Research Article

Asymptotic Study of the 2D-DQGE Solutions

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We study the regularity of the solutions of the surface quasi-geostrophic equation with subcritical exponent

\[ 1/2 < \alpha \leq 1 \]

We prove that if the initial data is small enough in the critical space

\[ \dot{H}^{2-2\alpha}(\mathbb{R}^2) \]

then the regularity of the solution is of exponential growth type with respect to time and its \[ \dot{H}^{2-2\alpha}(\mathbb{R}^2) \] norm decays exponentially fast. It becomes then infinitely differentiable with respect to time and has value in all homogeneous Sobolev spaces \[ \dot{H}^s(\mathbb{R}^2) \] for \[ s \geq 2-2\alpha \]. Moreover, we give some general properties of the global solutions.

1. Introduction

We consider the 2D dissipative quasi-geostrophic equation with subcritical exponent \( 1/2 < \alpha \leq 1 \):

\[
\partial_t \theta + (-\Delta)\alpha \theta + (u \cdot \nabla) \theta = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^2, \\
\theta(0, x) = \theta^0(x) \quad \text{in} \quad \mathbb{R}^2, \\
\tag{S_\alpha}

\]

where \( x \in \mathbb{R}^2, t > 0, \theta = \theta(x,t) \) is the unknown potential temperature, and \( u = (u_1, u_2) \) is the divergence free velocity which is determined by the Riesz transformation of \( \theta \) in the following sense:

\[
\begin{align*}
u_1 &= -\mathcal{R}_2 \theta = -\partial_2(-\Delta)^{-1/2} \theta, \\
u_2 &= \mathcal{R}_1 \theta = \partial_1(-\Delta)^{-1/2} \theta.
\end{align*}
\]

(1)

The critical homogeneous Sobolev space is \( \dot{H}^{2-2\alpha}(\mathbb{R}^2) \) and we have

\[
\| \lambda^{2\alpha-1} f(\lambda) \|_{\dot{H}^{2-2\alpha}} = \| f \|_{\dot{H}^{2-2\alpha}}, \quad \forall \lambda > 0.
\]

(2)

In [1], we studied the existence of global solutions of \( (S_\alpha) \) if the initial data \( \theta^0 \) is small in the critical space \( \dot{H}^{2-2\alpha}(\mathbb{R}^2) \) and the subcritical exponent \( \alpha \in (1/2, 1] \). In use of Theorem 4.2 in [2] with \( p = q = 2 \), we proved the following theorem.

Theorem 1 (see [1]). For \( \alpha \in (1/2, 1] \) and \( \theta^0 \in \dot{H}^{2-2\alpha}(\mathbb{R}^2) \), there exists a constant \( c_\alpha > 0 \) such that if

\[
\| \theta^0 \|_{\dot{H}^{2-2\alpha}} < c_\alpha,
\]

(3)

the initial value problem \( (S_\alpha) \) has a unique solution in \( \dot{C}_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2)). \) Moreover,

\[
\| \theta(t) \|_{\dot{H}^{2-2\alpha}}^2 + \int_0^t \| \theta(r) \|_{\dot{H}^{2-\alpha}}^2 \, dr \leq \| \theta^0 \|_{\dot{H}^{2-2\alpha}}^2, \quad \forall t \geq 0.
\]

(4)

We proved also the following result.

Theorem 2 (see [1]). Let \( 2/3 \leq \alpha \leq 1 \).

(i) If \( \theta \in \dot{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \) is a global solution of \( (S_\alpha) \), then

\[
\lim_{t \to \infty} \| \theta(t) \|_{\dot{H}^{2-2\alpha}} = 0.
\]

(5)

(ii) If \( \theta \in \dot{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \) is a global solution of \( (S_\alpha) \), then

\[
\lim_{t \to \infty} \| \theta(t) \|_{\dot{H}^{2-\alpha}} = 0.
\]

(6)

In this paper, we describe the long time behavior of these solutions with respect to the homogeneous Sobolev norm \( \| \cdot \|_{\dot{H}^s} \) for \( s \geq 2 - 2\alpha \). We prove the following.
Theorem 3. There exists $c_\alpha > 0$ such that, for all $\theta^0 \in H^{2-2\alpha}(\mathbb{R}^2)$, $\|\theta^0\|_{H^{2-2\alpha}} < c_\alpha$, and there exists a global solution $\theta \in C_b(\mathbb{R}, H^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^t, H^{2-\alpha}(\mathbb{R}^2))$ such that, for all $s > 2 - 2\alpha$, $\theta(t) \in H^s(\mathbb{R}^2)$ for all $t > 0$, and

$$\|\theta(t)\|_{H^s} = O\left((t^{-s-2+2\alpha})^{1/2}\right), \quad t \to \infty.$$ \hspace{1cm} (7)

When the initial data is in $H^{2-2\alpha}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and small enough in the homogeneous space $H^{2-2\alpha}(\mathbb{R}^2)$, we prove that the Leray solution is also in all Sobolev spaces $H^s(\mathbb{R}^2)$. Moreover, we describe the long time behavior of its homogeneous Sobolev norm $\|\cdot\|_{H^s}$, for $s \geq 0$. We state also the following.

Theorem 4. There exists $c_\alpha > 0$ such that, for all $\theta^0 \in H^{2-2\alpha}(\mathbb{R}^2)$, $\|\theta^0\|_{H^{2-2\alpha}} < c_\alpha$, and there exists a global solution $\theta \in C_b(\mathbb{R}, H^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^t, H^{2-\alpha}(\mathbb{R}^2))$ such that, for all $s \geq 0$,

$$\|\theta(t)\|_{H^s} = O\left((t^{-s/2+\alpha})\right), \quad t \to \infty.$$ \hspace{1cm} (8)

The paper is organized as follows. We start by recalling some preliminary background and stating useful preliminary results on Sobolev spaces. Sections 3 and 4 are devoted to the proof of the main results, Theorems 3 and 4. In Section 5, we give some general properties for any global solutions of the system $(\Delta_\alpha)$.

### 2. Notations and Preliminary Results

#### 2.1. Notations and Technical Lemmas

In this short section, we collect some notations and definitions that will be used later and we give some technical lemmas.

(i) The Fourier transformation in $\mathbb{R}^2$ is normalized as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} \exp(-ix \cdot \xi)f(x)dx,$$ \hspace{1cm} $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$ \hspace{1cm} (9)

(ii) The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \exp(ix \cdot \xi)g(\xi)d\xi,$$ \hspace{1cm} $x = (x_1, x_2) \in \mathbb{R}^2.$ \hspace{1cm} (10)

(iii) For $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ denotes the usual nonhomogeneous Sobolev space on $\mathbb{R}^2$ and $\langle \cdot, \cdot \rangle_{H^s}$ its scalar product.

(iv) For $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ denotes the usual homogeneous Sobolev space on $\mathbb{R}^2$ and $\langle \cdot, \cdot \rangle_{H^s}$ its scalar product.

(v) The convolution product of a suitable pair of functions $f$ and $g$ on $\mathbb{R}^2$ is given by

$$(f \ast g)(x) := \int_{\mathbb{R}^2} f(y)g(x-y)dy.$$ \hspace{1cm} (11)

(vi) For any Banach space $(B, \|\cdot\|)$, any real number $1 \leq p \leq \infty$, and any time $T > 0$, we will denote by $L^p_t(B)$ the space of measurable functions $t \in [0, T] \mapsto f(t) \in B$ such that $(t \mapsto \|f(t)\|) \in L^p([0, T])$.

(vii) If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are two vector fields, we set

$$f \circ g := (g_1f, g_2f),$$

$$\text{div}(f \circ g) := (\text{div}(g_1f), \text{div}(g_2f)).$$ \hspace{1cm} (12)

(viii) For any subset $X$ of a set $E$, $1_X$ denotes the characteristic function of $X$.

We recall a fundamental lemma concerning some product laws in homogeneous Sobolev spaces.

**Lemma 5** (see [3]). Let $s_1, s_2$ be two real numbers such that

$$s_1 < 1, \quad s_1 + s_2 > 0.$$ \hspace{1cm} (13)

There exists a constant $C := C(s_1, s_2)$, such that, for all $f, g \in H^{s_1}(\mathbb{R}^2) \cap H^{s_2}(\mathbb{R}^2)$,

$$\|fg\|_{H^{s_1+s_2-1}} \leq C \left(\|f\|_{H^{s_1}}\|g\|_{H^{s_2}} + \|f\|_{H^{s_1}}\|g\|_{H^{s_1}} + \|f\|_{H^{s_2}}\|g\|_{H^{s_2}}\right).$$ \hspace{1cm} (14)

If $s_1, s_2 < 1$ and $s_1 + s_2 > 0$, there exists a constant $C = C(s_1, s_2)$, such that, for all $f \in H^{s_1}(\mathbb{R}^2)$ and $g \in H^{s_2}(\mathbb{R}^2)$,

$$\|fg\|_{H^{s_1+s_2-1}} \leq \|f\|_{H^{s_1}}\|g\|_{H^{s_2}}.$$ \hspace{1cm} (15)

For the proof of the main results, we need the following lemma.

### 6. Under the same conditions as in Theorem 3, for all $\sigma \geq 0$ and $\epsilon \geq 0$,

$$\int_{\mathbb{R}^2} \left|\xi\right|^{2\sigma}e^{2\epsilon|\xi|}\left|\mathcal{F}((u \cdot \nabla)\theta)\mathcal{F}(w)\right|d\xi \leq C\|\theta\|_{H^{2-\alpha}}\|\theta\|_{H^{2}}\|w\|_{H^{2}}.$$ \hspace{1cm} (16)

where $\mathcal{F}(\theta) = e^{2\epsilon|\xi|}\mathcal{F}(\theta)$ and $\mathcal{F}(w) = e^{2\epsilon|\xi|}\mathcal{F}(w)(t, \xi)$ and $w \in H^\alpha(\mathbb{R}^2)$.

**Remarks 7.** (i) If $\sigma = 0$, formula (16) gives

$$\int_{\mathbb{R}^2} e^{2\epsilon|\xi|}\left|\mathcal{F}((u \cdot \nabla)\theta)\mathcal{F}(w)\right|d\xi \leq C\|\theta\|_{H^{2-\alpha}}\|\theta\|_{H^{2}}\|w\|_{H^{2}}.$$ \hspace{1cm} (17)

(ii) If $\epsilon = 0$ and $\sigma = 0$, formula (16) gives

$$\int_{\mathbb{R}^2} \left|\mathcal{F}((u \cdot \nabla)\theta)\mathcal{F}(w)\right|d\xi \leq C\|\theta\|_{H^{2-\alpha}}\|\theta\|_{H^{2}}\|w\|_{H^{2}}.$$ \hspace{1cm} (18)

(iii) If $\epsilon = 0$ and $\sigma = 2 - 2\alpha$, formula (16) gives

$$\int_{\mathbb{R}^2} \left|\xi\right|^{2(2-2\alpha)}e^{2\epsilon|\xi|}\left|\mathcal{F}((u \cdot \nabla)\theta)\right|d\xi \leq C\|\theta\|_{H^{2-\alpha}}\|\theta\|_{H^{2}}\|w\|_{H^{2-\alpha}}.$$ \hspace{1cm} (19)
Proof of Lemma 6. By the Cauchy-Schwarz inequality, we get
\[ \int |[\xi]^{2\sigma-a} e^{2\xi|l|} |\mathcal{F}((u \cdot \nabla) \theta) \, \bar{\omega}| d\xi \leq \int |[\xi]^{2\sigma-a} e^{2\xi|l|} |\mathcal{F}((u \cdot \nabla) \theta) \, |[\xi]^{\sigma-a} |\bar{\omega}| d\xi \]
\begin{equation}
\leq \left( \int |[\xi]^{2(\sigma-a)} e^{2\xi|l|} |\mathcal{F}((u \cdot \nabla) \theta) \right)^{1/2} \| \omega \|_{L^{2}}. \tag{20}
\end{equation}

Using the weak derivatives properties, the elementary inequality \( e^{\xi^2} \leq e^{\xi^2 - \xi^4} \), with \( \alpha \geq 0 \) and \( \xi, \eta \in \mathbb{R}^2 \), and the product laws (Lemma 5), with \( s_1 + s_2 = \sigma - \alpha + 2 > 0 \), \( s_1 = 2 - 2\alpha < 1 \), and \( s_2 = \sigma + \alpha \), we can eliminate the nonlinear part of (20) as follows:
\[ \int |[\xi]^{2(\sigma-a)} e^{2\xi|l|} \mathcal{F}((u \cdot \nabla) \theta) |^2 d\xi \leq \int |[\xi]^{2(\sigma-a)} e^{2\xi|l|} |\mathcal{F}((u \cdot \nabla) \theta) \mathcal{F}((u \cdot \nabla) \theta) \, |[\xi]^{\sigma-a} |\bar{\omega}| d\xi \]
\begin{equation}
\leq \left( \int |[\xi]^{2(\sigma-a)} e^{2\xi|l|} \mathcal{F}((u \cdot \nabla) \theta) \mathcal{F}((u \cdot \nabla) \theta) \right)^{1/2} \| \omega \|_{L^{2}}. \tag{21}
\end{equation}

3. Proof of Theorem 3

To prove Theorem 3, we need the following result.

Proposition 8. There exists \( c_\alpha > 0 \) such that, for all \( \theta^0 \in H^{2,2\alpha} (\mathbb{R}^2) \), \( \| \theta^0 \|_{H^{2,2\alpha} (\mathbb{R}^2) < c_\alpha} \), and there exists a global solution \( \theta \in \mathcal{C}^1((0,T^*) \cap L^2 (\mathbb{R}^2)) \) such that
\[ \int e^{(2\alpha-1)|l|} |\xi|^{2(2\alpha-2)} |\mathcal{F}(\theta(t,\xi))|^2 d\xi \]
\[ + \int_0^t \int e^{(2\alpha-1)|l|} |\xi|^{2(2\alpha-2)} |\mathcal{F}(\theta(t,\xi))|^2 d\xi d\tau \]
\begin{equation}
\leq 2 \| \theta^0 \|_{H^{2,2\alpha}}^2. \tag{22}
\end{equation}

Proof. The proof is done in two steps.

First Step. For a nonnegative integer \( n \), Friedrich’s operator \( J_n \) is defined by
\[ J_n (f) := \mathcal{F}^{-1} \left( \left| \chi_{|\xi|<n} \right| \mathcal{F}(f) \right). \tag{23}
\]
Consider the following approximate system \( \mathcal{S}_n^\alpha \) on \( \mathbb{R}_+ \times \mathbb{R}^2 \):
\[ \partial_t \theta + (-\Delta)^{1/2} J_n \theta + J_n (u_n \cdot \nabla) J_n \theta = 0, \]
\[ u = (-\Delta)^{-1/2} \theta, \tag{24}
\]
\[ \theta \mid t = 0 = J_n \theta^0. \]

Then, by the ordinary differential equations theory, the system \( \mathcal{S}_n^\alpha \) has a unique maximal solution \( \theta^0 \) in the space \( \mathcal{C}^1((0,T^*) \cap L^2 (\mathbb{R}^2)) \), \( T^* > 0 \). Using the uniqueness and the fact that \( J_n^2 = I_n \), we obtain \( \partial_t \theta_n = \theta_n \) and
\[ \partial_t \theta_n + (-\Delta)^{1/2} \theta_n + J_n (u_n \cdot \nabla \theta_n) = 0, \]
\[ u_n = (-\Delta)^{-1/2} \theta_n, \tag{25}
\]
\[ \theta_n \mid t = 0 = J_n \theta^0. \]

Taking the scalar product in \( L^2 (\mathbb{R}^2) \), we obtain, for \( t \in [0,T^*), \)
\[ \partial_t \| \theta_n \|_{L^2}^2 + 2 \| \theta_n \|_{L^2}^2 \leq 0. \tag{26}
\]

It follows that, for all \( t \in [0,T^*), \| \theta_n(t) \|_{L^2}^2 \leq \| J_n \theta^0 \|_{L^2}^2 \), which implies that \( T^* = +\infty \).

Now, taking scalar product in \( H^{2,2\alpha} (\mathbb{R}^2) \), we obtain
\[ \partial_t \| \theta_n \|_{H^{2,2\alpha}}^2 + 2 \| \theta_n \|_{H^{2,2\alpha}}^2 \leq 2 \langle J_n (u_n \cdot \nabla \theta_n), \theta_n \rangle_{H^{2,2\alpha}} \]
\[ \leq 2 \| J_n (u_n \cdot \nabla \theta_n) \|_{H^{2,2\alpha}} \| \theta_n \|_{H^{2,2\alpha}} \]
\[ = \| \theta_n \|_{H^{2,2\alpha}} \| \theta_n \|_{H^{2,2\alpha}} \]
\[ = \| \theta_n \|_{H^{2,2\alpha}} \]
\[ \leq \| \theta_n \|_{H^{2,2\alpha}} \]
\[ \leq \| \theta_n \|_{H^{2,2\alpha}}. \tag{27}
\]

By
\[ \| \theta_n \|_{H^{2,2\alpha}}^2 = \int \left| \xi \right|^{2(2\alpha-2)} |\mathcal{F}(\theta_n(t,\xi))|^2 \left( \left| i\xi \right|^{-1/2} - \left| \xi \right|^{-1} \right)^2 d\xi = \| \theta_n \|_{H^{2,2\alpha}}^2. \tag{28}
\]

But
\[ \| \theta_n \|_{H^{2,2\alpha}}^2 \leq \| \theta_n \|_{H^{2,2\alpha}}^2 \leq \| \theta_n \|_{H^{2,2\alpha}}^2. \tag{29}
\]

Then,
\[ \partial_t \| \theta_n \|_{H^{2,2\alpha}}^2 + 2 \| \theta_n \|_{H^{2,2\alpha}}^2 \leq c (\alpha) \| \theta_n \|_{H^{2,2\alpha}}^2. \tag{30}
\]

Let
\[ T_n := \sup \left\{ t \geq 0, \| \theta_n \|_{L^2 (\mathbb{R}^2)} < 2c_\alpha \right\}. \tag{31}
\]

For \( 0 \leq t < T_n \), by (30), we have
\[ \| \theta_n(t) \|_{H^{2,2\alpha}}^2 + 2 \int_0^t \| \theta_n \|_{H^{2,2\alpha}}^2 dt \leq \| \theta_n \|_{H^{2,2\alpha}}^2 + \int_0^t \| \theta_n \|_{H^{2,2\alpha}}^2 dt; \]
\[ \| \theta_n(t) \|_{H^{2,2\alpha}}^2 + 2 \int_0^t \| \theta_n \|_{H^{2,2\alpha}}^2 dt \leq \| \theta_n \|_{H^{2,2\alpha}}^2 < c_\alpha^2. \tag{32}
\]
then $T_n = \infty$, and, for all $t \geq 0$, we have
\[
\|\theta_n(t)\|_{\dot{H}^{2-2\alpha}}^2 + \int_0^t \|\theta_n\|_{\dot{H}^{2-2\alpha}}^2 \leq \|\theta_0\|_{\dot{H}^{2-2\alpha}}^2.
\] (33)

If we take the limit when $n$ goes to the infinity, we find a solution $\theta \in \mathcal{C}_p(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, L^{2-\alpha}(\mathbb{R}^2))$ which satisfies
\[
\|\theta(t)\|_{\dot{H}^{2-2\alpha}}^2 + \int_0^t \|\theta\|_{\dot{H}^{2-2\alpha}}^2 \leq \|\theta_0\|_{\dot{H}^{2-2\alpha}}^2.
\] (34)

which proves the first result of Theorem 3.

Second Step. Back to the approximate system,
\[
\partial_t \tilde{\theta}_n + [\xi]^{2\alpha} \tilde{\theta}_n + 1/n \xi(\theta_n u^\alpha \cdot \nabla \theta_n) = 0,
\]
\[
\partial_t \|\tilde{\theta}_n\|^2 + 2[\xi]^{2\alpha} \|\tilde{\theta}_n\|^2 + 2 \text{Re} \left( \mathcal{F}(u^\alpha \cdot \nabla \theta_n)(\xi) \cdot \tilde{\theta}_n(-\xi) \right) = 0.
\] (35)

For $\varepsilon > 0$, we define
\[
f_n = f_{n,\varepsilon} := \mathcal{F}^{-1}(e^{\varepsilon \xi} \tilde{\theta}_n),
\]
\[
\tilde{f}_n = e^{\varepsilon \xi} \tilde{\theta}_n.
\] (36)

Then,
\[
\partial_t \|f_n\|^2 + 2[\xi]^{2\alpha} \|f_n\|^2 =
\[
2\varepsilon \|\xi\|^2 - 2 \text{Re} e^{2\varepsilon \xi} \left( \mathcal{F}(u^\alpha \cdot \nabla \theta_n)(\xi) \cdot \tilde{\theta}_n(-\xi) \right)
\]
\[
= 2\varepsilon \|\xi\|^2 - 2 \text{Re} e^{2\varepsilon \xi} \left( \mathcal{F}(u^\alpha \cdot \nabla \theta_n)(\xi) \cdot e^{\varepsilon \xi} \tilde{\theta}_n(-\xi) \right)
\]
\[
= 2\varepsilon \|\xi\|^2 - 2 \text{Re} e^{2\varepsilon \xi} \left( \mathcal{F}(u^\alpha \cdot \nabla \theta_n)(\xi) \cdot f_n(-\xi) \right)
\]
\[
\leq 2\varepsilon \|\xi\|^2 + 2e^{\varepsilon \xi} \|\xi\| \cdot \|\mathcal{F}(\theta_n u^\alpha)\| \cdot \|\tilde{\theta}_n(\xi)\|
\]
\[
\leq 2\varepsilon \|\xi\|^2 + 2e^{\varepsilon \xi} \|\xi\| \cdot \|\tilde{\theta}_n(\xi)\|
\]
\[
\leq 2\varepsilon \|\xi\|^2 + 2e^{\varepsilon \xi} \|\xi\| \cdot \|\tilde{\theta}_n(\xi)\|.
\] (37)

Taking the norm in $\dot{H}^{2-2\alpha}$, we obtain
\[
\partial_t \|f_n\|^2_{\dot{H}^{2-2\alpha}} + 2\|f_n\|^2_{\dot{H}^{2-2\alpha}} \
\leq 2\varepsilon \|\xi\|^2_{\dot{H}^{2-2\alpha}} + 2 \int [\xi]^{2(2-\alpha)+1\alpha} \left( \|f_n\| \cdot \|\tilde{f}_n\| \right) d\xi
\]
\[
\leq 2\varepsilon \|\xi\|^2_{\dot{H}^{2-2\alpha}} + 2 \int [\xi]^{2(2-\alpha)+1\alpha} \left( \|f_n\| \cdot \|\tilde{f}_n\| \right) d\xi.
\] (40)

By Cauchy-Schwarz inequality, we have
\[
\partial_t \|f_n\|^2_{\dot{H}^{2-2\alpha}} + 2\|f_n\|^2_{\dot{H}^{2-2\alpha}} \
\leq 2\varepsilon \|\xi\|^2_{\dot{H}^{2-2\alpha}} + 2 \|g_n \cdot g_n\|_{\dot{H}^{2-2\alpha}} + 2\|f_n\|_{\dot{H}^{2-2\alpha}}
\]
\[
\leq 2\varepsilon \|\xi\|^2_{\dot{H}^{2-2\alpha}} + 2 \|g_n \cdot g_n\|_{\dot{H}^{2-2\alpha}} + 2\|f_n\|_{\dot{H}^{2-2\alpha}}.
\] (43)

Using product law (15) in the homogeneous Sobolev space with $s_1 = 2-2\alpha < 1$, $s_2 = -\alpha$, and $s_1 + s_2 = (2-2\alpha) + (-\alpha) > 0$, we obtain
\[
\partial_t \|f_n\|^2_{\dot{H}^{2-2\alpha}} + 2\|f_n\|^2_{\dot{H}^{2-2\alpha}} \
\leq 2\varepsilon \|\xi\|^2_{\dot{H}^{2-2\alpha}} + C_\alpha \|g_n\|_{\dot{H}^{2-2\alpha}} + \|f_n\|_{\dot{H}^{2-2\alpha}}
\]
\[
\leq 2\varepsilon \|\xi\|^2_{\dot{H}^{2-2\alpha}} + C_\alpha \|g_n\|_{\dot{H}^{2-2\alpha}} + \|f_n\|_{\dot{H}^{2-2\alpha}}.
\] (44)

To estimate the term $\|f_n\|_{\dot{H}^{2-2\alpha}}$, we use the Hölder inequality and we get
\[
\|f_n\|_{\dot{H}^{2-2\alpha}} \leq \|f_n\|^{1/\alpha}_{\dot{H}^{2-2\alpha}} \|f_n\|^\alpha_{\dot{H}^{2-2\alpha}}.
\] (45)

The convex inequality $ab \leq a^p / p + b^q / q$, with $p = 2\alpha/(2\alpha - 1)$ and $q = 2\alpha$, gives
\[
\varepsilon \|f_n\|^2_{\dot{H}^{2-2\alpha}} \leq C_\alpha \|f_n\|^2_{\dot{H}^{2-2\alpha}} + \|f_n\|^2_{\dot{H}^{2-2\alpha}}.
\] (46)

Thus,
\[
\partial_t \|f_n\|^2_{\dot{H}^{2-2\alpha}} + \frac{3}{2}\|f_n\|^2_{\dot{H}^{2-2\alpha}} \
\leq C_\alpha \|f_n\|^2_{\dot{H}^{2-2\alpha}} + C_\alpha \|f_n\|_{\dot{H}^{2-2\alpha}} + \|f_n\|^2_{\dot{H}^{2-2\alpha}}.
\] (47)

where $f_n$ is in $\mathcal{C}(\mathbb{R}^+, L^{2-2\alpha}(\mathbb{R}^2))$ and $\|f_n(0)\|_{\dot{H}^{2-2\alpha}} = \|\theta_n(0)\|_{\dot{H}^{2-2\alpha}} < C_\alpha$. 

Let $T > 0$ and $\epsilon = (\ln(2)/C_\alpha T)^{2(\alpha-1)/2\alpha}$; we set

$$T_n := \sup \left\{ t \geq 0; \| f_n \|_{H^{2(\alpha-1)}(t)} < 2c_\alpha \right\}. \tag{48}$$

For $0 \leq t < \inf(T, T_n)$, we have

$$\partial_t \| f_n \|_{H^{2(\alpha-1)}(t)}^2 + \| f_n \|_{H^{2\alpha-2\alpha}(t)}^2 \leq C_\alpha e^{2\| f_n \|_{H^{2\alpha-2\alpha}(t)}} \| f_n \|_{H^{2\alpha-2\alpha}(t)}^2. \tag{49}$$

By Gronwall lemma, we get that, for all $t \in [0, \inf(T, T_n))$,

$$\| f_n(t) \|_{H^{2\alpha-2\alpha}(t)}^2 \leq \| f_n(0) \|_{H^{2\alpha-2\alpha}(t)}^2 e^{C_\alpha e^{2\| f_n \|_{H^{2\alpha-2\alpha}(t)}} \inf(T, T_n)}, \tag{50}$$

which proves formula (22), and the proof of Proposition 8 is finished.

Now we intend to study the behavior of the solution at infinity. We claim to prove that, for all $s \geq 2 - 2\alpha$,

$$\| \phi(t) \|_{H^s} = O \left( \frac{1}{t^{(s-(2-2\alpha))/2\alpha}} \right), \quad t \to \infty. \tag{54}$$

We can suppose that $s > 2 - 2\alpha$. We have

$$\| \phi(t) \|_{H^s}^2 = \int |\hat{\phi}(t, \xi)|^2 d\xi = \int |\xi|^{2s-2(2\alpha)} e^{i\xi(t)} \frac{|\hat{f}(t, \xi)|^2}{|\xi|^{2s-2(2\alpha)}} d\xi$$

$$\leq t^{-(2s-2(2\alpha))/2\alpha} \frac{1}{|\xi|^{2s-2(2\alpha)}} \int |\xi|^{2(\alpha-1)} e^{i\xi(t)} \frac{|\hat{f}(t, \xi)|^2}{|\xi|^{2s-2(2\alpha)}} d\xi$$

$$\leq M t^{-(2s-2(2\alpha))/2\alpha} \frac{1}{|\xi|^{2s-2(2\alpha)}} \int |\xi|^{2(\alpha-1)} e^{i\xi(t)} \frac{|\hat{f}(t, \xi)|^2}{|\xi|^{2s-2(2\alpha)}} d\xi, \tag{55}$$

where $M := \sup_{s \geq 0} (\epsilon e^{-1/(2-2\alpha)\xi})^{2s-2(2s-2\alpha)}$.

Using (22), we get

$$\| \phi(t) \|_{H^s}^2 \leq M \| \phi \|_{H^{2\alpha-2\alpha}}^2 t^{-(s-(2-2\alpha))/\alpha} \tag{56}$$

and the proof of Theorem 3 is finished.

Remark 9. (i) Combining Theorems 2 and 3, we can obtain, for $2/3 \leq \alpha \leq 1$ and $s \geq 2 - 2\alpha$,

$$\| \phi(t) \|_{H^s}^2 = o \left( t^{-(s-(2-2\alpha))/2\alpha} \right), \quad t \to \infty. \tag{57}$$

Indeed, from (34), $\| \phi(t) \|_{H^{2\alpha-2\alpha}} < c_\alpha$, for all $t \geq 0$. For $T > 0$, we consider the following system:

$$\partial_t \nu + (-\Delta)^{s/2} \nu \in \mathbb{R}^+ \times \mathbb{R}^2,$$

$$V := (-\Delta)^{s/2} \nu \cdot \nu = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^2,$$

$$v(0, \cdot) = \theta \left( \frac{T}{2}, \cdot \right) \quad \text{in} \quad \mathbb{R}^2.$$

This system has a Leray unique solution $v_{T}$ that satisfies, for all $t > 0$,

$$\| v_T(t, \cdot) \|_{H^s} \leq \frac{\sqrt{M} \| v_T(0, \cdot) \|_{H^{2\alpha-2\alpha}}}{t^{(s-(2-2\alpha))/2\alpha}}. \tag{59}$$

From the uniqueness of the solution, we have $v_T(t, \cdot) = \theta(t + T/2, \cdot)$; then

$$\| \theta(t + T/2, \cdot) \|_{H^s} \leq \frac{\sqrt{M} \| \theta(T/2, \cdot) \|_{H^{2\alpha-2\alpha}}}{t^{(s-(2-2\alpha))/2\alpha}}. \tag{60}$$

For $t = T/2$, we have

$$\| \theta(T, \cdot) \|_{H^s} \leq \frac{2^{(s-(2-2\alpha))/2\alpha} \sqrt{M} \| \theta(T/2, \cdot) \|_{H^{2\alpha-2\alpha}}}{t^{(s-(2-2\alpha))/2\alpha}}. \tag{61}$$

Combining this inequality with the result of Theorem 2, we obtain the desired result.

(ii) If $\alpha \in (1/2, 2/3)$, we do not know if $\lim_{t \to \infty} \| \phi(t) \|_{H^{2\alpha-2\alpha}} = 0$ holds. But this result depends on the lower frequencies. Indeed, for $\delta > 0$ and $\epsilon > 0$, we have

$$\int_{|\xi| < \delta} |\xi|^{2(\alpha-2\alpha)} |\hat{\phi}(t, \xi)|^2 d\xi \leq M \| \phi \|_{H^{2\alpha-2\alpha}} \delta^{-2\alpha} t^{-\epsilon/\alpha} \tag{62}$$

$$\leq M \| \phi \|_{H^{2\alpha-2\alpha}} \delta^{-2\alpha} t^{-\epsilon/\alpha} \tag{63}$$

Then, for $\delta = t^{-1/2\alpha+\alpha}$, $\alpha > 0$, we have

$$\int_{|\xi| < \delta} |\xi|^{2(\alpha-2\alpha)} |\hat{\phi}(t, \xi)|^2 d\xi \leq M \| \phi \|_{H^{2\alpha-2\alpha}} t^{-\epsilon/\alpha} \tag{64}$$

$$\lim_{t \to \infty} \int_{|\xi| < \delta} |\xi|^{2(\alpha-2\alpha)} |\hat{\phi}(t, \xi)|^2 d\xi = 0.$$
Then, to prove the result,
\[
\lim sup_{t \to \infty} \|\theta(t)\|_{L^{p}(\mathbb{R}^{2})} = 0. \quad (65)
\]
It suffices to prove that
\[
\lim sup_{t \to \infty} \int_{|\xi| \leq t^{-1/(2-2\alpha)}} |\xi|^{2(2-2\alpha)} \|\theta(t, \xi)\|^{2} \, d\xi = 0. \quad (66)
\]

4. Proof of Theorem 4

First Step. Using the approximate system (24) and inequality (17),
\[
\partial_{t} \|f_{0}\|^{2} + 2 \|f_{n}\|^{2}_{p} \leq 2e\|f_{n}\|^{2}_{L^{1}+c} + C_{\alpha} \|f_{n}\|^{2}_{L^{2-\alpha}}. \quad (67)
\]
Then,
\[
\int_{\xi} e^{\xi^{2}/2} \|\theta(t, \xi)\|^{2} \, d\xi + \int_{t}^{0} \int_{\xi} e^{\xi^{2}/2} |\xi|^{2(2\alpha)} \|\theta(t, \xi)\|^{2} \, d\xi \, d\tau \leq 2 \|\theta^{0}\|^{2}_{L^{2}}. \quad (68)
\]

Second Step. From relation (22), we deduce that
\[
\|\theta\|_{L^{p}} \leq \int_{\xi} |\xi|^{2(2\alpha)} e^{\xi^{2}/2} |\xi|^{2\alpha} \|\theta(t, \xi)\|^{2} \, d\xi \leq t^{-3\alpha} \sup_{\xi \geq 0} x^{2\alpha} e^{-x} \cdot 2 \|\theta^{0}\|^{2}_{L^{2}} \leq C \alpha t^{-3\alpha} \|\theta^{0}\|^{2}_{L^{2}}. \quad (69)
\]
Then, the proof is achieved.

Remark 10. If \(2/3 \leq \alpha \leq 1\) and \(s \geq 0\), we have
\[
\|\theta(t)\|^{2}_{L^{p}} = o \left( t^{-s/(2\alpha)} \right), \quad t \to \infty. \quad (70)
\]

5. General Properties of Global Solutions

Theorem 11. Let \(\theta\) be a global solution of \((\mathcal{S}_{\alpha})\) such that
\[
\theta \in \mathcal{C}_{b}(\mathbb{R}^{+}, H^{2-2\alpha}(\mathbb{R}^{2})). \quad (71)
\]
Then, \(\theta \in \mathcal{C}_{b}(\mathbb{R}^{+}, H^{2-2\alpha}(\mathbb{R}^{2})) \cap L^{2}(\mathbb{T}, H^{2-\alpha}(\mathbb{R}^{2}))\) for some \(T > 0\). Moreover, for all \(s \geq 2 - 2\alpha\),
\[
\|\theta(t)\|_{H^{s}} = o \left( t^{-(1-s)/(2\alpha)} \right), \quad t \to \infty. \quad (72)
\]
Combining the energy estimate
\[
\|\theta_{t}\|^{2}_{L^{2}} + 2 \int_{0}^{t} \|\theta_{t}\|^{2}_{H^{s}} \, d\tau \leq \|\theta^{0}\|^{2}_{L^{2}}, \quad (73)
\]
and the conclusion of Theorems 4 and 11, we get the following.

Theorem 12. Let \(\theta\) be a global solution of \((\mathcal{S}_{\alpha})\) such that
\[
\theta \in \mathcal{C}_{b}(\mathbb{R}^{+}, H^{2-2\alpha}(\mathbb{R}^{2})). \quad (74)
\]
Then, \(\theta \in \mathcal{C}_{b}(\mathbb{R}^{+}, H^{2-2\alpha}(\mathbb{R}^{2})) \cap L^{2}(\mathbb{T}, H^{2-\alpha}(\mathbb{R}^{2}))\) for some \(T > 0\).
Moreover, for all \(s \geq 0\),
\[
\|\theta(t)\|_{H^{s}} = o \left( t^{-s/(2\alpha)} \right), \quad t \to \infty. \quad (75)
\]
Remarks 13. (a) Let \(\alpha \in (1/2, 1]\) and let \(\theta\) be a global solution of \((\mathcal{S}_{\alpha})\) such that
\[
\theta \in \mathcal{C}_{b}(\mathbb{R}^{+}, H^{2-2\alpha}(\mathbb{R}^{2})). \quad (76)
\]
Using the Sobolev injection,
\[
L^{p}(\mathbb{R}^{2}) \hookrightarrow H^{s}(\mathbb{R}^{2}), \quad \text{with} \quad \frac{1}{p} + \frac{s}{2} = \frac{1}{2}; \quad 0 < s < 1. \quad (77)
\]
We conclude that, for all \(p \in [2/(1 - (2 - 2\alpha)), \infty)\),
\[
\|\theta(t)\|_{L^{p}} = o \left( t^{-(1-2)/(2\alpha)} \right), \quad t \to \infty. \quad (78)
\]
(b) Let \(\theta\) be a global solution of \((\mathcal{S}_{\alpha})\) such that
\[
\theta \in \mathcal{C}_{b}(\mathbb{R}^{+}, H^{2-2\alpha}(\mathbb{R}^{2})); \quad (79)
\]
then, for all \(p \in (2, \infty)\),
\[
\|\theta(t)\|_{L^{p}} = o \left( t^{-(p-2)/2\alpha} \right), \quad t \to \infty. \quad (80)
\]
Using the classical interpolation inequality
\[
\|f\|_{L^{s}(\mathbb{R}^{3})} \leq C \|f\|_{L^{1}(\mathbb{R}^{3})}^{1/s} \|f\|_{L^{\infty}(\mathbb{R}^{3})}^{1/(3-s)}, \quad s \in (1, \infty), \quad (81)
\]
and Theorem 12, we get
\[
\|\theta(t)\|_{L^{\infty}} = o \left( t^{-1/(2\alpha)} \right), \quad t \to \infty. \quad (82)
\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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