Research Article

On Convergence of Fixed Points in G-Complete Fuzzy Metric Spaces

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We provide some results on the convergence of the sequence of fixed points for some different sequences of contraction mappings or fuzzy metrics in G-complete fuzzy metric spaces with the H-type t -norms. We improve the corresponding conclusions in the literature.

1. Introduction

In 1940, Ulam [1] proposed the following stability problem: “When is it true that a function which satisfies some functional equation approximately must be close to one satisfying the equation exactly?” This problem posed by Ulam has stimulated a long lasting interest in a particular stability of various types of equations and inequalities. That issue is sometimes called Ulam’s type stability. In the recent past, several Ulam stability results concerning the various functional equations were determined in [2–4], respectively, in the fuzzy and intuitionistic fuzzy normed spaces.

The concept of fuzzy sets was introduced initially by Zadeh [5] in 1965. After that, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. In the theory of fuzzy topological spaces, one of the main problems is to obtain an appropriate and consistent notion of a fuzzy metric space. This problem was investigated by many authors from different points of view [6–16]. Kramosil and Michalek [7] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [17]. Later, George and Veeramani [18] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek.

These fuzzy metric spaces have been widely accepted as an appropriate notion of metric fuzziness in the sense that it provides rich topological structures which can be obtained,

in many cases, from classical theorems. Further, these fuzzy metric spaces have very important applications in studying fixed point theorems [17, 19–28]. In [29], Shen et al. have shown that the convergence of the sequence of fixed points to some sequences of contraction mappings or fuzzy metrics satisfies certain conditions in fuzzy metric spaces. However, they have given the most of results in [29] only for the two specific t -norms, minimum t -norm and product t -norm.

In this paper, we will provide some improved results on convergence of fixed points in G-complete fuzzy metric spaces. Specifically, after redefining some basic concepts of [29] in Section 2, we will generalize the results in [29] to the G-complete fuzzy metric spaces with an H-type t -norm in Section 3. Since the H-type t -norms are very important and widely used in fuzzy fixed point theory [17, 20, 24, 30, 31], our results improve the corresponding conclusions in the literature.

2. Preliminaries

For the sake of completeness, we briefly recall some notions from the theory of fuzzy metric spaces used in this paper. Let $\mathbb{N}$ denote the set of all positive integers.

Definition 1 (see [32]). A triangular norm (t -norm for short) is a binary operation on the unit interval $[0, 1]$, that is, a function $*$: $[0, 1]^2 \rightarrow [0, 1]$, such that for all $a, b, c, d \in [0, 1]$ the following four axioms are satisfied:

(T-1) $a * 1 = a$ (boundary condition).

(T-2) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ (monotonicity).

(T-3) $a * b = b * a$ (commutativity).

(T-4) $a * (b * c) = (a * b) * c$ (associativity).

A t -norm $*$ is said to be continuous if it is a continuous function in $[0, 1]^2$. Examples of t -norms are $a *_P b = ab$ and $a *_M b = \min\{a, b\}$.

Definition 2 (see [30]). A t -norm $*$ is of H-type if the family $(x^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$, where $x^{(n)}$ is defined by

$$x^{(1)} = x, \quad x^{(n)} = x^{(n-1)} * x, \quad n \geq 2, \quad x \in [0, 1]. \quad (1)$$

It is obvious that $*_M$ is an H-type t -norm. In fact, there are innumerable H-type t -norms [30].

Lemma 3 (see [31]). Let $*$ be a t -norm. If $*$ is continuous and of H-type, then there exists a strictly increasing sequence $(b_n)_{n \in \mathbb{N}}$ from the interval $[0, 1)$ such that $\lim_{n \rightarrow \infty} b_n = 1$ and $b_n * b_n = b_n$.

In [29], Shen et al. redefined the notion of fuzzy metric space by appending the following condition (FM-6) based on the one in the sense of George and Veeramani [18].

Definition 4 (see [33]). The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t -norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $t, s > 0$:

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,

(FM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,

(FM-6) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$.

Definition 5 (see [7, 29]). Let $(X, M, *)$ be a fuzzy metric space. Then

- (a) a sequence (x_n) in X is said to converge to x in X if and only if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ for all $t > 0$; that is, for each $r \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$;
- (b) a sequence (x_n) in X is a G-Cauchy sequence if and only if $\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, t) = 1$ for any $p \in \mathbb{N}$ and $t > 0$;
- (c) the fuzzy metric space $(X, M, *)$ is called G-complete if every G-Cauchy sequence is convergent.

Definition 6 (see [7]). Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \rightarrow X$ is called a contraction mapping if there exists $k \in (0, 1)$ such that

$$M(Tx, Ty, kt) \geq M(x, y, t) \quad (2)$$

for every $x, y \in X$ and $t > 0$.

Lemma 7. Let $(X, M, *)$ be a G-complete fuzzy metric space. If $T : X \rightarrow X$ is a contraction mapping, then T has a unique fixed point.

Definition 8. Let $(X, M, *)$ be a fuzzy metric space and let (T_n) be a sequence of self-mappings of X . $T_0 : X \rightarrow X$ is a given mapping. The sequence (T_n) is said to converge pointwise to T_0 if $T_n x$ converge to $T_0 x$ for each $x \in X$; that is, for any $x \in X$, $r \in (0, 1)$, and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$M(T_n x, T_0 x, t) > 1 - r \quad (3)$$

for all $n \geq n_0$.

Definition 9 (see [29]). Let $(X, M, *)$ be a fuzzy metric space and let (T_n) be a sequence of self-mappings of X . $T_0 : X \rightarrow X$ is a given mapping. The sequence (T_n) is said to converge uniformly to T_0 if for each $r \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$M(T_n x, T_0 x, t) > 1 - r \quad (4)$$

for all $n \geq n_0$ and $x \in X$.

Definition 10. Let $(X, M, *)$ be a fuzzy metric space. A sequence of self-mappings (T_n) is uniformly equicontinuous if for each $r \in (0, 1)$ and $t > 0$, there exists an $\varepsilon \in (0, 1)$ such that $M(T_n x, T_n y, t) > 1 - r$ whenever $M(x, y, t) > 1 - \varepsilon$ for all $x, y \in X$ and $n \in \mathbb{N}$.

Definition 11 (see [18]). Let $(X, M, *)$ be a fuzzy metric space. The open ball $B_M(x, r, t)$ and the closed ball $B_M[x, r, t]$ with center $x \in X$ and radius $r \in (0, 1)$ and $t > 0$, respectively, are defined as follows:

$$\begin{aligned} B_M(x, r, t) &= \{y \in X : M(x, y, t) > 1 - r\}, \\ B_M[x, r, t] &= \{y \in X : M(x, y, t) \geq 1 - r\}. \end{aligned} \quad (5)$$

Lemma 12 (see [18]). Every open (closed) ball is an open (closed) set.

Definition 13. A fuzzy metric space $(X, M, *)$ is a compact space if (X, τ_M) is a compact topological space, where τ_M is a topology induced by the fuzzy metric M .

Lemma 14 (see [33]). Every closed subset A of a compact fuzzy metric space $(X, M, *)$ is compact.

Definition 15 (see [33]). A fuzzy metric space $(X, M, *)$ in which every point has a compact neighborhood is called locally compact.

Definition 16 (see [34, 35]). Let $*$ be a t -norm and $a, b \in [0, 1]$. Define the pseudodifference $a \ominus b$ by

$$a \ominus b = \sup \{c : b * c \leq a\}. \quad (6)$$

Definition 17. Let $(X, M_0, *)$ be a fuzzy metric space and let (M_n) be a sequence of fuzzy metrics with respect to the same t -norm $*$ on X . The sequence (M_n) is said to upper

semiconverge uniformly to M_0 if for each $r \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} M_n(x, y, t) &\geq M_0(x, y, t), \\ M_0(x, y, t) \ominus M_n(x, y, t) &> 1 - r \end{aligned} \quad (7)$$

for all $n \geq n_0$, $x, y \in X$.

3. Main Results

Theorem 18. Let $(X, M, *)$ be a fuzzy metric space and let (T_n) be a sequence of self-mappings of X . $T_0 : X \rightarrow X$ is a contraction mapping; that is, there exists a $k \in (0, 1)$ such that $M(T_0x, T_0y, kt) \geq M(x, y, t)$ for every $x, y \in X$ and $t > 0$. A is a compact subset of X . If (T_n) converges pointwise to T_0 in A and it is a uniformly equicontinuous sequence, then the sequence (T_n) converges uniformly to T_0 in A .

Proof. For each $\bar{r} \in (0, 1)$, we may choose an appropriate $r \in (0, 1)$ such that

$$(1 - r) * (1 - r) * (1 - r) > 1 - \bar{r}. \quad (8)$$

At the same time, for any fixed $t > 0$, since (T_n) is uniformly equicontinuous, there exists $\varepsilon \in (0, r)$ such that

$$M\left(T_nx, T_ny, \frac{t}{2+k}\right) > 1 - r, \quad (9)$$

whenever $M(x, y, t/(2+k)) > 1 - \varepsilon$, for all $x, y \in X$ and $n \in \mathbb{N}$. Now define the open covering $\mathcal{C}$ of A as

$$\mathcal{C} = \left\{ B_M\left(x, \varepsilon, \frac{t}{2+k}\right) : x \in A \right\}. \quad (10)$$

Since A is compact, there exist $x_i \in A$, $i = 1, 2, \dots, m$ such that

$$A \subseteq \bigcup_{i=1}^m B_M\left(x_i, \varepsilon, \frac{t}{2+k}\right). \quad (11)$$

Since T_n converges pointwise to T_0 in A , for each x_i , there exists $n_i \in \mathbb{N}$ such that

$$M\left(T_nx_i, T_0x_i, \frac{t}{2+k}\right) > 1 - r \quad (12)$$

for all $n \geq n_i$. Set $l = \max\{n_i, i = 1, 2, \dots, m\}$. For any $x \in A$, there exists an x_{i_0} such that $x \in B(x_{i_0}, \varepsilon, t/(2+k))$. Now for all $n \geq l$, we have that

$$\begin{aligned} M(T_nx, T_0x, t) &\geq M\left(T_nx, T_nx_{i_0}, \frac{t}{2+k}\right) \\ &\quad * M\left(T_nx_{i_0}, T_0x_{i_0}, \frac{t}{2+k}\right) \\ &\quad * M\left(T_0x_{i_0}, T_0x, \frac{kt}{2+k}\right) \\ &\geq (1 - r) * (1 - r) * M\left(x_{i_0}, x, \frac{t}{2+k}\right) \\ &\geq (1 - r) * (1 - r) * (1 - \varepsilon) \\ &\geq (1 - r) * (1 - r) * (1 - r) \\ &\geq 1 - \bar{r}. \end{aligned} \quad (13)$$

By the arbitrariness of $\bar{r}$ and t , we get that the sequence (T_n) converges uniformly to T_0 in A . $\square$

Theorem 19. Let $(X, M, *)$ be a G -complete fuzzy metric space and let (T_n) be a sequence of self-mappings of X , where the t -norm $*$ is of H -type. $T_0 : X \rightarrow X$ is a contraction mapping; that is, there exists a $k \in (0, 1)$ such that $M(T_0x, T_0y, kt) \geq M(x, y, t)$ for every $x, y \in X$ and $t > 0$, and then T_0 has a unique fixed point $x_0 \in X$. If there exists at least one fixed point x_n for each T_n and the sequence (T_n) converges uniformly to T_0 , then x_n converges to x_0 .

Proof. Since $*$ is of H -type, by Lemma 3 there exists a strictly increasing sequence (r_i) from the interval $[0, 1)$ such that $\lim_{i \rightarrow \infty} r_i = 1$ and $r_i * r_i = r_i$.

Suppose that x_n does not converge to x_0 . Thus, without loss of generality, we can suppose that there exists $t_0 > 0$ and $1 - r' \in (r_i)$ such that

$$1 - r' > M(x_n, x_0, t_0) \quad (14)$$

for all $n \in \mathbb{N}$. Let α be a given number in $(k, 1)$. According to the condition (FM-6) of Definition 4, for any x_n , we can find a $p(n) \in \mathbb{N}$ such that

$$M\left(x_n, x_0, t_0 \left(\frac{\alpha}{k}\right)^{p(n)}\right) > 1 - r'. \quad (15)$$

In addition, since the sequence (T_n) converges uniformly to T_0 , there exists an $n_0 \in \mathbb{N}$ such that

$$M(T_nx, T_0x, (1 - \alpha)t_0) > 1 - r' \quad (16)$$

for all $x \in X$ and $n \geq n_0$. Now for $n \geq n_0$, we get that

$$\begin{aligned} 1 - r' &> M(x_n, x_0, t_0) \\ &= M(T_nx_n, T_0x_0, t_0) \\ &\geq M(T_nx_n, T_0x_n, (1 - \alpha)t_0) * M(T_0x_n, T_0x_0, \alpha t_0) \\ &\geq M(T_nx_n, T_0x_n, (1 - \alpha)t_0) * M\left(x_n, x_0, \frac{\alpha}{k}t_0\right) \\ &\geq M(T_nx_n, T_0x_n, (1 - \alpha)t_0) \\ &\quad * M\left(T_nx_n, T_0x_n, \frac{\alpha}{k}(1 - \alpha)t_0\right) \\ &\quad * M\left(x_n, x_0, \left(\frac{\alpha}{k}\right)^2 t_0\right) \\ &\geq M(T_nx_n, T_0x_n, (1 - \alpha)t_0) \\ &\quad * M\left(T_nx_n, T_0x_n, \frac{\alpha}{k}(1 - \alpha)t_0\right) \\ &\quad * \dots * M\left(T_nx_n, T_0x_n, \left(\frac{\alpha}{k}\right)^{p(n)-1} (1 - \alpha)t_0\right) \\ &\quad * M\left(x_n, x_0, \left(\frac{\alpha}{k}\right)^{p(n)} t_0\right). \end{aligned} \quad (17)$$

Since

$$\left(\frac{a}{k}\right)^i (1-a)t_0 \geq (1-a)t_0, \quad i = 1, 2, \dots, p(n)-1, \quad (18)$$

by the monotonicity of fuzzy metric M with respect to t , we have that

$$\begin{aligned} M\left(T_n x_n, T_0 x_n, \left(\frac{a}{k}\right)^i (1-a)t_0\right) \\ \geq M(T_n x_n, T_0 x_n, (1-a)t_0) \\ > 1-r', \quad i = 1, 2, \dots, p(n)-1. \end{aligned} \quad (19)$$

Thus from inequality (15) and inequality (17), we get

$$1-r' > (1-r') * (1-r') * \dots * (1-r'). \quad (20)$$

But since $*$ is of H-type and $1-r' \in (r_n)$, inequality (20) is a contradiction. Thence, we get that x_n converges to x_0 . $\square$

Theorem 20. Let $(X, M, *)$ be a G -complete fuzzy metric space. If $T : X \rightarrow X$ is a self-mapping of X such that the iteration mapping T^m is a contraction mapping for a certain positive integer m , then T has a unique fixed point.

Proof. Without loss of generality, we suppose $m > 1$. Since T^m is a contraction mapping, by Lemma 7, T^m has a unique fixed point x . Thus $T^m x = x$ and $T(T^m x) = Tx$ which implies $T^m(Tx) = Tx$. It follows that $Tx = x$; that is, x is a fixed point of T . If $Ty = y$ for some $y \in X$, then we can get that $T^m y = y$ which implies $x = y$. Thus T has a unique fixed point. $\square$

Theorem 21. Let $(X, M, *)$ be a G -complete fuzzy metric space and let (T_n) be a sequence of self-mappings of X , where the t -norm $*$ is of H-type. Suppose that $T_0 : X \rightarrow X$ is a self-mapping such that T_0^m is a contraction mapping for a certain positive integer m . If there exists at least one fixed point x_n for each T_n and the sequence (T_n) converges uniformly to T_0 , then x_n converges to x_0 .

Proof. It is a corollary of Theorems 19 and 20. $\square$

Theorem 22. Let $(X, M, *)$ be a locally compact fuzzy metric space and let (T_n) be a sequence of self-mappings of X , where the t -norm $*$ is of H-type. $T_0 : X \rightarrow X$ is a contraction mapping; that is, there exists a $k \in (0, 1)$ such that $M(T_0 x, T_0 y, kt) \geq M(x, y, t)$ for every $x, y \in X$ and $t > 0$. If the following conditions are satisfied:

- (i) T_n^m is a contraction mapping for a certain number $m = m(n)$,
- (ii) (T_n) converges pointwise to T_0 and it is a uniformly equicontinuous sequence,
- (iii) $T_n x_n = x_n$, $n = 0, 1, 2, \dots$,

then x_n converges to x_0 .

Proof. Since $*$ is of H-type, by Lemma 3 there exists a strictly increasing sequence (r_i) from the interval $[0, 1)$ such that

$\lim_{i \rightarrow \infty} r_i = 1$ and $r_i * r_i = r_i$. Since X is a locally compact space, for the given point x_0 , we can choose a $t_0 > 0$ and an r' such that $1-r' \in (r_i)$ and the closed ball $B_M[x_0, r', t_0]$ is a compact set of X . Since T_n is uniformly equicontinuous and pointwise convergent on $B_M[x_0, r', t_0]$, by Theorem 18, T_n converges uniformly to T_0 on $B_M[x_0, r', t_0]$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$M(T_n x, T_0 x, (1-k)t_0) > 1-r' \quad (21)$$

for all $x \in B_M[x_0, r', t_0]$ and $n \geq n_0$. Consequently, for all $x \in B_M[x_0, r', t_0]$ and $n \geq n_0$, we have that

$$\begin{aligned} M(T_n x, x_0, t_0) \\ = M(T_n x, T_0 x_0, t_0) \\ \geq M(T_n x, T_0 x, (1-k)t_0) * M(T_0 x, T_0 x_0, kt_0) \\ \geq M(T_n x, T_0 x, (1-k)t_0) * M(x, x_0, t_0) \\ \geq 1-r' * 1-r' \\ = 1-r'. \end{aligned} \quad (22)$$

Then we have $T_n x \in B_M[x_0, r', t_0]$, and then by inductive reasoning $T_n^i x \in B_M[x_0, r', t_0]$ for all $i \in \mathbb{N}$ whenever $x \in B_M[x_0, r', t_0]$ and $n \geq n_0$. Especially, $B_M[x_0, r', t_0]$ is an invariant set for $T_n^{m(n)}$ whenever $n \geq n_0$. It follows that the fixed point x_n of T_n is contained in the set $B_M[x_0, r', t_0]$ whenever $n \geq n_0$. Now by Theorem 19, we get that x_n converges to x_0 . $\square$

Theorem 23. Let $(X, M_0, *)$ be a G -complete fuzzy metric space and let A be a compact subset of X . If (M_n) is a sequence of fuzzy metrics of X with respect to $*$ and (T_n) is a sequence of self-mappings of X satisfying the following conditions:

- (i) (M_n) upper semiconverges uniformly to M_0 ,
- (ii) T_n is a contraction mapping with respect to the fuzzy metric M_n for $n = 0, 1, 2, \dots$,
- (iii) (T_n) converges pointwise to T_0 with respect to M_0 ,

then T_n converges uniformly to T_0 in A with respect to the fuzzy metric M_0 .

Proof. For each $r \in (0, 1)$, we can choose an $\varepsilon \in (0, 1)$ such that

$$(1-\varepsilon) * (1-\varepsilon) > 1-r. \quad (23)$$

Since M_n upper semiconverges uniformly to M_0 , for $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n(\varepsilon, t)$ such that

$$\begin{aligned} M_n(x, y, t) &\geq M_0(x, y, t), \\ M_0(x, y, t) \ominus M_n(x, y, t) &> 1-\varepsilon \end{aligned} \quad (24)$$

for all $x, y \in X$ and $n \geq n(\varepsilon, t)$. It should be noted that

$$M_0(x, y, t) \geq (M_0(x, y, t) \ominus M_n(x, y, t)) * M_n(x, y, t) \quad (25)$$

because of the continuity of $*$.

Now for any $x, y \in X$ satisfying $M_0(x, y, t) > 1 - \varepsilon$ and $n \geq n(\varepsilon, t)$, we get that

$$\begin{aligned} M_0(T_n x, T_n y, t) &\geq (M_0(T_n x, T_n y, t) \ominus M_n(T_n x, T_n y, t)) \\ &\quad * M_n(T_n x, T_n y, t) \\ &\geq (1 - \varepsilon) * M_n(T_n x, T_n y, t) \\ &\geq (1 - \varepsilon) * M_n\left(x, y, \frac{t}{k_n}\right), \end{aligned} \quad (26)$$

where $k_n \in (0, 1)$ is the constant in Definition 6 for T_n . By the monotonicity of fuzzy metric M with respect to t , we have that

$$M_n\left(x, y, \frac{t}{k_n}\right) \geq M_n(x, y, t) > M_0(x, y, t) > 1 - \varepsilon. \quad (27)$$

Now from inequality (26), for $n \geq n(\varepsilon, t)$, we get that

$$M_0(T_n x, T_n y, t) \geq (1 - \varepsilon) * (1 - \varepsilon) > 1 - r, \quad (28)$$

whenever $M_0(x, y, t) > 1 - \varepsilon$. Thus T_n is uniformly equicontinuous in X with respect to the fuzzy metric M_0 . Since A is a compact subset of X and T_n converges pointwise to T_0 with respect to M_0 , it follows from Theorem 18 that T_n converges uniformly to T_0 in A with respect to the fuzzy metric M_0 . $\square$

Theorem 24. Let $(X, M_0, *)$ be a locally compact fuzzy metric space where the t -norm $*$ is of H-type. $T_0 : X \rightarrow X$ is a contraction mapping; that is, there exists a $k \in (0, 1)$ such that $M_0(T_0 x, T_0 y, kt) \geq M_0(x, y, t)$ for every $x, y \in X$ and $t > 0$. If (M_n) is a sequence of fuzzy metrics of X with respect to $*$ and (T_n) is a sequence of self-mappings of X satisfying the following conditions:

- (i) (M_n) upper semiconverges uniformly to M_0 ;
- (ii) T_n is a contraction mapping with respect to the fuzzy metric M_n and $T_n x_n = x_n$ for $n = 0, 1, 2, \dots$;
- (iii) (T_n) converges pointwise to T_0 with respect to M_0 .

Then the sequence of fixed points (x_n) converges to x_0 .

Proof. Since $*$ is of H-type, by Lemma 3 there exists a strictly increasing sequence (r_i) from the interval $[0, 1)$ such that $\lim_{i \rightarrow \infty} r_i = 1$ and $r_i * r_i = r_i$. Since X is a locally compact space, for the given point x_0 , we can choose a $t_0 > 0$ and an r' such that $1 - r' \in (r_i)$ and the closed ball $B_{M_0}[x_0, r', t_0]$ is a compact set of X .

It follows from Theorem 23 that T_n converges uniformly to T_0 in $B_M[x_0, r', t_0]$ with respect to the fuzzy metric M_0 . Thus there exists $n_0 \in \mathbb{N}$ such that

$$M_0(T_n x, T_0 x, (1 - k)t_0) > 1 - r' \quad (29)$$

for all $x \in B_M[x_0, r', t_0]$ and $n \geq n_0$.

Meantime, since M_n upper semiconverges uniformly to M_0 , for $r' \in (0, 1)$ and $t_0 > 0$, there exists n_1 such that

$$\begin{aligned} M_n(x, y, t_0) &\geq M_0(x, y, t_0), \\ M_0(x, y, t_0) \ominus M_n(x, y, t_0) &> 1 - r' \end{aligned} \quad (30)$$

for all $x, y \in X$ and $n \geq n_1$. Let $n' = \max\{n_0, n_1\}$. If $n \geq n'$ and $x \in B_{M_n}[x_0, r', t_0]$, then we have that $M_n(x, x_0, t_0) \geq 1 - r'$ and $M_0(x, x_0, t) \ominus M_n(x, x_0, t) > 1 - r'$. This implies that $M_0(x, x_0, t) \geq 1 - r'$ because $*$ is of H-type and

$$\begin{aligned} M_0(x, x_0, t_0) &\geq (M_0(x, x_0, t_0) \ominus M_n(x, x_0, t_0)) \\ &\quad * M_n(x, x_0, t_0). \end{aligned} \quad (31)$$

Thus, we have

$$B_{M_n}[x_0, r', t_0] \subseteq B_{M_0}[x_0, r', t_0], \quad (32)$$

for all $n \geq n'$. But for $n \geq n'$, it is easy to see that

$$B_{M_n}[x_0, r', t_0] \supseteq B_{M_0}[x_0, r', t_0], \quad (33)$$

because

$$M_n(x, y, t_0) \geq M_0(x, y, t_0) \geq 1 - r'. \quad (34)$$

Therefore, we get $B_{M_n}[x_0, r', t_0] = B_{M_0}[x_0, r', t_0]$, for all $n \geq n'$. Now for $x \in B_{M_0}[x_0, r', t_0]$ and $n \geq n'$, we have

$$\begin{aligned} M_n(T_n x, x_0, t_0) &\geq M_0(T_n x, x_0, t_0) \\ &= M_0(T_n x, T_0 x_0, t_0) \\ &\geq M_0(T_n x, T_0 x, (1 - k)t_0) * M_0(T_0 x, T_0 x_0, kt_0) \\ &\geq M_0(T_n x, T_0 x, (1 - k)t_0) * M_0(x, x_0, t_0) \\ &\geq 1 - r' * 1 - r' \\ &= 1 - r'. \end{aligned} \quad (35)$$

Then we have $T_n x \in B_{M_n}[x_0, r', t_0] = B_{M_0}[x_0, r', t_0]$, which implies that $B_{M_0}[x_0, r', t_0]$ is an invariant set in X with respect to M_n for $n \geq n'$. It follows that the fixed point x_n of T_n is contained in the set $B_{M_0}[x_0, r', t_0]$ whenever $n \geq n'$. Now by Theorem 19, we get that x_n converges to x_0 . $\square$

Theorem 25. Let $(X, M, *)$ be a compact fuzzy metric space where the t -norm $*$ is of H-type. (M_n) is a sequence of fuzzy metrics of X with respect to $*$ and (T_n) is a sequence of self-mappings of X satisfying the following conditions:

- (i) (M_n) upper semiconverges uniformly to M_0 ;
- (ii) T_n is a contraction mapping with respect to the fuzzy metric M_n for $n = 0, 1, 2, \dots$;
- (iii) (T_n) converges pointwise to T_0 with respect to M_0 .

If x_n is the fixed point of T_n and there is a subsequence (x_{n_k}) of (x_n) which converges to $x_0 \in X$, then $T_0 x_0 = x_0$.

Proof. Since $*$ is of H-type, by Lemma 3, there exists a strictly increasing sequence (r_i) from the interval $[0, 1)$ such that $\lim_{i \rightarrow \infty} r_i = 1$ and $r_i * r_i = r_i$.

Let K be the closure of the set $\{x_n : n \in \mathbb{N}\}$. By Lemma 14, we can easily know that K is a compact set. According to Theorem 23, it follows that the subsequence (T_{n_k}) converges uniformly to T_0 in K with respect to M_0 ; that is, for each $r \in (r_i)$ and $t > 0$, there exists an $k_0 \in \mathbb{N}$ such that $M(T_{n_k}x, T_0x, t/2) > r$ for all $x \in K$ and $k \geq k_0$. Now for $k \geq k_0$, we have that

$$\begin{aligned} M(T_{n_k}x_{n_k}, T_0x_0, t) &\geq M\left(T_{n_k}x_{n_k}, T_0x_{n_k}, \frac{t}{2}\right) \\ &\quad * M\left(T_0x_{n_k}, T_0x_0, \frac{t}{2}\right) \\ &\geq r * M\left(x_{n_k}, x_0, \frac{t}{2k_0}\right) \\ &\geq r * r \\ &= r, \end{aligned} \quad (36)$$

where $k_0 \in (0, 1)$ is the constant in Definition 6 for T_0 . Thus $T_{n_k}x_{n_k} = x_{n_k}$ converges to T_0x_0 . Then $T_0x_0 = x_0$. $\square$

Example 26. Let $(1 - 1/2^n)_{n \in \mathbb{N}}$ be a strictly increasing sequence from the interval $[0, 1)$, and let $(\alpha_n, \beta_n) = (1 - 1/2^n, 1 - 1/2^{n+1})$, for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} 1 - 1/2^n = 1$, by Lemma 3 in [36], the ordinal sum $*$ = $(\langle(\alpha_n, \beta_n), *_p\rangle)_{n \in \mathbb{N}}$ is an H-type t -norm. Let $X = [-1/2, 1/2]$. Define a fuzzy set M on $X^2 \times (0, \infty)$ by

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad (37)$$

for all $x, y \in X$ and $t > 0$. Then we can get that $(X, M, *)$ is a G-complete fuzzy metric space. Let T_0 be a mapping defined by $T_0(x) = x^2/4$ for all $x \in X$, and let (T_n) be a sequence of self-mappings of X defined by

$$T_n(x) = \left(\frac{1}{4} + \frac{1}{4^n}\right)x^2 - \frac{1}{4^n}, \quad (38)$$

for all $x \in X$ and $n \in \mathbb{N}$. By some simple calculations, we can get that T_0 is a contraction mapping and (T_n) converges uniformly to T_0 . Thus by Theorem 19, we obtain that the sequence (x_n) of fixed point of (T_n) converges to the fixed point x_0 of T_0 . In fact, we have that $x_0 = 0$ and

$$x_n = \frac{1 - \sqrt{1 + (1/4^n)(1 + (1/4^{n-1}))}}{(1/2) + (1/2^{2n-1})}, \quad (39)$$

for all $n \in \mathbb{N}$.

4. Conclusions

In this paper, we provide some results on the convergence of the sequence of fixed points for some different sequences of

contraction mappings or fuzzy metrics in G-complete fuzzy metric spaces with the H-type t -norms. We improve the corresponding conclusions in the literature. The first solution to the Ulam problem was obtained through a classical approach known from the fixed point theory, by iteration of a simple operator. Later it has been shown that some fixed point theorems can be directly applied in investigations of Ulam's type stability. Thus we hope our results would provide a background to ongoing work in the problems of those related fields.

Conflict of Interests

The authors have no conflicts of interests regarding this paper.

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References

- [1] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience, New York, NY, USA, 1960.
- [2] A. S. Al-Fhaid and S. A. Mohiuddine, "On the Ulam stability of mixed type QA mappings in IFN-spaces," *Advances in Difference Equations*, vol. 2013, article 203, 2013.
- [3] M. Eshaghi Gordji, H. Azadi Kenary, H. Rezaei, Y. W. Lee, and G. H. Kim, "Solution and Hyers-Ulam-Rassias stability of generalized mixed type additive-quadratic functional equations in fuzzy Banach spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 953938, 22 pages, 2012.
- [4] Z. Wang and W. Zhang, "Fuzzy stability of the pexiderized quadratic functional equation: a fixed point approach," *Fixed Point Theory and Applications*, vol. 2009, Article ID 460912, 10 pages, 2009.
- [5] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.
- [6] B. Ghil and Y. Kim, "Continuity of functions defined on the space of fuzzy sets," *Information Sciences*, vol. 157, pp. 155–165, 2003.
- [7] I. Kramosil and J. Michalek, "Fuzzy metrics and statistical metric spaces," *Kybernetika*, vol. 11, no. 5, pp. 326–334, 1975.
- [8] V. Gregori and S. Romaguera, "Some properties of fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 485–489, 2000.
- [9] H. Huang and C. Wu, "On the triangle inequalities in fuzzy metric spaces," *Information Sciences*, vol. 177, no. 4, pp. 1063–1072, 2007.
- [10] O. Kaleva and S. Seikkala, "On fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 12, no. 3, pp. 215–229, 1984.
- [11] M. L. Puri and D. A. Ralescu, "Fuzzy random variables," *Journal of Mathematical Analysis and Applications*, vol. 114, no. 2, pp. 409–422, 1986.
- [12] D. Qiu, C. Lu, W. Zhang, and Y. Mo, "On common fixed point theorems in the stationary fuzzy metric space of the bounded

- closed sets," *Abstract and Applied Analysis*, vol. 2013, Article ID 951259, 11 pages, 2013.
- [13] D. Qiu and W. Zhang, "The strongest t-norm for fuzzy metric spaces," *Kybernetika*, vol. 49, no. 1, pp. 141–148, 2013.
- [14] D. Qiu, W. Zhang, and C. Li, "Extension of a class of decomposable measures using fuzzy pseudometrics," *Fuzzy Sets and Systems*, vol. 222, pp. 33–44, 2013.
- [15] D. Qiu, W. Zhang, and C. Li, "On decomposable measures constructed by using stationary fuzzy pseudo-ultrametrics," *International Journal of General Systems*, vol. 42, no. 4, pp. 395–404, 2013.
- [16] R. Saadati, "On the L -fuzzy topological spaces," *Chaos, Solitons & Fractals*, vol. 37, no. 5, pp. 1419–1426, 2008.
- [17] D. Miheţ, "Fuzzy Ψ -contractive mappings in non-Archimedean fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 739–744, 2008.
- [18] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994.
- [19] H. M. Abu-Donia, "Common fixed point theorems for fuzzy mappings in metric space under ϕ -contraction condition," *Chaos, Solitons & Fractals*, vol. 34, no. 2, pp. 538–543, 2007.
- [20] I. Altun and D. Miheţ, "Ordered non-archimedean fuzzy metric spaces and some fixed point results," *Fixed Point Theory and Applications*, vol. 2010, Article ID 782680, 11 pages, 2010.
- [21] T. Bag and S. Samanta, "Fixed point theorems on fuzzy normed linear spaces," *Information Sciences*, vol. 176, no. 19, pp. 2910–2931, 2006.
- [22] T. Bag and S. K. Samanta, "Some fixed point theorems in fuzzy normed linear spaces," *Information Sciences*, vol. 177, no. 16, pp. 3271–3289, 2007.
- [23] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 90, no. 3, pp. 365–368, 1997.
- [24] D. Miheţ, "On fuzzy ε -contractive mappings in fuzzy metric spaces," *Fixed Point Theory and Applications*, vol. 2007, Article ID 87471, 5 pages, 2007.
- [25] D. Qiu, L. Shu, and J. Guan, "Common fixed point theorems for fuzzy mappings under ϕ -contraction condition," *Chaos, Solitons and Fractals*, vol. 41, no. 1, pp. 360–367, 2009.
- [26] A. Razani, "A contraction theorem in fuzzy metric spaces," *Fixed Point Theory and Applications*, vol. 3, pp. 257–265, 2005.
- [27] X. Zhang, "Common fixed point theorems for some new generalized contractive type mappings," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 2, pp. 780–786, 2007.
- [28] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988.
- [29] R. Vasuki and P. Veeramani, "Fixed point theorems and Cauchy sequences in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 135, no. 3, pp. 415–417, 2003.
- [30] E. Pap, O. Hadžić, and R. Mesiar, "A fixed point theorem in probabilistic metric spaces and an application," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 2, pp. 433–449, 1996.
- [31] V. Radu, *Lectures on Probabilistic Analysis*, vol. 2 of *Surveys, Lectures Notes and Monographs Series on Probability, Statistics & Applied Mathematics*, Universitatea de Vest din Timişoara, 1994.
- [32] E. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
- [33] Y. Shen, D. Qiu, and W. Chen, "On convergence of fixed points in fuzzy metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 135202, 6 pages, 2013.
- [34] D. Qiu, W. Zhang, and C. Li, "On nonlinear functional spaces based on triangular conorms," *Journal of Mathematical Analysis and Applications*, vol. 397, no. 2, pp. 671–677, 2013.
- [35] S. Weber, " $\perp$ -decomposable measures and integrals for Archimedean t -conorms $\perp$," *Journal of Mathematical Analysis and Applications*, vol. 101, no. 1, pp. 114–138, 1984.
- [36] O. Hadzic and E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.