F$ is expressed as in (17). Since $F \in L^2$ and $\|g_k\|_2 = 1$, then there exists $C > 0$ such that

$$|c_k| \leq C, \quad \forall k. \quad (27)$$

Then the function is $\sum_{k \in Z^d} \alpha_k g_k(x) \in C^\infty(\mathbb{R}^d)$.

Let us study the regularity of

$$\sum_{j=0}^\infty \sum_{k \in Z^d} \sum_{\alpha \in I_j} c_{\alpha, j, k, u} f_{\alpha, j, k, u}^\alpha(x). \quad (28)$$

Since the cardinality of $I_j$ is bounded independently of $j$, then from the localization of $f_{\alpha, j, k, u}$ it follows that

$$\forall j \geq 0, \quad \text{and all } x \quad |F_j(x)| \leq C 2^{-j\alpha}, \quad (29)$$

$$\forall j \geq 0, \quad \text{all } x, \quad \text{and all } I \quad \|\partial^I F_j(x)\| \leq C 2^{-j\alpha(|I| - d_u(I))}. \quad (30)$$

Let $x_0 \in \mathbb{R}^d$. Denote by $\delta_u$ the largest value of $d_u(I)$ such that $d_u(I) < s$. For $j \geq 0$, denote by $P_j(x - x_0)$ the Taylor polynomial of $F_j$ at $x_0$ of $u$-homogeneous degree $\delta_u$ (which was defined in (23)):

$$P_j(x - x_0) = \sum_{I \leq d_u(I)} \frac{\partial^I f_j(x_0)}{I!} (x - x_0)^I. \quad (31)$$

Then $P(x - x_0) := \sum_{j=0}^\infty P_j(x - x_0)$ is the $u$-Taylor polynomial of $F$. Its convergence follows from (30) and the fact that if $d_u(I) \leq \delta_u$ then $d_u(I) < s$ and $\sum_{j=0}^\infty 2^{-j\alpha(|I| - d_u(I))} < \infty$.

Let $j_0$ be the unique integer such that $2^{-j_0} \leq |x - x_0| < 2 \cdot 2^{-j_0}$. Then

$$\left| \int_F(x) - \sum_{k \in Z^d} \alpha_k g_k(x) - P(x - x_0) \right|$$

$$\leq \sum_{j=j_0}^{j_0} \left| F_j(x) - P_j(x - x_0) \right|$$

$$+ \sum_{j>j_0} \left| F_j(x) \right| + \sum_{j>j_0} \left| P_j(x - x_0) \right|. \quad (32)$$

It follows from (29) that

$$\sum_{j>j_0} \left| F_j(x) \right| \leq \sum_{j>j_0} C 2^{-j\alpha} \leq C 2^{-j\alpha} \leq C |x - x_0|_{u}^\alpha, \quad (33)$$
It follows from (30) that
\[
\sum_{j \geq j_0} |P_j (x - x_0)| \leq \sum_{j \geq j_0} \sum_{d \in \mathcal{A}(d)} |C^{2^{-j(s-d_u(J))}}(x - x_0)|. \tag{34}
\]

But from the definition of \(|\cdot|_u\)
\[
\left| (x - x_0)^J \right| \leq |x - x_0|^{d_u(J)}. \tag{35}
\]

Hence
\[
\sum_{j \geq j_0} |P_j (x - x_0)| \leq C 2^{-j\delta_s} |x - x_0|^{d_u(J)} \tag{36}
\]

Take \(\delta = \delta_u/u_1\) and \(l = [\delta]\). Then \(l < s/u_1 < [N]\) and \(l + 1 \leq [N]\). Since \(F_j\) is of class \(C^{(l+1)}\), then the \(u\)-Taylor inequality implies that
\[
\sum_{j=0}^{j_0} |F_j (x) - P_j (x - x_0)| \leq \sum_{j=0}^{j_0} |F_j (x) - P_j (x - x_0)| \tag{37}
\]

\[
\leq \sum_{j=0}^{j_0} C \delta \sum_{l \geq l+1, d(J) > \delta} |x - x_0|^{d_u(J)} \times \sup_{|h| \leq 2^{-l+1}|x - x_0|} |\partial^l F_j (x_0 + h)| \leq C \delta \sum_{l \geq l+1, d(J) > \delta} |x - x_0|^{d_u(J)} \lesssim \sum_{j=0}^{j_0} |F_j (x) - P_j (x - x_0)| \leq C |x - x_0|^{d_u(J)} \tag{38}
\]

\[
|F_j (x) - P_j (x - x_0)| \leq C |x - x_0|^{d_u(J)} \tag{39}
\]

\[
|F_j (x) - P_j (x - x_0)| \leq C |x - x_0|^{d_u(J)} \log \left( \frac{1}{|x - x_0|_{u}} \right). \tag{40}
\]

We conclude that \(F \in C^{u}_{\log} (\mathbb{R}^d)\).

(ii) If \(u \in \Delta\), then \(d(J) > \delta_u\) implies that \(d_u(J) > \delta_u\). From the definition of \(\delta_u\), we get \(d_u(J) > s\). Therefore
\[
\sum_{j=0}^{j_0} |F_j (x) - P_j (x - x_0)| \leq C |x - x_0|^{d_u(J)}. \tag{41}
\]

We conclude that \(F \in C^{u}_{\log} (\mathbb{R}^d)\). \(\Box\)

Let us now prove Theorem 8.

Proof. (1) Let \(F \in C^{u}_{\log} (\mathbb{R}^d)\) and \(N > s/u_1\); then as above
\[
|c^a_{j,k,u}| = \int \left| (F (x) - P (x - x_0)) f^{a}_{j,k,u} (x) \right| dx \leq C \int \left| x - x_0 \right|^{d_u(J)} f^{a}_{j,k,u} (x) \right| dx \leq C \int \left| x - x_0 \right|^{d_u(J)} f^{a}_{j,k,u} (x) \right| dx. \tag{42}
\]

From the localization of the wavelets, we obtain
\[
|c^a_{j,k,u}| \leq C 2^{-j(s-d_u(J))} + C |x_0 - 2^{-j} \left( D^a (J) \right)^{-1} k^u |. \tag{43}
\]

Therefore
\[
|c^a_{j,k,u}| \leq C 2^{-j(s-d_u(J))} \left( 1 + 2^j |x_0 - 2^{-j} \left( D^a (J) \right)^{-1} k^u | \right). \tag{44}
\]

(2) Conversely, assume that (44) holds; then
\[
|F_j (x)| \leq \sum_{k \in \mathbb{Z}^d} \sum_{\alpha \in \mathcal{I}_J} C 2^{-j(d_u(J))} \times \left( 2^{-j} |x_0 - 2^{-j} \left( D^a (J) \right)^{-1} k^u | \right) \times \left( 2^{-j} \left( 1 + 2^j |x_0 - 2^{-j} \left( D^a (J) \right)^{-1} k^u | \right) \right). \tag{45}
\]

Since the cardinality of \(I_j\) is bounded independently of \(j\), then from the localization of the wavelets it follows that
\[
|F_j (x)| \leq C \left( 2^{-j} + |x_0 - 2^{-j} \left( D^a (J) \right)^{-1} k^u | \right). \tag{46}
\]

Similarly we have
\[
|\partial^l F_j (x)| \leq C 2^{-j(s-d_u(J))} \left( 1 + 2^{j} |x_0 - 2^{-j} \left( D^a (J) \right)^{-1} k^u | \right). \tag{47}
\]

As in the first point above, the function is \(\sum_{k \in \mathbb{Z}^d} c_k g_k (x) \in C^N (\mathbb{R}^d)\).
Let $j_0$ be the unique integer such that $2^{-j_0} \leq |x-x_0|_u < 2 \cdot 2^{-j_0}$ and $j_1 = s j_0/\gamma$. As previously

\[
\begin{align*}
F(x) - \sum_{k \in \mathbb{Z}^d} c_k g_k(x) - P(x-x_0) &\leq \sum_{j=j_0}^{j_1} |F_j(x) - P_j(x-x_0)| + \sum_{j=j_0}^{j_1} |F_j(x)| \\
&+ \sum_{j>j_1} |F_j(x)| + \sum_{j>j_0} |P_j(x-x_0)|.
\end{align*}
\] 

Relation (46) implies that

\[
\sum_{j=j_0}^{j_1} |F_j(x)| \leq \sum_{j=j_0}^{j_1} C \left( 2^{-j \gamma} + |x-x_0|^l_{u, a} \right)
\leq C \left( 2^{-j \gamma} + (j_1 - j_0) |x-x_0|^l_{u, a} \right)
\leq C \left( |x-x_0|^l_{u, a} + |x-x_0|^l_{u, a} \log \frac{1}{|x-x_0|_a} \right)
\leq C |x-x_0|^l_{u, a} \log \frac{1}{|x-x_0|_a}.
\]  

The assumption $F \in C^\beta_{\Delta}(\mathbb{R}^d)$ for a $\beta > 0$ implies that

\[
\sum_{j>j_0} |P_j(x-x_0)| \leq \sum_{j>j_0} C 2^{-j \beta \gamma} \leq C 2^{-j \beta \gamma} \leq C |x-x_0|^l_{u, a}.
\]  

So (47) yields

\[
\sum_{j>j_0} |P_j(x-x_0)| \leq C \sum_{j>j_0} \sum_{|d| \leq |a|} 2^{-j(d-sd)(l)} u^j \sum_{s \in \Delta} \frac{1}{(s^d-2^j u^j)^k} |x-x_0|_{u, a}^l.
\]  

It follows from (35) that

\[
\sum_{j>j_0} |P_j(x-x_0)| \leq C \sum_{j>j_0} 2^{-j \gamma} \sum_{d \leq |a|} 2^{-j(d-sd)(l)} \leq C |x-x_0|^l_{u, a}.
\]  

As above, if $u_s \in \Delta$, then

\[
\sum_{j=0}^{j_s} |F_j(x) - P_j(x-x_0)| \leq C |x-x_0|^l_{u, a} \log \frac{1}{|x-x_0|_a}.
\]  

But if $u_s \notin \Delta$, then

\[
\sum_{j=0}^{j_s} |F_j(x) - P_j(x-x_0)| \leq C |x-x_0|^l_{u, a}.
\]  

We conclude that $F \in C^\beta_{\Delta}(\mathbb{R}^d)$.

As above, if $u_s \in \Delta$, then

\[
\sum_{j=0}^{j_s} |F_j(x) - P_j(x-x_0)| \leq C |x-x_0|^l_{u, a} \log \frac{1}{|x-x_0|_a}.
\]  

But if $u_s \notin \Delta$, then

\[
\sum_{j=0}^{j_s} |F_j(x) - P_j(x-x_0)| \leq C |x-x_0|^l_{u, a}.
\]  

We conclude that $F \in C^\beta_{\Delta}(\mathbb{R}^d)$.

\[\boxdot\]

### 4. Independence of the Wavelet Basis

We will first check that the definition of $D\cdot u$-anisotropic two-microlocal space $C^\beta_{D\cdot u}(x_0)$ does not depend on the chosen $D\cdot u$-anisotropic orthonormal wavelet basis. We will check a stronger (but simpler) requirement which implies that the condition considered has some additional stability; indeed, we will first prove that the matrix of the operator which maps a $D\cdot u$-anisotropic orthonormal wavelet basis to another $D\cdot u$-anisotropic orthonormal wavelet basis is invariant under the action of infinite matrices which belong to an algebra $\mathcal{M}_{D\cdot u}$ of almost diagonal matrices which was defined in [13] in order to prove the stability of anisotropic Besov spaces under changes of $D\cdot u$-anisotropic wavelet bases; therefore, we will then prove that condition (18) is also invariant under this action. Note that $\mathcal{M}_{D\cdot u\cdot 1, 2}$ is an anisotropic version of the class of almost diagonal matrices that have been considered by Frazier and Jawerth [25] in the isotropic setting and the corresponding isotropic operator algebras are in the book of Meyer [26] (resp., Coifman and Meyer [27, 28]).

Definition 9. Let $\gamma > d(u_d - 1)$. Denote by $\mathcal{M}_{D\cdot u}$ the set of infinite matrices $M(z, \lambda, \lambda'_u)$ indexed by $\lambda_u = (j, k, \alpha)$ and $\lambda' = (j', k', \alpha')$ (where $k, k' \in \mathbb{Z}^d$, $j, j' \in \mathbb{Z}, \alpha \in I_j, \alpha' \in I_{j'}$) and satisfying that

\[
\begin{align*}
&\text{there exists } C > 0 \text{ such that } V(\lambda, \lambda'_u), \\
&\quad |M(\lambda, \lambda'_u)| \leq C \omega(\lambda, \lambda'_u).
\end{align*}
\]
where
\[
\omega_j \left( \lambda_u, \lambda'_u \right)
= 2^{-\frac{d}{2}+\gamma|j-j'|}
\times \left( 1 + \left( j - j' \right)^2 \right)
\times \left( 1 + 2^{\text{inf}(j,j')} \left| 2^{-j} D_j^u k - 2^{-j'} D_j^{u'} k' \right|_{\lambda_u} \right)^{d+\gamma}^{-1}.
\]

(57)

The following propositions were proved in [13].

**Proposition 10.** For all \( \gamma > d(u_d - 1) \), \( \mathbb{M}_{D,u}^\gamma \) is an algebra of matrices.

**Proposition 11.** Assume that \( N \geq 1 \). Let \( \delta = \max[\delta' \in \Delta; \delta' < [N]] \).

If \( \{ f_{j,k,u}^\alpha \}_{j \in Z, \alpha} \) is a homogeneous \( D-u \)-anisotropic orthonormal wavelet bases of order \( N \), then the matrix, whose coefficients are given by \( \int f_{j,k,u}^\alpha (x) \widetilde{f}_{j,k',u}^\alpha (x) dx \), belongs to \( \mathbb{M}_{D,u}^{\gamma} \) for \( \gamma < u_i \delta \).

Let us recall the following definition [13].

**Definition 12.** For every \( \gamma > d(u_d - 1) \), the algebra \( \mathcal{O}(\mathbb{M}_{D,u}^\gamma) \) is the algebra of bounded operators on \( L^2(\mathbb{R}^d) \) whose matrices on a homogeneous \( D-u \)-anisotropic wavelet basis belong to \( \mathbb{M}_{D,u}^{\gamma} \).

Proposition 11 implies that this definition does not depend on the chosen homogeneous \( D-u \)-anisotropic wavelet basis of order \( N \), for \( \gamma < u_i \max[\delta' \in \Delta; \delta' < [N]] \), with \( N \geq 1 \).

Proposition 11 and the second point in Lemma 15 below yield the following theorem.

**Theorem 13.** Let \( s > 0 \). If \( f \in C_{D,u}^{s'}(x_0) \) with \(-s \leq s' < 0\) and \( T \in \mathcal{O}(\mathbb{M}_{D,u}^\gamma) \), then \( T f \in C_{D,u}^{s'}(x_0) \) for all \( \gamma > \max(s-s', d(u_d - 1)) \).

Anisotropic two-microlocal spaces \( C_{D,u}^{s'}(x_0) \) with \(-s \leq s' < 0\) are independent of the chosen \( D-u \)-anisotropic wavelet basis.

**Proof.** We will need the following version of Schur Lemma. \( \square \)

**Lemma 14.** Let \( m(p,q) \) be an infinite matrix and \( \omega(p) > 0 \). If \( \sum_q |m(p,q)| \omega(q) \leq \omega(p) \),
\[
\sum_p |m(p,q)| \omega(p) \leq \omega(q),
\]
then \( m \) is bounded on \( \ell^2 \), with norm bounded by 1.

**Proof.** Let \( (x(q)) \) be such that \( \sum |x(q)|^2 \leq 1 \) and \( y(p) = \sum_q m(p,q)x(q) \). Then
\[
|m(p,q)x(q)|
= |m(p,q)|^{1/2} \omega(q)^{-1/2} |m(p,q)|^{1/2} |x(q)|.
\]
(60)

Then, using both Cauchy-Schwartz inequality and (58),
\[
|y(p)|^2
\leq \left( \sum_q |m(p,q)| \omega(q) \right) \left( \sum_q |m(p,q)| \omega^{-1}(q) \right)^{-1/2}
\leq \omega(p) \left( \sum_q |m(p,q)| \omega^{-1}(q) \right)^{-1/2}.
\]
(61)

Using (59), we have
\[
\sum_p |y(p)|^2 \leq \sum_q \left( \omega^{-1}(q) \right)^{1/2} \left( \sum_p |m(p,q)| \omega(p) \right)^{1/2}
\leq \sum_q |x(q)|^2.
\]
(62)

**Lemma 15.** If \( 0 \leq s < \gamma \), then
\[
\sum_{\lambda_u} 2^{-j(d/2+s)} \omega_j \left( \lambda_u, \lambda'_u \right) \leq C_2^{-j(d/2+s)}.
\]
(63)

If \( s' \leq 0 \), \( s + s' \geq 0 \), and \( \gamma > s-s' \), then
\[
\sum_{\lambda_u} 2^{-j(d/2+s)} \left( 1 + \left| 2^{-j} D_j^u x - k \right|_{\lambda_u} \right)^{-s'} \omega_j \left( \lambda_u, \lambda'_u \right)
\leq C_2^{-j(d/2+s)} \left( 1 + \left| 2^{-j} D_j^u x - k \right|_{\lambda_u} \right)^{-s'}.
\]
(64)

**Proof.** We split the sum in the left term in (63) in two sums.

(i) If \( j' \geq j \)

\[
\sum_{\lambda_u} 2^{-j(d/2+s)}
\times \left( 2^{-j(d/2+s)} \right)|j-j'|
\times \left( \left( 1 + \left( j-j' \right)^2 \right)^{d+\gamma} \right)^{-1}
\times \left( 1 + 2^{\text{inf}(j,j')} \left| 2^{-j} D_j^u k - 2^{-j'} D_j^{u'} k' \right|_{\lambda_u} \right)^{d+\gamma}^{-1}.
\]
\[
= \sum_{k' \in \mathbb{Z}^d} 2^{-j'(d/2 + s)} \frac{2^{-(d/2 + \gamma)} |D_{j'}^x k - 2^{(j'-j)u} D_{j'}^x k'_{|u}|}{(1 + (j' - j)^2)}
\times \left(1 + \left|D_{j}^y k - 2^{(j-j')u} D_{j}^y k'_{|u}\right|^2 \right)^{-d - \gamma}
\times \sum_{k' \in \mathbb{Z}^d} \left(1 + \left|D_{j}^y k - 2^{(j-j')u} D_{j}^y k'_{|u}\right|^2 \right)^{-d - \gamma}
\leq \sum_{k' \in \mathbb{Z}^d} 2^{-j'(d/2 + s)} \frac{2^{-(d/2 + \gamma)} |D_{j'}^x k - 2^{(j'-j)u} D_{j'}^x k'_{|u}|}{(1 + (j' - j)^2)}
\leq C \sum_{j' < j} 2^{-j'(d/2 + s)} 2^{-d(j' - j)}
\leq C 2^{-j'(d/2 + s)} \sum_{j' < j} 2^{-d(j' - j)}
\leq C 2^{-j'(d/2 + s)} \text{ because } \gamma > s.
\]  

Hence (63) holds.

Let us now prove (64). Since \(s' < 0\), then
\[
(1 + 2^{j'u} D_{j'}^x x - k_{|u}\right|^{s' - s'}
\leq C \left(2^{j'u} D_{j'}^x x - 2^{j'u} D_{j'}^x 2^{-j'u}(D_{j'}^x)^{-1} k_{|u}\right)^{s' - s'}
\text{ because } \beta - d - \gamma - s < 0.
\]
\[ + \sum_{\lambda'_u} 2^{-j(d/2+s)} \omega_{j\gamma} (\lambda_u, \lambda'_u) \times \left(1 + 2^j \left| 2^{-j\alpha}(D^{\alpha}_{j\gamma})^{-1} k_{\lambda'_u} 2^{-j(d/2+s)} \right| \right)^{-s'} . \]

(69)

It follows from (63) and the assumption \( s + s' \geq 0 \) that the first term is bounded by \( C |x - 2^{-j\alpha}(D^{\alpha}_{j\gamma})^{-1} k|_{\lambda'_u} 2^{-j(d/2+s)} \) and the second term is bounded by

\[ \sum_{\lambda'_u} 2^{-j(d/2+s)} \omega_{j\gamma} (\lambda_u, \lambda'_u) \times \left(1 + 2^j \left| 2^{-j\alpha}(D^{\alpha}_{j\gamma})^{-1} k_{\lambda'_u} 2^{-j(d/2+s)} \right| \right)^{-s'} \]

\[ \leq C \sum_{\lambda'_u} 2^{-j(d/2+s)} \omega_{j\gamma} (\lambda_u, \lambda'_u) \]

\[ \leq C 2^{-j(d/2+s)} \text{ because } \gamma + s' > s. \]

(70)

Hence (64) holds. \( \square \)

Theorem 7 and the first result in Lemma 15 yield the following result.

Corollary 16. Let \( s > 0 \). If \( f \in C^s(\mathbb{R}^d) \) and \( T \in \mathcal{E} p_{\mathcal{M}^a_{D^{-1/2}}} \) then \( Tf \in C^s(\mathbb{R}^d) \) if \( u_1 s \notin \Delta \) and \( Tf \in C^s_{\log}(\mathbb{R}^d) \) if \( u_1 s \in \Delta, \text{ for all } \gamma \geq \max(s, d(u_1 - 1)). \)

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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