Research Article

Generalized Composition Operators on Zygmund-Orlicz Type Spaces and Bloch-Orlicz Type Spaces

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The boundedness and compactness of generalized composition operators on Zygmund-Orlicz type spaces and Bloch-Orlicz type spaces are established in this paper.

1. Introduction

Let $\mathbb{D}$ be a unit disk in complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The Bloch space, $\mathcal{B}$, consists of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty. \quad (1)$$

$\mathcal{B}$ becomes a Banach space when it is equipped with the norm $\|f\|_{\mathcal{B}} := |f(0)| + \|f\|_{\mathcal{B}}$ (see, e.g., [1]).

Let $\alpha > 0$, and the α -Bloch space, denoted by $\mathcal{B}^{\alpha}$, consists of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{\alpha} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |f'(z)| < \infty. \quad (2)$$

α -Bloch space is introduced and studied by numerous authors. The general theory of α -Bloch function spaces is referred to in [2]. Recently, many authors studied different class of Bloch-type spaces, where the typical weight function, $w(z) = 1 - |z|^2$ ($z \in \mathbb{D}$), is replaced by a bounded continuous positive function μ defined on $\mathbb{D}$. More precisely, a function $f \in H(\mathbb{D})$ is called a μ -Bloch function, denoted by $f \in \mathcal{B}^{\mu}$, if

$$\|f\|_{\mu} := \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty. \quad (3)$$

Clearly, if $\mu(z) = w(z)^{\alpha}$ with $\alpha > 0$, $\mathcal{B}^{\mu}$ is just the α -Bloch space $\mathcal{B}^{\alpha}$. It is readily seen that $\mathcal{B}^{\mu}$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^{\mu}} := |f(0)| + \|f\|_{\mu}. \quad (4)$$

$\mathcal{B}^{\mu}$ spaces appear in the literature of a natural way when one studies the properties of some operators of holomorphic functions in a certain space. For instance, Attele in [3] proves that the Hankel operator induced by a function f in the Bergman space is bounded if and only if $f \in \mathcal{B}^{\mu_1}$ (where $\mu_1 = w(z) \log(w(z))$, with $z \in \mathbb{D}$). The space $\mathcal{B}^{\mu_1}$ is known as the log-Bloch space or the weighted Bloch space. The logarithmic Bloch-type space is introduced by Krantz and Stević [4, 5], where some properties of this space are studied. In the last decade, there was a big interest in the investigation of Bloch-type spaces and various concrete linear operators $L : X \rightarrow Y$, where at least one of the spaces X and Y is Bloch space. For some other recent results in the areas, see, for example, [4, 6–10] and a lot of references therein.

Recently, Fernández in [8] uses Young's functions to define the Bloch-Orlicz space as a generalization of Bloch space. More precisely, let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$ -function; that is, φ is a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} (t/\varphi(t)) = \lim_{t \rightarrow 0} (\varphi(t)/t) = 0$. The Bloch-Orlicz space associated with the function φ , denoted by $\mathcal{B}^{\varphi}$, is the class of all analytic functions $f \in \mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(\lambda |f'(z)|) < \infty, \quad (5)$$

for some $\lambda > 0$ depending on f . Without loss of generality, we can suppose that φ^{-1} is continuous and differentiable. In fact, if φ^{-1} is not differentiable everywhere, we set the function

$$\psi(t) = \int_0^t \frac{\varphi(x)}{x} dx \quad (t \geq 0); \quad (6)$$

then φ is differentiable, whence ψ^{-1} is differentiable everywhere on $[0, \infty)$. Furthermore, φ is a strictly increasing and convex function satisfying $\varphi(0) = 0$; then the function $\varphi(t)/t$, $t > 0$, is increasing and

$$\varphi(t) \geq \psi(t) \geq \int_{t/2}^t \frac{\varphi(x)}{x} dx \geq \varphi\left(\frac{t}{2}\right), \quad (7)$$

for all $t \geq 0$. Hence $\mathcal{B}^\varphi = \mathcal{B}^\psi$. By the convexity of φ , it is not difficult to see that the Minkowski's functional

$$\|f\|_{b^\varphi} = \inf \left\{ k > 0 : S_\varphi\left(\frac{f'}{k}\right) \leq 1 \right\} \quad (8)$$

defines a seminorm for $\mathcal{B}^\varphi$, which in this case is known as Luxemburg's seminorm, where

$$S_\varphi(f) := \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(|f(z)|). \quad (9)$$

In fact, it can be shown that $\mathcal{B}^\varphi$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_{b^\varphi}. \quad (10)$$

We observe that, for any $f \in \mathcal{B}^\varphi \setminus \{0\}$, the following relation

$$S_\varphi\left(\frac{f'}{\|f\|_{\mathcal{B}^\varphi}}\right) \leq 1 \quad (11)$$

holds.

The inequality above allows us to obtain

$$|f'(z)| \leq \varphi^{-1}\left(\frac{1}{1 - |z|^2}\right) \|f\|_{\mathcal{B}^\varphi}, \quad (12)$$

for all $f \in \mathcal{B}^\varphi$ and for all $z \in \mathbb{D}$. This last inequality implies that the evaluation function is defined as $T_z(f) := f(z)$, where $z \in \mathbb{D}$ is fixed and $f \in \mathcal{B}^\varphi$ is continuous on $\mathcal{B}^\varphi$. In fact, let $z \in \mathbb{D}$ be fixed and any $f \in \mathcal{B}^\varphi$; we have

$$\begin{aligned} |T_z(f)| &= |f(z)| \leq |f(0)| + \int_{[0,z]} |f'(s)| |ds| \\ &\leq \left(1 + \int_0^1 \varphi^{-1}\left(\frac{1}{1 - |z|^2 t^2}\right) dz\right) \|f\|_{\mathcal{B}^\varphi}. \end{aligned} \quad (13)$$

From the definition of Luxemburg seminorm and the expression (11), we have

$$S_\varphi(f') \leq 1 \iff \|f\|_{\mathcal{B}^\varphi} \leq 1, \quad (14)$$

for any $f \in \mathcal{B}^\varphi$.

As an easy consequence of (11), we have that the Bloch-Orlicz space is isometrically equal to μ -Bloch space when

$$\mu(z) = \frac{1}{\varphi^{-1}\left(1/(1 - |z|^2)\right)}, \quad (15)$$

with $z \in \mathbb{D}$. Thus, for any $f \in \mathcal{B}^\varphi$, we have

$$\|f\|_{\mathcal{B}^\varphi} = \sup_{z \in \mathbb{D}} \mu(z) |f'(z)|. \quad (16)$$

Denote by $\mathcal{X}$ the class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty, \quad (17)$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. From the theorem of Zygmund (see [11, Theorem 5.3]) and the closed graph theorem, we see that $f \in \mathcal{X}$ if and only if $\sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty$. It is easy to see that $\mathcal{X}$ is a Banach space under the norm $\|\cdot\|_{\mathcal{X}}$, where

$$\|f\|_{\mathcal{X}} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)|. \quad (18)$$

From (18) it is easy to obtain

$$|f'(z) - f'(0)| \leq \frac{1}{2} \|f\|_{\mathcal{X}} \log \frac{1 + |z|}{1 - |z|} \quad \text{for } f \in \mathcal{X}. \quad (19)$$

For some other information and operators on this space, see, for example, [6, 12, 13].

Inspired by the way Bloch-Orlicz spaces were defined (see [8, 14]), we define the Zygmund-Orlicz space, which is denoted by $\mathcal{Z}^\varphi$, as the class of all analytic functions f in $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(\lambda |f''(z)|) \leq \infty, \quad (20)$$

for some $\lambda > 0$ depending on f . Same as the Bloch-Orlicz space, since φ is convex, it is not difficult to see that the Minkowski functional

$$\|f\|_{\mathcal{Z}^\varphi} = \inf \left\{ k > 0 : S_\varphi\left(\frac{f''}{k}\right) \leq 1 \right\} \quad (21)$$

defines a seminorm for $\mathcal{Z}^\varphi$. Furthermore, it can be shown that $\mathcal{Z}^\varphi$ is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\varphi} = |f(0)| + |f'(0)| + \|f\|_{\mathcal{Z}^\varphi}. \quad (22)$$

The following useful lemmas are easily obtained.

Lemma 1 (see [15]). *If $f \in \mathcal{X}$, then*

$$(i) \quad |f(z)| \leq \|f\|_{\mathcal{X}} \quad \text{for every } z \in \mathbb{D}, \quad (23)$$

$$(ii) \quad |f'(z)| \leq \log \left(\frac{e}{1 - |z|^2} \right) \|f\|_{\mathcal{X}} \quad \text{for every } z \in \mathbb{D}. \quad (24)$$

Also we can observe the following property.

Lemma 2. *For any $f \in \mathcal{X}^\varphi \setminus \{0\}$, the following relation*

$$S_\varphi \left(\frac{f''}{\|f\|_{\mathcal{X}^\varphi}} \right) \leq 1 \quad (25)$$

holds.

Proof. In the same way as the case of Bloch-Orlicz space, so the details are omitted here. $\square$

Lemma 2 allows us to obtain that

$$|f''(z)| \leq \varphi^{-1} \left(\frac{1}{1-|z|^2} \right) \|f\|_{\mathcal{X}^\varphi}, \quad (26)$$

for all $f \in \mathcal{X}^\varphi$ and all $z \in \mathbb{D}$.

From the definition of Luxemburg seminorm and expression (25), we have

$$S_\varphi(f'') \leq 1 \iff \|f\|_{\mathcal{X}^\varphi} \leq 1. \quad (27)$$

Also, as an easy consequence of (25), we have that Zygmund-Orlicz space is isometrically equal to μ -Zygmund space, where $\mu(z) = 1/\varphi^{-1}(1/(1-|z|^2))$ with $z \in \mathbb{D}$. For more information about $\mathcal{X}^\mu$, see [7, 10, 16].

Specially, if φ is an $\mathcal{N}$ -function such that

$$I_\varphi(z) = \int_0^1 \varphi^{-1} \left(\frac{1}{1-|z|^2 t^2} \right) dt \quad (28)$$

is bounded for all $z \in \mathbb{D}$, then we get $\mathcal{X}^\varphi \subset H^\infty$, the space of all bounded analytic functions on $\mathbb{D}$. However, there exists $\mathcal{N}$ -functions for which $I_\varphi(z)$ is not a bounded function; for instance, consider $\varphi(t) = t \log(1+t)$ with $t \geq 0$.

Let ϕ be an analytic self-map of $\mathbb{D}$; then the composition operator on $H(\mathbb{D})$ is given by

$$C_\phi f = f \circ \phi. \quad (29)$$

Composition operators acting on various spaces of analytic functions have been the object for recent years. In particular, the problems of relating operator-theoretic properties of C_ϕ to function-theoretic properties of ϕ are interesting and have been widely discussed. See the book of Cowen and MacCluer [17] and Shapiro [9] for discussions of composition operators classical spaces of analytic functions.

Assume that $g : \mathbb{D} \rightarrow \mathbb{C}$ is a holomorphic map of the disk $\mathbb{D}$, for $f \in H(\mathbb{D})$; we define a linear operator as follows:

$$(C_\phi^g f)(z) = \int_0^z f'(\phi(\xi)) g(\xi) d\xi, \quad z \in \mathbb{D}. \quad (30)$$

The operator C_ϕ^g is called the generalized composition operator, when $g = \phi'$. We see that this operator is essentially composition operator, since the difference $C_\phi^g - C_\phi$ is constant. Therefore, C_ϕ^g is a generalization of the composition operator, which was introduced in [6].

Recall that if X and Y are Banach spaces and $L : X \rightarrow Y$ is a linear operator, then L is said to be compact if, for every bounded sequence $\{x_n\}$ in X , the sequence $(L\{x_n\})$ has a convergent subsequence. The book [17] contains plenty of information on this topic. By the standard arguments (see, e.g., Proposition 3.11 in [17]), the following lemma follows.

Lemma 3. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$ -function, $g \in H(\mathbb{D})$, and ϕ an analytic self-map of $\mathbb{D}$. Let $X = \mathcal{X}$ and $Y = \mathcal{B}^\varphi$ or $\mathcal{X}^\varphi$. Then $C_\phi^g : X \rightarrow Y$ is compact if and only if $C_\phi^g : X \rightarrow Y$ is bounded and, for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in X which converges to zero uniformly on $\mathbb{D}$ as $k \rightarrow \infty$, one has $\|C_\phi^g f_k\|_Y \rightarrow 0$ as $k \rightarrow \infty$.*

Some characterization of the boundedness and compactness of the composition operator, as well as Volterra type operator, on Bloch-Orlicz-type space and Zygmund space can be found in [2, 18–22]. In [6], the boundedness and compactness of the generalized composition operator on Zygmund space and Bloch-type spaces are characterized by Li and Stević.

In this paper, we are devoted to investigating the boundedness and compactness of generalized composition operators between Zygmund-Orlicz type spaces and Bloch-Orlicz type spaces. The paper is organized as follows. In Section 2 we give the necessary and sufficient conditions for the boundedness and compactness of the operator $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$. In Section 3 we obtain the necessary and sufficient conditions for the boundedness and compactness of the operator C_ϕ^g on Zygmund type spaces. Throughout this paper, we use the letter C to denote a generic positive constant that can change its value at each occurrence. The notation $a \leq b$ means that there is a positive constant C such that $a \leq Cb$. If $a \leq b$ and $b \leq a$ hold, then one says that $a \asymp b$.

2. The Boundedness and Compactness of

$$C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$$

Now, we are ready to state and prove the main results in this section.

Theorem 4. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$ -function, $g \in H(\mathbb{D})$, and ϕ an analytic self-map of $\mathbb{D}$. Then $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is bounded if and only if*

$$L := \sup_{z \in \mathbb{D}} \left(\varphi^{-1} \left(\frac{1}{1-|z|^2} \right) \right)^{-1} |g(z)| \log \left(\frac{e}{1-|\phi(z)|^2} \right) < \infty. \quad (31)$$

Proof. Suppose that (31) holds. For arbitrary $z \in \mathbb{D}$ and $f \in \mathcal{X}$, by Lemma 1, we have the following estimate:

$$S_\varphi \left(\frac{(C_\phi^g f)'(z)}{L \|f\|_{\mathcal{X}}} \right)$$

$$\begin{aligned}
&= \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{|f'(\phi(z))| |g(z)|}{L \|f\|_{\mathcal{X}}} \right) \\
&\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \\
&\quad \times \left(\varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \frac{|f'(\phi(z))|}{\log(e/(1 - |\phi(z)|^2)) \|f\|_{\mathcal{X}}} \right) \\
&\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \right) \leq 1,
\end{aligned} \tag{32}$$

where we use Lemma 1 (relation (ii)) in the last inequality. From here, we can conclude that $\|C_\phi^g\|_\varphi \leq L \|f\|_{\mathcal{X}}$. Since $C_\phi^g f(0) = 0$, it follows that the generalized composition operator $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is bounded.

Now, suppose that there exists a constant $C > 0$ such that

$$\|C_\phi^g\|_\varphi \leq C \|f\|_{\mathcal{X}} \quad \forall f \in \mathcal{X}. \tag{33}$$

Let

$$h(z) = (z - 1) \left[\left(1 + \log \frac{e}{1 - z} \right)^2 + 1 \right], \tag{34}$$

and put

$$f_a(z) = \frac{h(\bar{a}z)}{\bar{a}} \left(\log \frac{e}{1 - |a|^2} \right)^{-1}, \tag{35}$$

for any $a \in \mathbb{D}$ such that $1/\sqrt{2} < |a| < 1$. Then we have

$$f'_a(z) = \left(\log \frac{e}{1 - \bar{a}z} \right)^2 \left(\log \frac{e}{1 - |a|^2} \right)^{-1}, \tag{36}$$

$$f''_a(z) = \frac{2\bar{a}}{1 - \bar{a}z} \left(\log \frac{e}{1 - \bar{a}z} \right) \left(\log \frac{e}{1 - |a|^2} \right)^{-1},$$

which implies that

$$\begin{aligned}
|f''_a(z)| &\leq \frac{2}{1 - |z|} \left(C + \log \frac{e}{1 - |a|} \right) \left(\log \frac{e}{1 - |a|^2} \right)^{-1} \\
&\leq \frac{2}{1 - |z|},
\end{aligned} \tag{37}$$

for $1/\sqrt{2} < |a| < 1$ and $\sup_{1/\sqrt{2} < |a| < 1} \|f_a\|_{\mathcal{X}} < \infty$. Therefore we have

$$\|C_\phi^g f_a\|_\varphi \leq C \|f_a\|_{\mathcal{X}} \leq L < \infty. \tag{38}$$

It follows that

$$S_\varphi \left(\frac{(C_\phi^g f_a)'(z)}{L} \right) \leq 1. \tag{39}$$

That is,

$$(1 - |z|^2) \varphi \left(\frac{(C_\phi^g f_a)'(z)}{L} \right) \leq 1, \tag{40}$$

for any $a \in \mathbb{D}$. But, on the other hand,

$$\varphi \left(\frac{|g(z)|}{L} |f'_a(\phi(z))| \right) \leq \frac{1}{1 - |z|^2}, \tag{41}$$

and then

$$\frac{|g(z)|}{L} |f'_a(\phi(z))| \leq \varphi^{-1} \left(\frac{1}{1 - |z|^2} \right). \tag{42}$$

So we have

$$\left(\varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \right)^{-1} |g(z)| |f'_a(\phi(z))| \leq L, \tag{43}$$

for all $a \in \mathbb{D}$. In particular, for $a = \phi(z)$, we have

$$\left(\varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \right)^{-1} |g(z)| \log \frac{e}{1 - |\phi(z)|^2} \leq L. \tag{44}$$

This concludes the proof of the theorem. $\square$

Theorem 5. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$ -function, $g \in H(\mathbb{D})$, and ϕ an analytic self-map of $\mathbb{D}$. Then $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is compact if and only if $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is bounded and

$$\lim_{|\phi(z)| \rightarrow 1} \mu(z) |g(z)| \log \left(\frac{e}{1 - |\phi(z)|^2} \right) = 0. \tag{45}$$

Proof. First assume that $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is bounded and (45) holds. By the boundedness of C_ϕ^g with $f(z) = z$, we see that

$$L := \sup_{z \in \mathbb{D}} \mu(z) |g(z)| < \infty. \tag{46}$$

Let $\{f_n\}$ be a sequence in $\mathcal{X}$ such that $\sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{X}} \leq K$ and $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. Then, by Lemma 3, it suffices to show that $\|C_\phi^g f_n\|_\mu \rightarrow 0$ as $n \rightarrow \infty$. By (45), we have that, for every $\varepsilon > 0$, there is a constant $r \in (0, 1)$, such that $r < |\phi(z)| \leq 1$, which implies

$$\mu(z) |g(z)| \log \left(\frac{e}{1 - |\phi(z)|^2} \right) < \frac{\varepsilon}{k}, \tag{47}$$

for any $z \in \mathbb{D}$. From here, we have that

$$\begin{aligned}
&\mu(z) |(C_\phi^g f_n)'(z)| \\
&= \mu(z) |g(z)| |f'_n(\phi(z))| \\
&\leq \mu(z) |g(z)| \log \left(\frac{e}{1 - |\phi(z)|^2} \right) \|f_n\|_{\mathcal{X}} \leq \varepsilon,
\end{aligned} \tag{48}$$

where $r < |\phi(z)| < 1$.

On the other hand, let $\Omega = \{\omega \in \mathbb{D} : |\omega| \leq r\}$; by the Cauchy estimate, if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence converging to zero on compact subsets of $\mathbb{D}$, then the sequence $\{f'_n\}_{n \in \mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. In particular, since Ω is compact, it follows that $\lim_{n \rightarrow \infty} \sup_{\omega \in \Omega} |f'_n(\omega)| = 0$. Using these facts and letting $n \rightarrow \infty$, we obtain

$$\begin{aligned}
&\sup_{|\phi(z)| \leq r} \mu(z) |(C_\phi^g f_n)'(z)| \\
&= \sup_{|\phi(z)| \leq r} \mu(z) |g(z)| |f'_n(\phi(z))| \\
&\leq L \sup_{\omega \in \Omega} |f'_n(\omega)| \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{49}$$

Since $(C_\phi^g f_n)(0) \rightarrow 0$ as $n \rightarrow \infty$, for given $\varepsilon > 0$, there exists an $n \in \mathbb{N}$, such that

$$\|C_\phi^g f_n\|_\mu = |C_\phi^g f_n(0)| + \sup_{z \in \mathbb{D}} \mu(z) |(C_\phi^g f_n)'(z)| \leq 3\varepsilon, \quad (50)$$

whenever $n \geq N$, which means that $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is a compact operator.

To prove the converse, suppose that there exists an $\varepsilon_0 > 0$ such that

$$\sup_{|\phi(z)| \geq r} \mu(z) |g(z)| \log \left(\frac{e}{1 - |\phi(z)|^2} \right) \geq \varepsilon_0, \quad (51)$$

for any $r \in (0, 1)$. Given a sequence of real number $\{r_n\} \subset (0, 1)$ such that $r_n \rightarrow 1$ as $n \rightarrow \infty$, we can find a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\phi(z_n)| \geq r_n$ and

$$\mu(z_n) |g(z_n)| \log \left(\frac{e}{1 - |\omega|^2} \right) \geq \frac{1}{2} \varepsilon_0, \quad (52)$$

where $\omega_n = \phi(z_n)$, and, if necessary, we may suppose that $|\omega_n| \rightarrow 1$ as $n \rightarrow \infty$.

Note that f_a , defined by (35), converges to zero uniformly on compact subset of $\mathbb{D}$ as $|a| \rightarrow 1$ and

$$f'_a(a) = \log \frac{e}{1 - |a|^2}, \quad \text{for } a \in \mathbb{D} \setminus \{0\}. \quad (53)$$

Now, we choose functions $\{f_n\}_{n \in \mathbb{N}}$ defined by

$$\begin{aligned} f_n(z) &= \frac{\bar{w}_n z - 1}{\bar{w}} \left[\left(1 + \log \frac{e}{1 - \bar{w}_n z} \right)^2 + 1 \right] \left[\log \frac{e}{1 - |w_n|^2} \right]^{-1}. \end{aligned} \quad (54)$$

From the proof of Theorem 4, we see that

$$\sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{X}} \leq C. \quad (55)$$

Moreover, we can see that $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ and satisfies

$$\begin{aligned} \|C_\phi^g f_n\|_\mu &\geq \mu(z_n) |g(z_n)| |f'_n(w_n)| \\ &= \mu(z_n) |g(z_n)| \log \left(\frac{e}{1 - |w_n|^2} \right) \geq \frac{1}{2} \varepsilon_0 > 0. \end{aligned} \quad (56)$$

Therefore, $C_\phi^g : \mathcal{X} \rightarrow \mathcal{B}^\varphi$ is not a compact operator. This completes the proof of the theorem. $\square$

3. The Boundedness and Compactness of

$$C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$$

In this section, we characterize the boundedness and compactness of the operator C_ϕ^g on Zygmund-Orlicz type spaces.

Theorem 6. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$ -function, $g \in H(\mathbb{D})$, and ϕ an analytic self-map of $\mathbb{D}$. Then $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is bounded if and only if

$$\begin{aligned} k_1 &= \sup_{z \in \mathbb{D}} \frac{|\phi'(z)| |g(z)|}{\varphi^{-1}(1/(1 - |z|^2)) (1 - |\phi(z)|^2)} < \infty, \\ k_2 &= \sup_{z \in \mathbb{D}} \frac{|g'(z)| \log(e/(1 - |\phi(z)|^2))}{\varphi^{-1}(1/(1 - |z|^2)) (1 - |\phi(z)|^2)} < \infty. \end{aligned} \quad (57)$$

Proof. Suppose first that (57) holds. For each $f \in \mathcal{X}$, we have the following estimate:

$$\begin{aligned} S_\varphi \left(\frac{(C_\phi^g f)''(z)}{\|f\|_{\mathcal{X}}} \right) &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \\ &\quad \times \left(\frac{|f''(\phi(z))| |g(z)| + |f'(\phi(z))| |g'(z)|}{C \|f\|_{\mathcal{X}}} \right) \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \\ &\quad \times \left[\left(\frac{\varphi^{-1}(1/(1 - |\phi(z)|^2)) |f''(\phi(z))| |\phi'(z)| |g(z)|}{\varphi^{-1}(1/(1 - |z|^2)) C \|f\|_{\mathcal{X}}} \right) \right. \\ &\quad \left. + \left(\frac{\varphi^{-1}(1/(1 - |\phi(z)|^2)) |f'(\phi(z))| |g'(z)|}{\varphi^{-1}(1/(1 - |z|^2)) C \|f\|_{\mathcal{X}}} \right) \right] \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \\ &\quad \times \left(\frac{k_1 \varphi^{-1}(1/(1 - |z|^2)) (1 - |\phi(z)|^2) |f''(\phi(z))|}{C \|f\|_{\mathcal{X}}} \right. \\ &\quad \left. + \frac{k_2 \varphi^{-1}(1/(1 - |z|^2)) |f'(\phi(z))|}{\log(e/(1 - |\phi(z)|^2)) C \|f\|_{\mathcal{X}}} \right) \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{k_1 C_1 + k_2 C_2}{C} \varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \right) \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \right) = 1, \end{aligned} \quad (58)$$

where C_1 and C_2 are constants, such that $k_1 C_1 + k_2 C_2 \leq C$. Here we use relations (26) and (24) in the last seconded inequality. Now, we can conclude that

$$\|C_\phi^g f\|_{\mathcal{X}^\varphi} \leq C \|f\|_{\mathcal{X}}, \quad (59)$$

and the generalized composition operator $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}$ is bounded.

Now assume that $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}$ is bounded; that is, there exists a constant C such that $\|C_\phi^g f\|_{\mathcal{X}} \leq C\|f\|_{\mathcal{X}}$ for all $f \in \mathcal{X}$. Taking the functions $f(z) = z$ and $f(z) = z^2$, respectively, we obtain

$$(1 - |z|^2) \varphi \left(\frac{|g'(z)|}{C} \right) \leq 1, \quad (60)$$

which implies

$$\sup_{z \in \mathbb{D}} \frac{|g'(z)|}{\varphi^{-1}(1/(1 - |z|^2))} < \infty, \quad (61)$$

$$(1 - (|z|^2)) \varphi \left(\frac{|\phi'(z)| |g(z)| + |\phi(z)| |g'(z)|}{C(1 - |z|^2)} \right) \leq 1. \quad (62)$$

Then

$$\frac{|\phi'(z)| |g(z)| + |\phi(z)| |g'(z)|}{\varphi^{-1}(1/(1 - |z|^2))} \leq C. \quad (63)$$

So we obtain

$$\sup_{z \in \mathbb{D}} \frac{|\phi'(z)| |g(z)| + |\phi(z)| |g'(z)|}{\varphi^{-1}(1/(1 - |z|^2))} \leq \infty. \quad (64)$$

Using these facts and the boundedness of the function $\phi(z)$ we get

$$\sup_{z \in \mathbb{D}} \frac{|\phi'(z)| |g(z)|}{\varphi^{-1}(1/(1 - |z|^2))} < \infty. \quad (65)$$

Let $h(z) = (z - 1)[(1 + \log(e/(1 - z)))^2 + 1]$ and put $f_a(z) = (h(\bar{a}z)/\bar{a})(\log(e/(1 - |a|^2)))^{-1}$ for $a \in \mathbb{D}$ such that $|a| > 1/2$. We have

$$\begin{aligned} (1 - |a|^2) \varphi \left(\left(|g'(\lambda)| \log \frac{e}{1 - |\phi(z)|^2} - 2 |\phi'(\lambda)| |g(\lambda)| \frac{|\phi(\lambda)|}{1 - |\phi(z)|^2} \right) \right. \\ \left. \times (C \|C_\phi^g\|_{\mathcal{X} \rightarrow \mathcal{X}^\varphi})^{-1} \right) \leq 1. \end{aligned} \quad (66)$$

Then

$$\begin{aligned} & \left(\left(|g'(\lambda)| \log \frac{e}{1 - |\phi(z)|^2} - 2 |\phi'(\lambda)| |g(\lambda)| \frac{|\phi(\lambda)|}{1 - |\phi(z)|^2} \right) \right. \\ & \quad \left. \times (C \|C_\phi^g\|_{\mathcal{X} \rightarrow \mathcal{X}^\varphi})^{-1} \right) \\ & \leq \varphi^{-1} \left(\frac{1}{1 - |\lambda|^2} \right), \\ & \frac{|g'(\lambda)| \left| \log(e/(1 - |\phi(\lambda)|^2)) \right|}{\varphi^{-1}(1/(1 - |\lambda|^2))} \\ & \quad - 2 \frac{|\phi'(\lambda)| |g(\lambda)| |\phi(\lambda)|}{\varphi^{-1}(1/(1 - |\lambda|^2))(1 - |\phi(\lambda)|^2)} \leq C \|C_\phi^g\|_{\mathcal{X} \rightarrow \mathcal{X}^\varphi}. \end{aligned} \quad (67)$$

So we obtain

$$\begin{aligned} & \frac{|g'(\lambda)| \left| \log(e/(1 - |\phi(\lambda)|^2)) \right|}{\varphi^{-1}(1/(1 - |\lambda|^2))} \\ & \leq C \|C_\phi^g\|_{\mathcal{X} \rightarrow \mathcal{X}^\varphi} + 2 \frac{|\phi'(\lambda)| |g(\lambda)| |\phi(\lambda)|}{\varphi^{-1}(1/(1 - |\lambda|^2))(1 - |\phi(\lambda)|^2)}. \end{aligned} \quad (68)$$

Set

$$h_a(z) = \frac{h(\bar{a}z)}{\bar{a}} \left(\log \frac{1}{1 - |a|^2} \right)^{-1} - \int_0^z \log \frac{1}{1 - \bar{a}w} dw, \quad (69)$$

for $a \in \mathbb{D}$, such that $|a| > 1/2$. Then,

$$\begin{aligned} h'_a(z) &= \left(\log \frac{1}{1 - \bar{a}z} \right)^2 \left(\log \frac{1}{1 - |a|^2} \right)^{-1} - \log \frac{e}{1 - \bar{a}z}, \\ h''_a(z) &= \frac{2\bar{a}}{1 - \bar{a}z} \left(\log \frac{1}{1 - \bar{a}z} \right) \left(\log \frac{1}{1 - |a|^2} \right)^{-1} - \frac{\bar{a}}{1 - \bar{a}z}. \end{aligned} \quad (70)$$

Similar to the case of f_a , we have $h_a \in \mathcal{X}$ and $M_1 = \sup_{1/2 < |a| < 1} \|h_a\|_{\mathcal{X}} < \infty$. From this and by the facts that $h'_a(a) = 0$ and $h''_a(a) = \bar{a}/(1 - |a|^2)$, it follows that

$$C \|C_\phi^g\|_{\mathcal{X} \rightarrow \mathcal{X}^\varphi} \geq \frac{|\phi'(\lambda)| |g(\lambda)| |\phi(\lambda)|}{\varphi^{-1}(1/(1 - |\lambda|^2))(1 - |\phi(\lambda)|^2)}. \quad (71)$$

From (71) we have

$$\begin{aligned} & \sup_{|\phi(\lambda)| > 1/2} \frac{|\phi'(\lambda)| |g(\lambda)| |\phi(\lambda)|}{\varphi^{-1}(1/(1 - |\lambda|^2))(1 - |\phi(\lambda)|^2)} \\ & < 2 \frac{|\phi'(\lambda)| |g(\lambda)| |\phi(\lambda)|}{\varphi^{-1}(1/(1 - |\lambda|^2))(1 - |\phi(\lambda)|^2)} \\ & \leq \sup_{|\phi(\lambda)| > 1/2} C \|C_\phi^g\|_{\mathcal{X} \rightarrow \mathcal{X}^\varphi} < \infty. \end{aligned} \quad (72)$$

By (65) we see that

$$\begin{aligned} & \sup_{|\phi(\lambda)| \leq 1/2} \frac{|\phi'(\lambda)| |g(\lambda)|}{\varphi^{-1}(1/(1-|\lambda|^2))(1-|\phi(\lambda)|^2)} \\ & \leq \sup_{|\phi(\lambda)| \leq 1/2} \frac{4}{3} \frac{|\phi'(\lambda)| |g(\lambda)|}{\varphi^{-1}(1/(1-|\lambda|^2))} < \infty. \end{aligned} \quad (73)$$

From (72) and (73) we obtain the first inequality of (57). Similarly, from (61) and (68) the second inequality in (57) follows as desired. $\square$

Theorem 7. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an $\mathcal{N}$ -function, $g \in H(\mathbb{D})$, and ϕ an analytic self-map of $\mathbb{D}$. Then $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is compact if and only if $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is bounded and

$$\begin{aligned} & \lim_{|\phi(z)| \rightarrow 1} \frac{|\phi'(z)| |g(z)|}{\varphi^{-1}(1/(1-|z|^2))(1-|\phi(z)|^2)} = 0, \\ & \lim_{|\phi(z)| \rightarrow 1} \frac{|g'(z)| \log(e/(1-|\phi(z)|^2))}{\varphi^{-1}(1/(1-|z|^2))} = 0. \end{aligned} \quad (74)$$

Proof. Suppose that $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is compact. It is clear that $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is bounded. Let $\{Z_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\phi(Z_n)| \rightarrow 1$ as $n \rightarrow \infty$ and $(f_n)_{n \in \mathbb{N}}$ defined by

$$\begin{aligned} f_n(z) &= \frac{\phi(z_n)z - 1}{\bar{\phi}(z_n)} \\ & \times \left[1 + \log \frac{e}{1 - \bar{\phi}(z_n)z} \right] \left(\log \frac{e}{1 - |\phi(z_n)|^2} \right)^{-1}. \end{aligned} \quad (75)$$

Then $\sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{X}} < \infty$, and $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. Since $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is compact, it gives $\lim_{n \rightarrow \infty} \|C_\phi^g f_n\|_{\mathcal{X}^\varphi} = 0$. Note that

$$\begin{aligned} f_n'(\phi(z_n)) &= \log \frac{e}{1 - |\phi(z_n)|^2}, \\ f_n''(\phi(z_n)) &= \frac{2\bar{\phi}(z_n)}{1 - |\phi(z_n)|^2}. \end{aligned} \quad (76)$$

We have

$$\begin{aligned} \|C_\phi^g f_n\|_{\mathcal{X}^\varphi} &\geq \frac{|\phi'(z_n)| |g(z_n)| |\phi(z_n)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} \\ & - \frac{|g'(z_n)| \log(e/(1-|\phi(z_n)|^2))}{\varphi^{-1}(1/(1-|z_n|^2))}, \end{aligned} \quad (77)$$

and, consequently,

$$\begin{aligned} & \lim_{|\phi(z_n)| \rightarrow 1} \frac{|\phi'(z_n)| |g(z_n)| |\phi(z_n)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} \\ & = \lim_{|\phi(z_n)| \rightarrow 1} \frac{|g'(z_n)| \log(e/(1-|\phi(z_n)|^2))}{\varphi^{-1}(1/(1-|z_n|^2))}. \end{aligned} \quad (78)$$

If one of these two limits exists, set

$$\begin{aligned} h_n(z) &= \frac{h(\bar{\phi}(z_n)z)}{\phi(z_n)} \left(\log \frac{e}{1 - |\phi(z_n)|^2} \right)^{-1} \\ & - \left(\log \frac{e}{1 - |\phi(z_n)|^2} \right)^{-2} \int_0^z \log^3 \frac{e}{1 - \bar{\phi}(z_n)w} dw; \end{aligned} \quad (79)$$

then $h_n'(\phi(z_n)) = 0$, $\sup_{n \in \mathbb{N}} \|h_n\|_{\mathcal{X}} \leq C$, and h_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. Since $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is compact, we have

$$\lim_{n \rightarrow \infty} \|C_\phi^g h_n\|_{\mathcal{X}^\varphi} = 0. \quad (80)$$

On the other hand,

$$\frac{|\phi'(z_n)| |g(z_n)| |\phi(z_n)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} \leq \|C_\phi^g h_n\|_{\mathcal{X}^\varphi}. \quad (81)$$

Hence

$$\lim_{n \rightarrow \infty} \frac{|\phi'(z_n)| |g(z_n)| |\phi(z_n)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} = 0. \quad (82)$$

Therefore

$$\begin{aligned} & \lim_{|\phi(z_n)| \rightarrow 1} \frac{|\phi'(z_n)| |g(z_n)| |\phi(z_n)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} \\ & = \lim_{k \rightarrow \infty} \frac{|\phi'(z_n)| |g(z_n)| |\phi(z_n)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} = 0. \end{aligned} \quad (83)$$

This together with (78) implies

$$\lim_{|\phi(z_n)| \rightarrow 1} \frac{|g'(z_n)| \log(e/(1-|\phi(z_n)|^2))}{\varphi^{-1}(1/(1-|z_n|^2))} = 0. \quad (84)$$

The implication follows from the last two equalities.

Conversely, assume that $C_\phi^g : \mathcal{X} \rightarrow \mathcal{X}^\varphi$ is bounded and (74) holds. From the proof of Theorem 6 we have

$$\begin{aligned} C_1 &= \sup_{z \in \mathbb{D}} \frac{|g'(z)|}{\varphi^{-1}(1/(1-|z|^2))} < \infty, \\ C_2 &= \sup_{z \in \mathbb{D}} \frac{|\phi'(z)| |g'(z)|}{\varphi^{-1}(1/(1-|z|^2))} < \infty. \end{aligned} \quad (85)$$

On the other hand, from (74), we have that, for every $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\begin{aligned} \frac{|\phi'(z)| |g(z)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} &< \varepsilon, \\ \frac{|g'(z)| \log(e/(1-|\phi(z)|^2))}{\varphi^{-1}(1/(1-|z_n|^2))} &< \varepsilon, \end{aligned} \quad (86)$$

when $\delta < |\phi(z)| < 1$.

Assume that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{F}$ such that $\sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{F}} \leq 1$ and f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. Let $k = \{z \in \mathbb{D} : |\phi(z)| \leq \delta\}$. Then by (85) and (86) it follows that

$$\begin{aligned} &\|C_\phi^g f_n\|_{\mathcal{F}^\varphi} \\ &= \sup_{z \in \mathbb{D}} \frac{|(C_\phi^g f_n)''(z)|}{\varphi^{-1}(1/(1-|z|^2))} \\ &= \sup_{z \in \mathbb{D}} \left(\frac{|f_n''(\phi(z))| |\phi'(z)| |g(z)| + |f_n'(\phi(z))| |g'(z)|}{\varphi^{-1}(1/(1-|z|^2))} \right) \\ &\leq \sup_{z \in K} \left(\frac{|f_n''(\phi(z))| |\phi'(z)| |g(z)| + |f_n'(\phi(z))| |g'(z)|}{\varphi^{-1}(1/(1-|z|^2))} \right) \\ &\quad + \sup_{z \in \mathbb{D} \setminus K} \left(\frac{|f_n''(\phi(z))| |\phi'(z)| |g(z)| + |f_n'(\phi(z))| |g'(z)|}{\varphi^{-1}(1/(1-|z|^2))} \right) \\ &\leq C_2 \sup_{z \in K} |f_n''(\phi(z))| + C_1 \sup_{z \in K} |f_n'(\phi(z))| \\ &\quad + C \sup_{z \in \mathbb{D} \setminus K} \frac{|\phi'(z)| |g(z)|}{\varphi^{-1}(1/(1-|z_n|^2))(1-|\phi(z_n)|^2)} \|f_n\|_{\mathcal{F}} \\ &\quad + C \sup_{z \in \mathbb{D} \setminus K} \frac{|g'(z)| \log(e/(1-|\phi(z)|^2))}{\varphi^{-1}(1/(1-|z_n|^2))} \|f_n\|_{\mathcal{F}} \\ &\leq C_2 \sup_{|w| \leq \delta} |f_n''(w)| + C_1 \sup_{|w| \leq \delta} |f_n'(w)| + 2C\varepsilon L. \end{aligned} \quad (87)$$

So we obtain

$$\begin{aligned} \|C_\phi^g f_n\|_{\mathcal{F}^\varphi} &= |f_n'(\phi(0))| |f(0)| + \sup_{z \in \mathbb{D}} \frac{|(C_\phi^g f_n)''(z)|}{\varphi^{-1}(1/(1-|z|^2))} \\ &\leq C_2 \sup_{|w| \leq \delta} |f_n''(w)| + C_1 \sup_{|w| \leq \delta} |f_n'(w)| + 2C\varepsilon \|f_n\|_{\mathcal{F}} \\ &\quad + |f_n'(\phi(0))| |g(0)|. \end{aligned} \quad (88)$$

By the Cauchy estimate, if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence converging to zero on compact subsets of $\mathbb{D}$, then the sequence $\{f_n\}_{n \in \mathbb{N}}$

and $\{f_n''\}_{n \in \mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. In particular, since K is compact it follows that $\lim_{n \rightarrow \infty} \sup_{w \in K} |f_n'(w)| = 0$ and $\lim_{n \rightarrow \infty} \sup_w |f_n''(w)| = 0$. Using these facts and letting $n \rightarrow \infty$ in the last inequality, we obtain that

$$\lim_{n \rightarrow \infty} \|C_\phi^g f_n\|_{\mathcal{F}^\varphi} \leq 2C\varepsilon L. \quad (89)$$

Since ε is an arbitrary positive number it follows that the last limit is equal to zero. Employing Lemma 3 the implication follows. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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