Research Article

Compactness of the Commutator of Multilinear Fourier Multiplier Operator on Weighted Lebesgue Space

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Let T_σ be the multilinear Fourier multiplier operator associated with multiplier σ satisfying the Sobolev regularity that $\sup_{\ell \in \mathbb{Z}} \|\sigma_\ell\|_{W^{s_1, \dots, s_m}(\mathbb{R}^{mm})} < \infty$ for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$). The authors prove that if $b_1, \dots, b_m \in \text{BMO}(\mathbb{R}^n)$ and $\vec{w} \in \prod_{k=1}^m A_{p_k/t_k}(t_k = n/s_k)$, then the commutator $T_{\sigma, \Sigma b}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \dots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$. Moreover, the authors also prove that if $b_1, \dots, b_m \in \text{VMO}(\mathbb{R}^n)$ and $\vec{w} \in \prod_{k=1}^m A_{p_k/t_k}(t_k = n/s_k)$, then the commutator $T_{\sigma, \Sigma b}$ is compact operator from $L^{p_1}(\mathbb{R}^n, w_1) \times \dots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

1. Introduction

The study of the multilinear Fourier multiplier operator was originated by Coifman and Meyer in their celebrated work [1, 2]. Let $\sigma \in L^\infty(\mathbb{R}^{mm})$; the multilinear Fourier multiplier operator T_σ is defined by

$$T_\sigma(\vec{f})(x) := \int_{\mathbb{R}^{mm}} \exp(2\pi i x(\xi_\ell + \dots + \xi_m)) \times \sigma(\xi_\ell, \dots, \xi_m) \hat{f}_1(\xi_\ell) \dots \hat{f}_m(\xi_m) d\vec{\xi} \quad (1)$$

for all $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^n)$, where $d\vec{\xi} = d\xi_1 \dots d\xi_m$ and $\hat{f}$ is the Fourier transform of f . Coifman and Meyer [2] proved that if $\sigma \in C^s(\mathbb{R}^{mm} \setminus \{0\})$ satisfies

$$\begin{aligned} & |\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_m}^{\alpha_m} \sigma(\xi_1, \dots, \xi_m)| \\ & \leq C_{\alpha_1, \dots, \alpha_m} (|\xi_1| + \dots + |\xi_m|)^{-(|\alpha_1| + \dots + |\alpha_m|)} \end{aligned} \quad (2)$$

for all $|\alpha_1| + \dots + |\alpha_m| \leq s$ with $s \geq 2mn + 1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p$, $p_1, \dots, p_m < \infty$ with $1/p = \sum_{k=1}^m 1/p_k$. For the case of $s \geq mn + 1$, Kenig and Stein [3] and Grafakos and Torres [4] improved Coifman and Meyer's multiplier theorem to the indices $1/m \leq p \leq 1$ by the multilinear Calderón-Zygmund

operator theory. In the last several years, considerable attention has been paid to the behavior on function spaces for T_σ when the multiplier satisfies certain Sobolev regularity condition. Let $\Phi \in \mathcal{S}(\mathbb{R}^{mm})$ satisfy

$$\text{supp } \Phi \subset \left\{ (\xi_1, \dots, \xi_m) : \frac{1}{2} \leq \sum_{k=1}^m |\xi_k| \leq 2 \right\}, \quad (3)$$

$$\sum_{\ell \in \mathbb{Z}} \Phi(2^\ell \xi_1, \dots, 2^\ell \xi_m) = 1 \quad \forall (\xi_1, \dots, \xi_m) \in \mathbb{R}^{mm} \setminus \{0\}.$$

For $\ell \in \mathbb{Z}$, set

$$\sigma_\ell(\xi_1, \dots, \xi_m) := \sigma(2^{-\ell} \xi_1, \dots, 2^{-\ell} \xi_m) \Phi(\xi_1, \dots, \xi_m),$$

$$\begin{aligned} \|\sigma_\ell\|_{W^s(\mathbb{R}^{mm})} &:= \left(\int_{\mathbb{R}^{2n}} (1 + |\xi_1|^2 + \dots + |\xi_m|^2)^s \right. \\ & \quad \times |\hat{\sigma}(\xi_1, \dots, \xi_m)|^2 d\vec{\xi} \Big)^{1/2}. \end{aligned} \quad (4)$$

Tomita [5] proved that if

$$\sup_{\ell \in \mathbb{Z}} \|\sigma_\ell\|_{W^s(\mathbb{R}^{mm})} < \infty \quad (5)$$

for some $s \in (mn/2, \infty)$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $1 < p$, $p_1, \dots, p_m < \infty$

and $1/p = \sum_{k=1}^m 1/p_k$. Grafakos and Si [6] considered the mapping properties from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when $p \leq 1$. Let σ satisfy the Sobolev regularity that

$$\|\sigma_\ell\|_{W^{s_1, \dots, s_m}(\mathbb{R}^{mn})} := \left(\int_{\mathbb{R}^{mn}} \langle \xi_1 \rangle^{2s_1} \cdots \langle \xi_m \rangle^{2s_m} \times |\hat{\sigma}_\ell(\xi_1, \dots, \xi_m)|^2 d\vec{\xi} \right)^{1/2}, \quad (6)$$

where $\langle \xi_k \rangle := (1 + |\xi_k|^2)^{1/2}$. Miyachi and Tomita [7] proved that if

$$\sup_{\ell \in \mathbb{Z}} \|\sigma_\ell\|_{W^{s_1, \dots, s_m}(\mathbb{R}^{mn})} < \infty \quad (7)$$

for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$), then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $1 < p_1, \dots, p_m < \infty$ with $1/p = \sum_{k=1}^m 1/p_k$.

As well known, when σ satisfies (3) for some $s \geq mn + 1$, then T_σ is a standard multilinear Calderón-Zygmund operator, and then by the weighted estimates with multiple weights for multilinear Calderón-Zygmund operators, which were estimated by Lerner et al. [8], we know that, for any $p_1, \dots, p_m \in [1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$ and weights $w_1, \dots, w_m$ such that $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{p}}(\mathbb{R}^{mn})$,

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^{p, \infty}(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim \prod_{k=1}^m \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)}. \quad (8)$$

By a suitable kernel estimate and the theory of multilinear singular integral operator, Bui and Duong [9] established the weighted estimates with multiple weights for T_σ when σ satisfies (3) for $m = 2$ and $s \in (n, 2n]$. Hu and Yi [10] considered the behavior on $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ for $T_{\sigma, \Sigma b}$ when σ satisfies (5) for $s_1, \dots, s_m \in (n/2, n]$ and showed that $T_{\sigma, \Sigma b}$ enjoys the same $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ mapping properties as that of the operator T_σ .

Now, considerable attention has been paid to the behavior on the compactness of multilinear Fourier multipliers operator with Sobolev regularity. Let $VMO(\mathbb{R}^n)$ be the closure of C_c^∞ in the $BMO(\mathbb{R}^n)$ topology, which coincides with the space of functions of vanishing mean oscillation (see [11, 12]). Bényi and Torres [13] proved that if $b_1, \dots, b_m \in VMO(\mathbb{R}^n)$ and T is multilinear Calderón-Zygmund operator, then, for $p_1, \dots, p_m \in (1, \infty)$, $p \in (1, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, the commutator $T_{\sigma, \Sigma b}$ is compact operator from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Hu [14] proved that if σ is a multilinear multiplier which satisfies (5) for some $s \in (mn/2, mn]$, $t_1, \dots, t_m \in [1, 2]$, $t_k = n/s_k$, $b_1, \dots, b_m \in VMO(\mathbb{R}^n)$, and $p_k \in (t_k, \infty)$ for $k = 1, \dots, m$ and $p \in (1, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, then $T_{\sigma, \Sigma b}$ is compact operators from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Bényi et al. [15] proved that if $\vec{b} \in VMO(\mathbb{R}^n) \times \cdots \times VMO(\mathbb{R}^n)$ and T is multilinear Calderón-Zygmund operator, $\vec{w} \in A_p \times \cdots \times A_p$, then, for $p_1, \dots, p_m \in (1, \infty)$, $p \in (1, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, the commutator $T_{\sigma, \Sigma b}$ is compact operator from $L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

Inspired by the above, we consider the weighted compactness of the commutator $T_{\sigma, \Sigma b}$ of the multilinear Fourier multiplier operator on $L^p(\mathbb{R}^n)$.

Given a multilinear Fourier multiplier operator T_σ , $b_1, \dots, b_m \in BMO(\mathbb{R}^n)$ and $\vec{b} = (b_1, \dots, b_m)$, the commutator $T_{\sigma, \Sigma b}(\vec{f})(x)$ is defined by

$$T_{\sigma, \Sigma b}(\vec{f})(x) := \sum_{k=1}^m [b_k, T_\sigma]_k(f_1, \dots, f_m)(x), \quad (9)$$

with

$$\begin{aligned} & [b_k, T_\sigma]_k(f_1, \dots, f_k, \dots, f_m)(x) \\ &:= b_k(x) T_\sigma(f_1, \dots, f_k, \dots, f_m)(x) \\ &\quad - T_\sigma(f_1, \dots, b_k f_k, \dots, f_m)(x). \end{aligned} \quad (10)$$

Our main results are stated as follows.

Theorem 1. Suppose that σ be a multilinear multiplier which satisfies (7) for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$) and $t_1, \dots, t_m \in [1, 2]$. Let $t_k = n/s_k$, $p_k \in (t_k, \infty)$ for $k = 1, \dots, m$ and $1/p = \sum_{k=1}^m 1/p_k$ with $p > 1$. If the weights $w_1, \dots, w_m$ satisfy $\vec{w} \in \prod_{k=1}^m A_{p_k/t_k}$, then, for any $b_1, \dots, b_m \in BMO(\mathbb{R}^n)$, $T_{\sigma, \Sigma b}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

Theorem 2. Suppose that σ be a multilinear multiplier which satisfies (7) for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$) and $t_1, \dots, t_m \in [1, 2]$. Let $t_k = n/s_k$, $p_k \in (t_k, \infty)$ for $k = 1, \dots, m$ and $1/p = \sum_{k=1}^m 1/p_k$ with $p > 1$. If the weights $w_1, \dots, w_m$ satisfy $\vec{w} \in \prod_{k=1}^m A_{p_k/t_k}$, then, for any $b_1, \dots, b_m \in VMO(\mathbb{R}^n)$, $T_{\sigma, \Sigma b}$ is a compact operator from $L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

Because the regularity condition $\|\sigma_\ell\|_{W^s(\mathbb{R}^{mn})} < \infty$ is stronger than $\|\sigma_\ell\|_{W^{s_1, \dots, s_m}(\mathbb{R}^{mn})} < \infty$, we have the following corollaries.

Corollary 3. Suppose that σ be a multilinear multiplier which satisfies (5) for some $s \in (mn/2, mn]$. Let $r = mn/s$, $p_k \in (mn/s, \infty)$ for $k = 1, \dots, m$ and $1/p = \sum_{k=1}^m 1/p_k$ with $p > 1$. If the weights $w_1, \dots, w_m$ satisfy $\vec{w} \in \prod_{k=1}^m A_{p_k/r}$, then, for any $b_1, \dots, b_m \in BMO(\mathbb{R}^n)$, $T_{\sigma, \Sigma b}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

Corollary 4. Suppose that σ be a multilinear multiplier which satisfies (5) for some $s \in (mn/2, mn]$. Let $r = mn/s$, $p_k \in (mn/s, \infty)$ for $k = 1, \dots, m$ and $1/p = \sum_{k=1}^m 1/p_k$ with $p > 1$. If the weights $w_1, \dots, w_m$ satisfy $\vec{w} \in \prod_{k=1}^m A_{p_k/r}$, then, for any $b_1, \dots, b_m \in VMO(\mathbb{R}^n)$, $T_{\sigma, \Sigma b}$ is a compact operator from $L^{p_1}(\mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, w_m)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

The paper is organized as follows. In Section 2, we give some necessary notion and lemmas. In Section 3, we prove our main results, Theorems 1 and 2. Throughout the paper, C always denotes a positive constant that may vary from line to line but remains independent of the main variables. We use the symbol $A \lesssim B$ to indicate that there exists a positive constant C such that $A \leq CB$. We use $B(x, R)$ to denote a ball

centered at x with radius R . For a ball $B \subset \mathbb{R}^n$ and $\lambda > 0$, we use λB to denote the ball concentric with B whose radius is λ times of B 's. As usual, $|E|$ denotes the Lebesgue measure of a measurable set E in $\mathbb{R}^n$ and χ_E denotes the *characteristic function* of E . For $p \geq 1$, we denote by $p' = p/(p-1)$ the dual exponent of p .

2. Some Notations and Lemmas

Let us first introduce some definitions below.

Definition 5. Let $m \geq 1$ be an integer, and let $w_1, \dots, w_m$ be weights, $p_1, \dots, p_m, p \in (0, \infty)$, with $1/p = \sum_{k=1}^m 1/p_k$, $r_k \in (0, p_k]$ ($1 \geq k \geq m$). Set $\vec{w} = (w_1, \dots, w_m)$ and $v_{\vec{w}} := \prod_{k=1}^m w_k^{p/p_k}$. Then

$$\begin{aligned} \vec{w} &\in A_{p_1/r_1} \times \dots \times A_{p_m/r_m} \\ \implies &\begin{cases} \vartheta_k := w_k^{1-(p_k/r_k)'} \in A_{m(p_k/r_k)'}, & k = 1, \dots, m, \\ v_{\vec{w}} \in A_{mp/r}. \end{cases} \end{aligned} \quad (11)$$

Definition 6. For any $\vec{f} := (f_1, \dots, f_m)$, $\vec{r} := (r_1, \dots, r_m)$ and $p \geq 1$, $\mathcal{M}$ is defined by

$$\mathcal{M}_{\vec{r}}(\vec{f})(x) := \sup_{x \in Q} \prod_{k=1}^m \left(\frac{1}{|Q|} \int_Q |f_k(y_k)|^{r_k} dy_k \right)^{1/r_k}. \quad (12)$$

For $\delta > 0$, M_δ is the maximal function

$$\begin{aligned} M_\delta(f)(x) &:= M(|f|^\delta)^{1/\delta}(x) \\ &= \left(\sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|^\delta dy \right)^{1/\delta}. \end{aligned} \quad (13)$$

The sharp maximal function $M^\#$ of Fefferman-Stein is defined by

$$\begin{aligned} M^\#(f)(x) &:= \sup_{x \in Q} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| dy \\ &= \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \end{aligned} \quad (14)$$

where $f_Q := (1/|Q|) \int_Q f(y) dy$.

Next, we give some symbols.

Let $\sigma \in L^\infty(\mathbb{R}^{mn})$ and $\Phi \in \mathcal{S}(\mathbb{R}^{mn})$ satisfy (3). For $\ell \in \mathbb{Z}$, define

$$\tilde{\sigma}_\ell(\xi_1, \dots, \xi_m) := \Phi(2^\ell \xi_1, \dots, 2^\ell \xi_m) \sigma(\xi_1, \dots, \xi_m). \quad (15)$$

Then

$$\begin{aligned} \tilde{\sigma}_\ell(\xi_1, \dots, \xi_m) &= \Phi(2^\ell \xi_1, \dots, 2^\ell \xi_m), \\ \check{\sigma}_\ell &= 2^{-\ell mn} \tilde{\sigma}_\ell(2^{-\ell} \xi_1, \dots, 2^{-\ell} \xi_m), \end{aligned} \quad (16)$$

where $\check{f}$ denotes the inverse Fourier transform of f .

For $N \in \mathbb{N}$, let

$$\sigma^N(\xi_1, \dots, \xi_m) := \sum_{|\ell| \leq N} \tilde{\sigma}_\ell(\xi_1, \dots, \xi_m) \quad (17)$$

and denote by T_{σ^N} the multiplier operator associated with σ^N . It is obvious that T_{σ^N} is an m -linear singular operator with kernel

$$K^N(x; y_1, \dots, y_m) := \check{\sigma}^N(x - y_1, \dots, x - y_m). \quad (18)$$

For an integer k with $1 \leq k \leq 2$ and $x, x', y_1, \dots, y_m \in \mathbb{R}^n$, let

$$\begin{aligned} W^N(x, x'; y_1, \dots, y_m) &:= K^N(x; y_1, \dots, y_m) \\ &\quad - K^N(x'; y_1, \dots, y_m). \end{aligned} \quad (19)$$

Assume that T is a multilinear operator initially defined on the m -fold product of Schwartz spaces, and, taking values in the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \dots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (20)$$

By the associated kernel $K(x, y_1, \dots, y_m)$, we mean that K is a function defined off the diagonal $x = y_1 = \dots = y_m$ in $\mathbb{R}^{(m+1)n}$, satisfying

$$\begin{aligned} T(f_1, \dots, f_m)(x) &:= \int_{\mathbb{R}^{mn}} K(x; y_1, \dots, y_m) f_1(y_1) \cdots f_m \\ &\quad \times (y_m) d\vec{y}, \end{aligned} \quad (21)$$

for all functions $f_k \in \mathcal{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{k=1}^m \text{supp } f_k$. It is easy to see that the associated kernel $K(x, y_1, \dots, y_m)$ to Fourier multiplier operator T_σ is given by

$$K(x, y_1, \dots, y_m) := \check{\sigma}(x - y_1, \dots, x - y_m). \quad (22)$$

To prove main results, we need the following lemmas.

By the reverse Hölder inequality, we have the first lemma.

Lemma 7. Assume that $\vec{w} \in \prod_{k=1}^m A_{p_k/t_k}$, with $t_1, \dots, t_m \in [1, 2]$, $p_k \in (t_k, \infty)$ ($k = 1, \dots, m$), and $1/p = \sum_{k=1}^m 1/p_k$ with $p > 1$. Let $s_k \in (n/2, n]$; then there exists a constant $\epsilon_k \in (1, \min\{p_k/t_k, s_k/(s_k - 1), 2s_k/n\})$ such that $\vec{w}_k \in A_{p_k/(t_k \epsilon_k)}$.

For $p_1, \dots, p_m \in (0, \infty)$ and $s_1, \dots, s_m \in \mathbb{R}$, the weighted Lebesgue space of mixed type $L^{(p_1, \dots, p_m)}(\omega_{(s_1, \dots, s_m)})$ is defined by the norm

$$\begin{aligned} \|F\|_{L^{(p_1, \dots, p_m)}(\omega_{(s_1, \dots, s_m)})} &= \left(\int_{\mathbb{R}^n} \cdots \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(x)|^{p_1} \langle x_1 \rangle^{s_1} dx_1 \right)^{p_2/q_1} \right. \right. \\ &\quad \left. \left. \times \langle x_2 \rangle^{s_2} dx_2 \right)^{p_3/q_2} \cdots \langle x_m \rangle^{s_m} dx_m \right)^{1/p_m}, \end{aligned} \quad (23)$$

where $x := (x_1, \dots, x_m) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$.

Lemma 8 (see [16]). *Let $r > 0$, $2 \leq p_j < \infty$, and $s_j \geq 0$ for $1 \leq j \leq m$. Then there exists a constant $C > 0$ such that*

$$\|\hat{F}\|_{L^{(p_1, \dots, p_m)}(\omega_{(s_1, \dots, s_m)})} \leq C \|F\|_{W^{s_1/p_1, \dots, s_m/p_m}} \quad (24)$$

for all $F \in W^{s_1/p_1, \dots, s_m/p_m}(\mathbb{R}^{mn})$ with $\text{supp } F \subset \{|x_1|^2 \dots + |x_m|^2 \leq r\}$.

The following lemma is the key to our main lemma.

Lemma 9. *Suppose that σ be a multilinear multiplier which satisfies (7) for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$). Let $0 < \delta < r$, $1/r = 1/r_1 + \dots + 1/r_m$, $r_j = \epsilon_j t_j$, and ϵ_j is the same as that appears in Lemma 7. Then for all $\vec{f} \in L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ with $r_j \leq p_j < \infty$ for $1 \leq j \leq m$,*

$$M_\delta^\#(T_\sigma(\vec{f}))(x) \leq C \mathcal{M}_{\vec{r}}(\vec{f})(x), \quad (25)$$

where $\vec{r} = (r_1, \dots, r_m)$.

Proof. By Lemma 8, $1 < t_j \epsilon_j \leq 2$; then $r_j/m \leq 1$. Fix a point x and a cube Q such that $x \in Q$. It suffices to prove

$$\left(\frac{1}{|Q|} \int_Q |T_\sigma(\vec{f})(z) - c_Q|^\delta dz \right)^{1/\delta} \leq C \mathcal{M}_{\vec{r}}(\vec{f})(x), \quad (26)$$

for some constant c_Q . We decompose $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j \chi_{Q^*}$ for all $j = 1, \dots, m$ and $Q^* = 4\sqrt{n}Q$. Then

$$\begin{aligned} \prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m (f_j^0(y_j) + f_j^\infty(y_j)) \\ &= \sum_{\alpha_1, \dots, \alpha_m \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m) \\ &= \prod_{j=1}^m f_j^0(y_j) + \sum_{\alpha_1, \dots, \alpha_m \in \mathcal{F}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m), \end{aligned} \quad (27)$$

where $\mathcal{F} = \{\alpha_1, \dots, \alpha_m : \text{there is at least one } \alpha_j \neq 0\}$. Then we can write

$$\begin{aligned} T_\sigma(\vec{f})(z) &= T_\sigma(\vec{f}^0)(z) + \sum_{\alpha_1, \dots, \alpha_m \in \mathcal{F}} T_\sigma(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(z) \\ &:= I + II. \end{aligned} \quad (28)$$

Applying Kolmogorov's inequality to I , we have

$$\begin{aligned} &\left(\frac{1}{|Q|} \int_Q |T_\sigma(\vec{f}^0)(z)|^\delta dz \right)^{1/\delta} \\ &\leq C \|T_\sigma(\vec{f}^0)\|_{L^{r, \infty}(Q, dx/|Q|)} \\ &\leq C \prod_{j=1}^m \left(\frac{1}{|Q^*|} \int_{Q^*} |f_j(y_j)|^{r_j} dy_j \right)^{r_j} \\ &\leq C \mathcal{M}_{\vec{r}}(\vec{f})(x), \end{aligned} \quad (29)$$

since T_σ is bounded from $L^{r_1} \times \dots \times L^{r_m}$ to L^r .

Take

$$c_Q = \sum_{\alpha_1, \dots, \alpha_m \in \mathcal{F}} T_\sigma(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x). \quad (30)$$

We claim that, for any $z \in Q$,

$$\begin{aligned} &\sum_{\alpha_1, \dots, \alpha_m \in \mathcal{F}} |T_\sigma(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(z) - T_\sigma(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x)| \\ &\leq C \mathcal{M}_{\vec{r}}(\vec{f})(x). \end{aligned} \quad (31)$$

Let

$$\begin{aligned} W^\ell(x, z; y_1, \dots, y_m) &= \check{\sigma}_\ell(x - y_1, \dots, x - y_m) \\ &\quad - \check{\sigma}_\ell(z - y_1, \dots, z - y_m). \end{aligned} \quad (32)$$

At first, we consider the case $\alpha_1 = \dots = \alpha_m$. We get

$$\begin{aligned} &|T_\sigma(f_1^\infty \cdots f_m^\infty)(z) - T_\sigma(f_1^\infty \cdots f_m^\infty)(x)| \\ &\leq \sum_{l \in \mathbb{Z}} |T_{\sigma_l}(f_1^\infty \cdots f_m^\infty)(z) - T_{\sigma_l}(f_1^\infty \cdots f_m^\infty)(x)| \\ &\leq \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^{mm} \setminus (Q^*)^m} |W^\ell(x, z; y_1, \dots, y_m)| \prod_{j=1}^m f_j(y_j) d\vec{y} \\ &\leq \sum_{l \in \mathbb{Z}} \sum_{k=0}^\infty \iint_{(2^{k+1}Q^*) \setminus (2^kQ^*)} |W^\ell(x, z; y_1, \dots, y_m)| \prod_{j=1}^m f_j(y_j) d\vec{y} \\ &\leq \sum_{k=0}^\infty \sum_{l \in \mathbb{Z}} \prod_{j=1}^m \left(\int_{(2^{k+1}Q^*)^m} |f_j(y_j)|^{r_j} dy_j \right)^{1/r_j} \\ &\quad \times \left(\int_{(2^{k+1}Q^*) \setminus (2^kQ^*)} \left(\int_{(2^{k+1}Q^*) \setminus (2^kQ^*)} \cdots \left(\int_{(2^{k+1}Q^*) \setminus (2^kQ^*)} \right. \right. \right. \\ &\quad \left. \left. \left. |W^\ell(x, z; y_1, \dots, y_m)|^{r'_1} dy_1 \right)^{r'_2/r'_1} \cdots \right)^{r'_m/r'_{m-1}} dy_m \right)^{1/r'_m} \\ &:= \sum_{k=0}^\infty \sum_{l \in \mathbb{Z}} \prod_{j=1}^m \left(\int_{(2^{k+1}Q^*)^m} |f_j(y_j)|^{r_j} dy_j \right)^{1/r_j} \Pi_{k,l}^{\infty, \dots, \infty}. \end{aligned} \quad (33)$$

Denote $h = z - x$, $\tilde{Q} = x - Q^*$, and $\ell(Q)$ the side length of a cube Q ; it follows from Lemma 7 that

$$\begin{aligned} &\Pi_{k,\ell}^{\infty, \dots, \infty} \\ &= \left(\int_{(2^{k+1}\tilde{Q}) \setminus (2^k\tilde{Q})} \left(\int_{(2^{k+1}\tilde{Q}) \setminus (2^k\tilde{Q})} \cdots \left(\int_{(2^{k+1}\tilde{Q}) \setminus (2^k\tilde{Q})} \right. \right. \right. \\ &\quad \left. \left. \left. |\check{\sigma}_\ell(h + y_1, \dots, h + y_m) - \check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} dy_1 \right)^{r'_2/r'_1} \right)^{r'_m/r'_{m-1}} \right)^{1/r'_m} \end{aligned}$$

$$\begin{aligned}
& \cdots \Big)^{r'_m/r'_{m-1}} dy_m \Big)^{1/r'_m} \\
& \leq 2 \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left. |\check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} dy_1 \right)^{r'_2/r'_1} \\
& \quad \cdots \Big)^{r'_m/r'_{m-1}} dy_m \Big)^{1/r'_m} \\
& \leq C(2^k \ell(Q))^{-(s_1 + \dots + s_m)} \\
& \quad \times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left. |\check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} \langle y_1 \rangle^{s_1 r'_1} dy_1 \right)^{r'_2/r'_1} \\
& \quad \cdots \Big)^{r'_m/r'_{m-1}} \langle y_m \rangle^{s_m r'_m} dy_m \Big)^{1/r'_m} \\
& \leq C(2^k \ell(Q))^{-(s_1 + \dots + s_m)} 2^{\ell(s_1 + \dots + s_m)} \\
& \quad \times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left. |2^{-\ell m n} \check{\sigma}_\ell(2^{-\ell} y_1, \dots, 2^{-\ell} y_m)|^{r'_1} \right. \\
& \quad \times \langle 2^{-\ell} y_1 \rangle^{s_1 r'_1} dy_1 \Big)^{r'_2/r'_1} \\
& \quad \cdots \Big)^{r'_m/r'_{m-1}} \langle 2^{-\ell} y_m \rangle^{s_m r'_m} dy_m \Big)^{1/r'_m}
\end{aligned}$$

$$\begin{aligned}
& \leq C(2^k \ell(Q))^{-(s_1 + \dots + s_m)} 2^{\ell(s_1 + \dots + s_m)} 2^{-l m n} 2^{-l(n/r'_1 + \dots + n/r'_m)} \\
& \quad \times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} |\check{\sigma}_\ell(z_1, \dots, z_m)|^{r'_1} \right. \\
& \quad \times \langle z_1 \rangle^{s_1 r'_1} dz_1 \Big)^{r'_2/r'_1} \cdots \Big)^{r'_m/r'_{m-1}} \langle z_m \rangle^{s_m r'_m} dz_m \Big)^{1/r'_m} \\
& \leq C(2^k \ell(Q))^{-(s_1 + \dots + s_m)} 2^{-\ell(n/r_1 + \dots + n/r_m - s_1 - \dots - s_m)} \|\sigma_\ell\|_{W^{s_1, \dots, s_m}}. \tag{34}
\end{aligned}$$

Given that $2^{\ell_0} \leq \ell(Q) \leq 2^{\ell_0+1}$, we have that

$$\begin{aligned}
& \sum_{\ell < \ell_0} II_{k, \ell}^{\infty, \dots, \infty} \\
& \leq C \sup_{\ell} \|\sigma_\ell\|_{W^{s_1, \dots, s_m}} \\
& \quad \times \sum_{\ell < \ell_0} (2^k \ell(Q))^{-(s_1 + \dots + s_m)} 2^{-\ell(n/r_1 + \dots + n/r_1 - s_1 - \dots - s_m)} \\
& \leq C \sup_{\ell} \|\sigma_\ell\|_{W^{s_1, \dots, s_m}} 2^{-k(s_1 + \dots + s_m)} \ell(Q)^{-(n/r_1 + \dots + n/r_m)}. \tag{35}
\end{aligned}$$

On the other hand, a similar process follows that in [17]; we get that

$$\begin{aligned}
& II_{k, \ell}^{\infty, \dots, \infty} \\
& = \left(\int_{(2^{k+1} \bar{Q}) \setminus (2^k \bar{Q})} \left(\int_{(2^{k+1} \bar{Q}) \setminus (2^k \bar{Q})} \cdots \left(\int_{(2^{k+1} \bar{Q}) \setminus (2^k \bar{Q})} \right. \right. \right. \\
& \quad \left. \left. |\check{\sigma}_\ell(h + y_1, \dots, h + y_m) - \check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} dy_1 \right)^{r'_2/r'_1} \right. \\
& \quad \left. \cdots \right)^{r'_m/r'_{m-1}} dy_m \Big)^{1/r'_m}
\end{aligned}$$

$$\begin{aligned}
& \leq \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \dots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_0^1 |\vec{h} \cdot \nabla \check{\sigma}_\ell(y_1 + \theta h, \dots, y_m + \theta h)| d\theta \right)^{r'_1/r'_1} dy_1 \\
& \quad \dots \left. \right)^{r'_m/r'_{m-1}} dy_m \left. \right)^{1/r'_m} \\
& \leq \int_0^1 \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad |\vec{h} \cdot \nabla \check{\sigma}_\ell(y_1 + \theta h, \dots, y_m + \theta h)|^{r'_1} dy_1 \\
& \quad \dots \left. \right)^{r'_m/r'_{m-1}} dy_m \left. \right)^{1/r'_m} d\theta \\
& \leq \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \dots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad |\vec{h} \cdot \nabla \check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} dy_1 \\
& \quad \dots \left. \right)^{r'_m/r'_{m-1}} dy_m \left. \right)^{1/r'_m}, \\
\end{aligned} \tag{36}$$

where $\vec{h} = (h, \dots, h) \in (\mathbb{R}^n)^m$. Since

$$\vec{h} \cdot \nabla \check{\sigma}_\ell(y_1, \dots, y_m) = \sum_{j=1}^m h_j \partial_j \nabla \check{\sigma}_\ell(y_1, \dots, y_m), \tag{37}$$

we have

$$\begin{aligned}
& II_{k,\ell}^{\infty, \dots, \infty} \\
& \leq \sum_{j=1}^m \ell(Q) \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \dots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad |\partial_j \cdot \check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} dy_1 \\
& \quad \dots \left. \right)^{r'_m/r'_{m-1}} dy_m \left. \right)^{1/r'_m} \\
& \leq \sum_{j=1}^m \ell(Q) (2^k \ell(Q))^{-(s_1 + \dots + s_m)} \\
& \quad \times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \dots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad |\partial_j \check{\sigma}_\ell(y_1, \dots, y_m)|^{r'_1} \langle y_1 \rangle^{s_1 r'_1} dy_1 \\
& \quad \dots \left. \right)^{r'_m/r'_{m-1}} \langle y_m \rangle^{s_m r'_m} dy_m \left. \right)^{1/r'_m} \\
& \leq C (2^k \ell(Q))^{-(s_1 + \dots + s_m)} 2^{\ell(s_1 + \dots + s_m)} \\
& \quad \times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \dots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad |2^{-\ell m n} \cdot \partial_j \check{\sigma}_\ell(2^{-\ell} y_1, \dots, 2^{-\ell} y_m)|^{r'_1} \\
& \quad \times \langle 2^{-\ell} y_1 \rangle^{s_1 r'_1} dy_1 \left. \right)^{r'_2/r'_1} \dots \left. \right)^{r'_m/r'_{m-1}}
\end{aligned}$$

$$\begin{aligned}
& \times \left\langle 2^{-\ell} y_m \right\rangle^{s_m r'_m} dy_m \Bigg)^{1/r'_m} \\
& \leq C \left(2^k \ell(Q) \right)^{-(s_1 + \dots + s_m)} 2^{\ell(s_1 + \dots + s_m)} 2^{-\ell m n} 2^{-\ell(n/r'_1 + \dots + n/r'_m)} \\
& \quad \times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \dots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \right. \\
& \quad \left| \partial_j \tilde{\sigma}_\ell(z_1, \dots, z_m) \right|^{r'_1} \langle z_1 \rangle^{s_1 r'_1} dz_1 \Bigg)^{r'_2/r'_1} \\
& \quad \dots \Bigg)^{r'_m/r'_{m-1}} \left\langle z_m \right\rangle^{s_m r'_m} dz_m \Bigg)^{1/r'_m} \\
& \leq C \left(2^k \ell(Q) \right)^{-(s_1 + \dots + s_m)} 2^{-\ell(n/r_1 + \dots + n/r_m + 1 - s_1 - \dots - s_m)} \\
& \quad \times \|\sigma_\ell\|_{W^{s_1, \dots, s_m}}.
\end{aligned} \tag{38}$$

From Lemma 8, $n/r_1 + \dots + n/r_m > s_1 + \dots + s_m - 1$. It is deduced that

$$\begin{aligned}
& \sum_{\ell \geq \ell_0} \Pi_{k, \ell}^{\infty, \dots, \infty} \\
& \leq C \sup_{\ell} \|\sigma_\ell\|_{W^{s_1, \dots, s_m}} 2^{-\ell(s_1 + \dots + s_m)} \ell(Q)^{-(n/r_1 + \dots + n/r_m)}.
\end{aligned} \tag{39}$$

So

$$\begin{aligned}
& \sum_{\alpha_1, \dots, \alpha_m \in \mathcal{F}} |T_\sigma(f_1^{\alpha_1} \dots f_m^{\alpha_m})(z) - T_\sigma(f_1^{\alpha_1} \dots f_m^{\alpha_m})(x)| \\
& \leq C \sum_{k=0}^{\infty} 2^{-k(s_1 + \dots + s_m - n/r_1 - \dots - n/r_m)} \mathcal{M}_{\vec{r}}(\vec{f})(x) \\
& \leq C \mathcal{M}_{\vec{r}}(\vec{f})(x).
\end{aligned} \tag{40}$$

It remains to consider the case that there exists a proper subset $\{j_1, \dots, j_\gamma\}$ of $\{1, \dots, m\}$, $1 \leq \gamma < m$, such that $\alpha_{j_1} = \dots = \alpha_{j_\gamma} = 0$. We have

$$\begin{aligned}
& |T_\sigma(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z) - T_\sigma(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x)| \\
& \leq \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} \prod_{j=1}^m \left(\int_{2^{k+1} Q^*} |f_j(y_j)|^{r_j} d\vec{y} \right)^{1/r_j} \\
& \quad \times \left(\int_{2^{k+1} Q^* \setminus 2^k Q^*} \dots \left(\int_{2^{k+1} Q^* \setminus 2^k Q^*} \left(\int_{Q^*} \right. \right. \right. \\
& \quad \dots \left. \left. \left(\int_{Q^*} |W_{0,l}(x, z; y_1, \dots, y_m)|^{r'_1} dy_1 \right)^{r'_2/r'_1} \right. \right. \\
& \quad \dots dy_{\gamma} \Bigg)^{r'_{\gamma+1}/r'_\gamma} dy_{\gamma+1} \Bigg)^{1/r'_m} \dots dy_m \Bigg)^{1/r'_m}.
\end{aligned} \tag{41}$$

By the same argument as that of the case $\alpha_1 = \dots = \alpha_m$, we have that

$$\begin{aligned}
& |T_\sigma(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z) - T_\sigma(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x)| \\
& \leq C \mathcal{M}_{\vec{r}}(\vec{f})(x).
\end{aligned} \tag{42}$$

This completes the proof. $\square$

Lemma 10. Suppose that σ be a multilinear multiplier which satisfies (7) for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$). Let $0 < \delta < r$, $1/r = 1/r_1 + \dots + 1/r_m$, and $r_j = e_j t_j$, and e_j is the same as that appears in Lemma 7. Then, for $\vec{b} \in (BMO(\mathbb{R}^n))^m$ and any $\vec{\gamma} > \vec{r}$, that is, $\gamma_j > r_j$, $j = 1, \dots, m$, there exists some constant $C > 0$ such that

$$\begin{aligned}
& M_\delta^\#(T_{\sigma, \Sigma \mathbf{b}}(\vec{f}))(x) \\
& \leq C \|\vec{b}\|_{BMO^m} (M_\varepsilon(T_\sigma(\vec{f}))(x) + \mathcal{M}_{\vec{\gamma}}(\vec{f})(x)),
\end{aligned} \tag{43}$$

for all m -tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support.

The proof of the above lemma is standard. A statement similar to Lemma 2.7 in [17] with minor modifications deduces the estimates. We omit the details here.

Lemma 11. Let σ be a multilinear multiplier which satisfies (7) for some $s_k \in (n/2, n]$ ($k = 1, \dots, m$), $r_1, \dots, r_m \in [1, 2)$ such

that $r_j s_j > n$ ($j = 2, \dots, m$). Then, for every $\iota \in (0, s_1]$, $R > 0$ and $x \in \mathbb{R}^n \setminus 2R$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \int_{|y_1| < R} |\tilde{\sigma}_l(x - y_1, \dots, x - y_m) \\ & \quad \times f_1(y_1) \cdots f_m(y_m)| dy_1 \cdots dy_m \\ & \leq \frac{1}{|x|^\iota} 2^{-l(\iota - n/r_1)} \|f_1\|_{L^{p_1}(\mathbb{R}^n, w_1)} \prod_{i=2}^m M_{r_k} f_k \\ & \quad \times (x) \vartheta(R)^{(1/r_1) - (1/p_1)}. \end{aligned} \quad (44)$$

By a similar way in the proof of the Lemma 2.4 in [14], with slight changes, we can get the conclusion of Lemma 11 and we omit the details.

About the proof of compactness, as in [18] we will rely on the Fréchet-Kolmogorov theorem characterizing the precompactness of a set in L^p . More precisely, see Yosida's book [19]. For more about compactness, we refer to [20, 21].

Lemma 12. *A set $\mathcal{H}$ is precompact in L^p , $p < \infty$ if and only if*

- (i) $\sup_{h \in \mathcal{H}} \|h\|_{L^p} < \infty$,
- (ii) $\lim_{A \rightarrow \infty} \|h\|_{L^p(|x| > A)} = 0$ uniformly in $h \in \mathcal{H}$,
- (iii) $\lim_{t \rightarrow 0} \|h(\cdot + t) - h(\cdot)\|_{L^p} = 0$ uniformly in $h \in \mathcal{H}$.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We only present the case that $m = 2$. We have by Lemma 10 and Theorem 3.2 in [8]

$$\begin{aligned} & \|T_{\Sigma b}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \\ & \leq \|M_\delta(T_{\Sigma b}(\vec{f}))\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \\ & \leq \|M_\delta^\#(T_{\Sigma b}(\vec{f}))\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \\ & \leq \|\vec{b}\|_{\text{BMO}^2} \left(\|\mathcal{M}_{\vec{\gamma}}(\vec{f})(x)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \right. \\ & \quad \left. + \|M_\epsilon(T)(\vec{f})(x)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \right) \\ & \leq \|\vec{b}\|_{\text{BMO}^2} \left(\|\mathcal{M}_{\vec{\gamma}}(\vec{f})(x)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \right. \\ & \quad \left. + \|M_\epsilon^\#(T)(\vec{f})(x)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \right) \\ & \leq \|\vec{b}\|_{\text{BMO}^2} \left(\|\mathcal{M}_{\vec{\gamma}}(\vec{f})(x)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \right. \\ & \quad \left. + \|\mathcal{M}(\vec{f})(x)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \right) \\ & \leq \|\vec{b}\|_{\text{BMO}^2} \prod_{k=1}^m \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)}, \end{aligned} \quad (45)$$

which completes the proof of Theorem 1. $\square$

Proof of Theorem 2. We will employ some ideas of Bényi and Torres [13]. Without loss of the generality, we only prove the case $m = 2$. Let $p_k \in (r_k, \infty)$ ($k = 1, 2$), $p \in (1, \infty)$, with $1/p = \sum 1/p_k$, and $b_1, b_2 \in C_c^\infty(\mathbb{R}^n)$. Note that, for any $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$,

$$\lim_{N \rightarrow \infty} T_{\sigma^N, \Sigma b}(f_1, f_2)(x) = T_{\sigma, \Sigma b}(f_1, f_2)(x). \quad (46)$$

It is enough to prove that the following conditions hold:

- (a) $T_{\sigma^N, \Sigma b}$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times L^{p_2}(\mathbb{R}^n, w_2)$ to $L^p(\mathbb{R}^n, \gamma_{\tilde{w}})$;
- (b) for each fixed $\epsilon > 0$, there exists a constant $A = A(\epsilon)$ which is independent of N , f_1 , and f_2 such that

$$\left(\int_{|x| > A} |T_{\sigma^N, \Sigma b}(f_1, f_2)|^p \gamma_{\tilde{w}}(x) dx \right)^{1/p} \leq \epsilon \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)}; \quad (47)$$

- (c) for each fixed $\epsilon > 0$, there exists a constant $\rho = \rho(\epsilon)$ which is independent of N , f_1 , and f_2 such that, for all t with $0 < |t| < \rho$,

$$\begin{aligned} & \|T_{\sigma^N, \Sigma b}(f_1, f_2)(\cdot) - T_{\sigma^N, \Sigma b}(f_1, f_2)(\cdot + t)\|_{L^p(\mathbb{R}^n, \gamma_{\tilde{w}})} \\ & \leq \epsilon \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)}. \end{aligned} \quad (48)$$

Then by the Fatou Lemma, the conditions (a), (b), and (c) still hold true if $T_{\sigma^N, \Sigma b}(f_1, f_2)$ is replaced by $T_{\sigma, \Sigma b}(f_1, f_2)$.

It is clear that the first condition (a) holds according to Theorem 1.

Then, we prove the conclusion (b). Let $D > 0$ be large enough such that $\text{supp } b_1 \subset B_D(0) := B(0, D)$ and let $A \geq \max\{2D, 1\}$. Then for every x with $|x| > 2A$, we have by Lemma 11 that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\text{supp } b_1} |K^N(x; y_1, y_2) f_1(y_1) f_1(y_1)| dy_1 dy_2 \\ & = \int_{\mathbb{R}^n} \int_{\text{supp } b_1} \left| \sum_{|\ell| \leq N} \tilde{\sigma}^N(x - y_1, x - y_2) \right. \\ & \quad \left. \times f_1(y_1) f_1(y_1) \right| dy_1 dy_2 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{0 \leq \ell \leq N} \frac{1}{|x|^{s_1}} 2^{-\ell(s_1 - n/r_1)} \|f_1\|_{L^{p_1}(\mathbb{R}^n, w_1)} \\
&\quad \times M_{r_2} f_2(x) \vartheta(B_D(0))^{(1/r_1) - (1/p_1)} \\
&+ \sum_{-N \leq \ell < 0} \frac{1}{|x|^\theta} 2^{-\ell(\theta - n/r_1)} \|f_1\|_{L^{p_1}(\mathbb{R}^n, w_1)} \\
&\quad \times M_{r_2} f_2(x) \vartheta(B_D(0))^{(1/r_1) - (1/p_1)} \\
&\leq \left(\frac{1}{|x|^{s_1}} + \frac{1}{|x|^\theta} \right) \|f_1\|_{L^{p_1}(\mathbb{R}^n, w_1)} \\
&\quad \times M_{r_2} f_2(x) \vartheta(B_D(0))^{(1/r_1) - (1/p_1)},
\end{aligned} \tag{49}$$

if we choose $\iota = s_1$ and $\iota = \theta \in (n/(p_1 \epsilon_1), n/r_1)$ in Lemma 11 respectively. Therefore,

$$\begin{aligned}
&\left(\int_{|x|>A} |T_{\sigma^N, b_1}(f_1, f_2)|^p \nu_{\tilde{w}}(x) dx \right)^{1/p} \\
&\leq \|b_1\|_{L^\infty(\mathbb{R}^n)} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)} \vartheta(B_D(0))^{(1/r_1) - (1/p_1)} \\
&\quad \times \left(\left(\int_{|x|>A} \frac{w_1(x)}{|x|^{s_1 p_1}} dx \right)^{1/p_1} + \left(\int_{|x|>A} \frac{w_1(x)}{|x|^{\theta p_1}} dx \right)^{1/p_1} \right) \\
&\leq \|b_1\|_{L^\infty(\mathbb{R}^n)} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)} \vartheta(B_D(0))^{(1/r_1) - (1/p_1)} \\
&\quad \times \left(\left(\int_{|x|>A} \frac{w_1(x)}{|x|^{n(p_1/t_1)}} dx \right)^{1/p_1} \right. \\
&\quad \left. + \left(\int_{|x|>A} \frac{w_1(x)}{|x|^{n(p_1/(t_1 \epsilon_1))}} dx \right)^{1/p_1} \right) \\
&\leq \epsilon \|b_1\|_{L^\infty(\mathbb{R}^n)} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)},
\end{aligned} \tag{50}$$

where the last inequality holds by the fact (see [22, 23]) that for $v \in A_p$, $p > 1$

$$\int_{\mathbb{R}^n} \frac{v(x)}{(1 + |x|)^{np}} dx < \infty. \tag{51}$$

This in turn leads to conclusion (b) directly.

We turn our attention to conclusion (c). We write

$$T_{\sigma^N, b_1}(f_1, f_2)(x) - T_{\sigma^N, b_1}(f_1, f_2)(x+t) = \sum_{j=1}^4 E_j(x, t), \tag{52}$$

with

$$\begin{aligned}
E_1(x, t) &= \int_{\max_{1 \leq k \leq 2} |x - y_k| \geq \delta_t} K^N(x; y_1, y_2) \\
&\quad \times (b_1(x+t) - b_1(x)) f_1(y_1) f_2(y_2) dy_1 dy_2, \\
E_2(x, t) &= \int_{\max_{1 \leq k \leq 2} |x - y_k| \geq \delta_t} (K^N(x; y_1, y_2) - K^N(x+t; y_1, y_2)) \\
&\quad \times (b_1(y_1) - b_1(x+t)) f_1(y_1) f_2(y_2) dy_1 dy_2, \\
E_3(x, t) &= \int_{\max_{1 \leq k \leq 2} |x - y_k| < \delta_t} K^N(x; y_1, y_2) \\
&\quad \times (b_1(y_1) - b_1(x)) f_1(y_1) f_2(y_2) dy_1 dy_2, \\
E_4(x, t) &= \int_{\max_{1 \leq k \leq 2} |x - y_k| < \delta_t} K^N(x+t; y_1, y_2) \\
&\quad \times (b_1(x+t) - b_1(y_1)) f_1(y_1) f_2(y_2) dy_1 dy_2,
\end{aligned} \tag{53}$$

with $\delta_t > 4|t|$ a convenient choice to be determined later.

In a completely same way in the proof of Theorem 1.1 in [14], we can obtain the estimate of $|E_j(x, t)|$ ($j = 1, \dots, 4$). We only list the results and omit the details.

$$\begin{aligned}
|E_1(x, t)| &\leq |t| \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \times \left(\prod_{k=1}^2 M_{r_k} f_k(x) + M_\delta(T(f_1, f_2))(x) \right), \\
|E_2(x, t)| &\leq (|t| \delta_t^{-1})^q \|b_1\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \times \prod_{k=1}^2 (M_{r_k} f_k(x) + M_{r_k} f_k(x+t)),
\end{aligned} \tag{54}$$

$$|E_3(x, t)| \leq \delta_t \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)} \prod_{k=1}^2 M_{r_k} f_k(x),$$

$$|E_4(x, t)| \leq \delta_t \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)} \prod_{k=1}^2 M_{r_k} f_k(x+t).$$

Fix each $\epsilon > 0$, set

$$\rho = \frac{A\epsilon}{2(1 + \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)})}, \tag{55}$$

with

$$A = \min \left\{ 1, \left(\frac{\epsilon}{2(1 + \|b_1\|_{L^\infty(\mathbb{R}^n)})} \right)^{1/q} \right\} \tag{56}$$

and $\delta_t = |t|A^{-1}$ for each $t \in \mathbb{R}^n$, where constant $\varrho > 0$. Our estimates for those terms of E_j ($j = 1, \dots, 4$) then lead to that when $0 < |t| < \rho$,

$$\begin{aligned} & \|T_{\sigma^N, b_1}(f_1, f_2)(\cdot) - T_{\sigma^N, \Pi b}(f_1, f_2)(\cdot + t)\|_{L^p(\mathbb{R}^n, \nu_{\tilde{w}})} \\ & \leq ((|t| + \delta_t) \|\nabla b_1\|_{L^\infty(\mathbb{R}^n)} + (|t| \delta_t^{-1})^\varrho \|b_1\|_{L^\infty(\mathbb{R}^n)}) \\ & \quad \times \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)} \\ & \leq \epsilon \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, w_k)}; \end{aligned} \quad (57)$$

this establishes conclusion (c) and we conclude that T_{σ^N, b_1} is compact.

In a completely analogous way, if $b_2 \in C_c^\infty$, then T_{σ^N, b_2} is compact. Moreover, then $T_{\sigma^N, \Sigma b}$ is compact, thus we complete the proof of Theorem 2. $\square$

Conflict of Interests

The authors declare that they do not have any commercial or associative interest that represents a conflict of interests in connection with the work submitted.

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