Research Article

Stability of Exponential Functional Equations with Involutions

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Let S be a commutative semigroup if not otherwise specified and $f : S \rightarrow \mathbb{R}$. In this paper we consider the stability of exponential functional equations $|f(x + \sigma(y)) - g(x)f(y)| \leq \phi(x)$ or $\phi(y)$, $|f(x + \sigma(y)) - f(x)g(y)| \leq \phi(x)$ or $\phi(y)$ for all $x, y \in S$ and where $\sigma : S \rightarrow S$ is an involution. As main results we investigate bounded and unbounded functions satisfying the above inequalities. As consequences of our results we obtain the Ulam-Hyers stability of functional equations (Chung and Chang (in press); Chávez and Sahoo (2011); Houston and Sahoo (2008); Jung and Bae (2003)) and a generalized result of Albert and Baker (1982).

1. Introduction

Throughout this paper we denote by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, $\mathbb{R}^n$ the set of real numbers, nonnegative real numbers, complex numbers, and the n -dimensional Euclidean space, respectively, and $\phi : S \rightarrow \mathbb{R}^+$, $\epsilon \geq 0$. A function $m : S \rightarrow \mathbb{C}$ is called *exponential* provided that $m(x + y) = m(x)m(y)$ for all $x, y \in S$ and $\sigma : S \rightarrow S$ is called an *involution* provided that $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. In [1], Baker proved the stability of the exponential functional equation: let $f : S \rightarrow \mathbb{C}$ satisfy the exponential functional inequality

$$|f(x + y) - f(x)f(y)| \leq \epsilon \quad (1)$$

for all $x, y \in S$. Then, either f is a bounded function satisfying

$$|f(x)| \leq \frac{1 + \sqrt{1 + 4\epsilon}}{2} \quad (2)$$

for all $x \in S$, or f is an unbounded exponential function (see also Baker et al. [2]). In particular, if $S = V$, where V is a vector space over the field $\mathbb{Q}$ of rational numbers and $f : V \rightarrow \mathbb{R}$ is a bounded function satisfying (1) for $0 \leq \epsilon < 1/4$, then f satisfies either

$$\frac{1 + \sqrt{1 - 4\epsilon}}{2} \leq f(x) \leq \frac{1 + \sqrt{1 + 4\epsilon}}{2} \quad (3)$$

for all $x \in V$, or else

$$-\epsilon \leq f(x) \leq \frac{1 - \sqrt{1 - 4\epsilon}}{2} \quad (4)$$

for all $x \in V$ (see Albert and Baker [3]). In [4–6], some functional equations arising from number theory or product of matrices are introduced. The equations therein can be reduced to the functional equations with involution $\sigma : S \rightarrow S$:

$$f(x + \sigma(y)) - f(x)f(y) = 0 \quad (5)$$

for all $x, y \in S$ (see Section 4). As we see in Section 4, the equations in [4–6] can be reduced to the equation of the form (5) when $S = \mathbb{C}$ with operation of multiplication and $\sigma(z) = \bar{z}$ for all $z \in \mathbb{C}$ or $\sigma(z) = z$ for all $z \in \mathbb{C}$ and the equations in [6] are reduced to those forms (5) in the complex numbers with $\sigma(z) = \bar{z}$ and in the set of quaternions with $\sigma(q) = q^*$, where q^* denotes the conjugate of the quaternion q .

As main results of the paper, we consider, in Section 2, the Ulam-Hyers stability of exponential functional equations with involution $\sigma : S \rightarrow S$:

$$|f(x + \sigma(y)) - g(x)f(y)| \leq \phi(x),$$

$$|f(x + \sigma(y)) - f(x)g(y)| \leq \phi(y),$$

$$\begin{aligned}
|f(x + \sigma(y)) - g(x)f(y)| &\leq \phi(y), \\
|f(x + \sigma(y)) - f(x)g(y)| &\leq \phi(x)
\end{aligned} \tag{6}$$

for all $x, y \in S$. In Section 3, as a consequence of our main results we obtain the behaviors of bounded functions f satisfying each of the inequalities

$$\begin{aligned}
|f(x + \sigma(y)) - f(x)f(y)| &\leq \phi(x), \\
|f(x + \sigma(y)) - f(x)f(y)| &\leq \phi(y)
\end{aligned} \tag{7}$$

for all $x, y \in S$. Also, as a direct consequence of the stability of (7) we obtain a generalized version of the result of Albert and Baker [3]. Our method of proof also works for investigation of bounded solutions of some other functional inequalities (see [7–9]). We refer to [7–12] for the exponential functional equations, inequalities, and related results.

In Section 4, as an application of our result, we obtain the stability of the functional equations:

$$f(x_1, y_1)f(x_2, y_2) = f(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1) \tag{8}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$, and

$$\begin{aligned}
&f(x_1, y_1, u_1, v_1)f(x_2, y_2, u_2, v_2) \\
&= f(x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2, \\
&\quad x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2, \\
&\quad x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2, \\
&\quad x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2)
\end{aligned} \tag{9}$$

for all $x_1, y_1, u_1, v_1, x_2, y_2, u_2, v_2 \in \mathbb{R}$, where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^4 \rightarrow \mathbb{R}$. As stated in [6], (8) and (9) arise from a well-known theorem in number theory.

Reducing (8) and (9) to those in the complex numbers and the quaternions, we obtain the stability of (8) and (9); that is, we investigate bounded and unbounded functions f satisfying each of the following functional inequalities:

$$|f(x_1, y_1)f(x_2, y_2) - f(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1)| \leq \phi(x_2, y_2), \tag{10}$$

$$|f(x_1, y_1)f(x_2, y_2) - f(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1)| \leq \phi(x_1, y_1) \tag{11}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and for some $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$, and we investigate bounded and unbounded functions g satisfying each of the following functional inequalities:

$$\begin{aligned}
&|g(x_1, y_1, u_1, v_1)g(x_2, y_2, u_2, v_2) \\
&\quad - g(x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2, \\
&\quad x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2, \\
&\quad x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2, \\
&\quad x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2)| \\
&\leq \psi(x_2, y_2, u_2, v_2),
\end{aligned} \tag{12}$$

$$\begin{aligned}
&|g(x_1, y_1, u_1, v_1)g(x_2, y_2, u_2, v_2) \\
&\quad - g(x_1x_2 + y_1y_2 + u_1u_2 + v_1v_2, \\
&\quad x_1y_2 - y_1x_2 + u_1v_2 - v_1u_2, \\
&\quad x_1u_2 - y_1v_2 - u_1x_2 + v_1y_2, \\
&\quad x_1v_2 + y_1u_2 - u_1y_2 - v_1x_2)| \\
&\leq \psi(x_1, y_1, u_1, v_1)
\end{aligned} \tag{13}$$

for all $x_1, y_1, u_1, v_1, x_2, y_2, u_2, v_2 \in \mathbb{R}$ and for some $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}^+$. We also refer the reader to [5] for another proof of finding general solutions of (8) and refer to [13] for the stability of inequalities (10)–(13).

2. Stability of (6)

Throughout this section we assume that S is a commutative semigroup if not otherwise specified. An exponential function $m : S \rightarrow \mathbb{C}$ is called σ -exponential if $m(\sigma(x)) = m(x)$ for all $x \in S$. We denote σ -exponential functions m by m_σ . It is easy to see that if S is uniquely 2-divisible (i.e., for each $x \in S$ there exists a unique $y \in S$ such that $2y = x$; we write $y = x/2$), then m_σ is a σ -exponential function if and only if

$$m_\sigma(x) = m\left(\frac{x + \sigma(x)}{2}\right), \quad x \in S \tag{14}$$

for some exponential function $m : S \rightarrow \mathbb{C}$. Throughout this section we denote by $\sigma : S \rightarrow S$ an involution. In the following we exclude the trivial cases when $f = 0$ or $g = 0$.

Theorem 1. *Let $f, g : S \rightarrow \mathbb{C}$ satisfy*

$$|f(x + \sigma(y)) - g(x)f(y)| \leq \phi(x) \tag{15}$$

for all $x, y \in S$. If S is a group and f is bounded, then

$$|1 - |g(x)|| \leq \frac{\phi(x)}{M_f} \tag{16}$$

for all $x \in S$, where $M_f = \sup_{x \in S} |f(x)|$. If f is unbounded, then there exists a σ -exponential function $m_\sigma : S \rightarrow \mathbb{C}$ such that

$$g(x) = g(0)m_\sigma(x), \tag{17}$$

$$|f(x) - f(0)g(0)m_\sigma(x)| \leq \phi(x)$$

for all $x \in S$, where $g(0) = \pm 1$. In particular, if $\phi(x) = \epsilon \geq 0$ for all $x \in S$, then we have

$$g(x) = m_\sigma(x), \quad f(x) = f(0)m_\sigma(x) \tag{18}$$

for all $x \in S$.

Proof. First, we assume that f is bounded (in this case we assume that S is a group). Using the triangle inequality with (15) we have

$$|g(x)f(y)| \leq |f(x + \sigma(y))| + \phi(x) \leq M_f + \phi(x) \tag{19}$$

for all $x, y \in S$. Taking the supremum of the left-hand side of (19) with respect to y we have

$$|g(x)| M_f \leq M_f + \phi(x) \quad (20)$$

for all $x \in S$, which implies

$$M_f (|g(x)| - 1) \leq \phi(x) \quad (21)$$

for all $x \in S$. Replacing y by $\sigma(-x + y)$ in (15) and using the triangle inequality with the result we have

$$|f(y)| \leq |g(x) f(\sigma(-x + y))| \leq |g(x)| M_f + \phi(x) \quad (22)$$

for all $x, y \in S$. Taking the supremum of the left-hand side of (22) with respect to y we have

$$M_f \leq |g(x)| M_f + \phi(x) \quad (23)$$

for all $x \in S$, which implies

$$M_f (1 - |g(x)|) \leq \phi(x) \quad (24)$$

for all $x \in S$. From (21) and (24) we get (16). Now, we assume that f is unbounded. Putting $x = 0$ in (15) we have

$$|f(\sigma(y)) - g(0) f(y)| \leq \phi(0) \quad (25)$$

for all $y \in S$. Now, using the triangle inequality, (15), and (25) we have

$$\begin{aligned} & |f(z)| |g(y) g(x) - g(0) g(x + \sigma(y))| \\ & \leq |g(y) g(x) f(z) - g(y) f(x + \sigma(z))| \\ & \quad + |g(y) f(x + \sigma(z)) - f(y + \sigma(x) + z)| \\ & \quad + |f(y + \sigma(x) + z) - g(0) f(x + \sigma(y) + \sigma(z))| \\ & \quad + |g(0) f(x + \sigma(y) + \sigma(z)) - g(0) g(x + \sigma(y)) f(z)| \\ & \leq |g(y)| \phi(x) + \phi(y) + \phi(0) + |g(0)| \phi(x + \sigma(y)) \end{aligned} \quad (26)$$

for all $x, y, z \in S$. Since f is unbounded, it follows from (26) that

$$g(x) g(y) = g(0) g(x + \sigma(y)) \quad (27)$$

for all $x, y \in S$. Dividing (27) by $g(0)^2$ we have

$$g_0(x) g_0(y) = g_0(x + \sigma(y)) \quad (28)$$

for all $x, y \in S$, where $g_0(x) = g(x)/g(0)$. Thus, g_0 is σ -exponential and $g(x) = g(0)m_\sigma(x)$ for some σ -exponential m_σ . Replacing y by $\sigma(y)$ in (25) and multiplying $g(0)$ in the result we have

$$|g(0) f(y) - g(0)^2 f(\sigma(y))| \leq |g(0)| \phi(0) \quad (29)$$

for all $y \in S$. Using the triangle inequality with (25) and (29) we have

$$|1 - g(0)^2| |f(\sigma(y))| \leq (1 + |g(0)|) \phi(0) \quad (30)$$

for all $y \in S$. Since f is unbounded, we have $g(0)^2 = 1$. Putting $y = 0$ in (15) we have

$$|f(x) - f(0) g(x)| \leq \phi(x) \quad (31)$$

for all $x \in S$. Thus, we get (17). For the particular case when $\phi(x) = \epsilon$ for all $x \in S$, replacing (x, y) by $(\sigma(x), y)$ and (x, y) by $(\sigma(y), x)$ in (15) and using triangle inequality with the resulting two inequalities, we have

$$|f(x) g(\sigma(y)) - f(y) g(\sigma(x))| \leq 2\epsilon \quad (32)$$

for all $x, y \in S$, which is

$$|f(x) g(y) - f(y) g(x)| \leq 2\epsilon \quad (33)$$

for all $x, y \in S$, since $g(x) = g(\sigma(x))$ for all $x \in S$. Dividing both sides of (33) by $|g(y)|$ we have

$$|f(x) - \alpha(y) g(x)| \leq \frac{2\epsilon}{|g(y)|} \quad (34)$$

for all $y \in J := \{y : g(y) \neq 0\}$, where $\alpha(y) = f(y)/g(y)$. Since f is unbounded, we have $\alpha(y) \neq 0$ for all $y \in J$ and hence g is unbounded. Replacing y by $y_1 \in J$ and y by $y_2 \in J$ in (34) we have two inequalities. Using the triangle inequality with these two resulting inequalities we have

$$|\alpha(y_1) - \alpha(y_2)| |g(x)| \leq 2\epsilon \left(\frac{1}{|g(y_1)|} + \frac{1}{|g(y_2)|} \right) \quad (35)$$

for all $x \in S$. Since g is unbounded, from (35) we have $\alpha(y_1) = \alpha(y_2)$ and hence $\alpha(y) := \alpha$ is independent of $y \in J$. From (34) we have

$$|f(x) - \alpha g(x)| \leq \frac{2\epsilon}{|g(y)|} \quad (36)$$

for all $x \in S$ and $y \in J$. Since g is unbounded we have

$$f(x) = \alpha g(x) = \alpha g(0) m_\sigma(x) \quad (37)$$

for all $x \in S$. Putting (37) in (15) we have

$$|\alpha g(0) (1 - g(0))| |m_\sigma(x) m_\sigma(y)| \leq \epsilon \quad (38)$$

for all $x, y \in S$. Since $\alpha \neq 0$, $g(0) \neq 0$, and m_σ is unbounded, we have $g(0) = 1$. Thus, we get (18). This completes the proof. $\square$

Remark 2. We have no idea if the case $g(x) = -m_\sigma(x)$ in (17) occurs or not. For example, it can be verified that if there exists a sequence $x_n \in S$, $n = 1, 2, 3, \dots$, such that $(1 + \phi(x_n))/|g(x_n)| \rightarrow 0$ as $n \rightarrow \infty$, the case $g(x) = -m_\sigma(x)$ in (17) does not occur; in particular (g, f) satisfies (18).

Remark 3. In general, the inequality (16) cannot be replaced by the weaker inequality:

$$|1 - g(x)| \leq \frac{\epsilon}{M_f} \quad (39)$$

for all $x \in S$. Indeed, let $\phi(x) = \epsilon > 0$, $\sigma(y) = y$, $f(x) = \epsilon \sin x$, $g(x) = \cos x$ for all $x, y \in S$. Then we have

$$|f(x+y) - f(x)g(y)| = |\epsilon \cos x \sin y| \leq \epsilon \quad (40)$$

for all $x, y \in S$. However, the inequality (39) fails since

$$|1 - g(\pi)| = |1 - \cos \pi| = 2 > 1 = \frac{\epsilon}{M_f}. \quad (41)$$

As a direct consequence of Theorem 1 we have the following.

Corollary 4. *Let $f : S \rightarrow \mathbb{C}$ satisfy*

$$|f(x + \sigma(y)) - f(x)f(y)| \leq \phi(x) \quad (42)$$

for all $x, y \in S$. If S is a group and f is bounded, then

$$|f(y)| |1 - |f(x)|| \leq \phi(x) \quad (43)$$

for all $x, y \in S$. If f is unbounded, then f is σ -exponential.

Proof. The inequality (16) implies (43). Now, by Theorem 1, we have

$$f(x) = f(0)m_\sigma(x), \quad x \in S \quad (44)$$

for some σ -exponential function m_σ . Putting (44) to (42) and letting $x = 0$ we have

$$|f(0)(1 - f(0))| |m_\sigma(y)| \leq \phi(0) \quad (45)$$

for all $y \in S$. Since f is unbounded, from (45) we have $f(0) = 1$. This completes the proof. $\square$

Theorem 5. *Let $f, g : S \rightarrow \mathbb{C}$ satisfy*

$$|f(x + \sigma(y)) - f(x)g(y)| \leq \phi(y) \quad (46)$$

for all $x, y \in S$. If S is a group and f is bounded, then

$$|1 - |g(x)|| \leq \frac{\phi(x)}{M_f} \quad (47)$$

for all $x \in S$, where $M_f = \sup_{x \in S} |f(x)|$. If f is unbounded, then there exists an exponential function $m : S \rightarrow \mathbb{C}$ such that

$$g(x) = m(x), \quad |f(x) - f(0)m(\sigma(x))| \leq \phi(\sigma(x)) \quad (48)$$

for all $x \in S$. In particular, if $\phi(x) = \epsilon \geq 0$ for all $x \in S$, then we have

$$g(x) = m(x), \quad f(x) = f(0)m(\sigma(x)) \quad (49)$$

for all $x \in S$.

Proof. Using the same method as in the proof of Theorem 1, we can show that if f is bounded and S is a group, then f

satisfies (47). Assume that f is unbounded. Using the triangle inequality and (46) we have

$$\begin{aligned} & |f(z)| |g(x)g(y) - g(y+x)| \\ & \leq |f(z)g(x)g(y) - f(z+\sigma(x))g(y)| \\ & \quad + |f(z+\sigma(x))g(y) - f(z+\sigma(x)+\sigma(y))| \\ & \quad + |f(z+\sigma(y+x)) - f(z)g(y+x)| \\ & \leq \phi(x)|g(y)| + \phi(y) + \phi(x+y) \end{aligned} \quad (50)$$

for all $x, y, z \in S$. Since f is unbounded, it follows from (50) that

$$g(x+y) = g(x)g(y) \quad (51)$$

for all $x, y \in S$. Therefore, g is an exponential function: say $g = m$. Putting $x = 0$ and replacing y by $\sigma(x)$ in (46) we get (48). Assume that $\phi(y) = \epsilon$ for all $y \in S$. Replacing (x, y) by $(x, \sigma(y))$ and (x, y) by $(y, \sigma(x))$ in (46) using triangle inequality with the resulting two inequalities we have

$$|f(x)g(\sigma(y)) - f(y)g(\sigma(x))| \leq 2\epsilon \quad (52)$$

for all $x, y \in S$. Now, using the same method as in the proof of Theorem 1 (after the inequality (33)) we can show that

$$f(x) = \alpha g(\sigma(x)) \quad (53)$$

for all $x \in S$. Since g is an exponential function, we have $g(\sigma(0)) = g(0) = 1$ and hence $\alpha = f(0)$. This completes the proof. $\square$

As a direct consequence of Theorem 5 we have the following.

Corollary 6. *Let $f : S \rightarrow \mathbb{C}$ satisfy*

$$|f(x + \sigma(y)) - f(x)f(y)| \leq \phi(y) \quad (54)$$

for all $x, y \in S$. If S is a group and f is bounded, then

$$|f(y)| |1 - |f(x)|| \leq \phi(x) \quad (55)$$

for all $x, y \in S$. If f is unbounded, then f is σ -exponential.

Proof. If f is bounded, the inequality (55) follows from (47). If f is unbounded, then, by Theorem 5, f is exponential. Thus, from (54) we have

$$|f(x)| |f(\sigma(y)) - f(y)| \leq \phi(y) \quad (56)$$

for all $x, y \in S$. Since f is unbounded, it follows from (56) that $f(y) = f(\sigma(y))$ for all $y \in S$. This completes the proof. $\square$

Theorem 7. *Let $f, g : S \rightarrow \mathbb{C}$ satisfy*

$$|f(x + \sigma(y)) - g(x)f(y)| \leq \phi(y) \quad (57)$$

for all $x, y \in S$. If $f(0) = 0$, then

$$|f(x)| \leq \phi(0), \quad |g(x)| \leq \inf_{y \in S} \frac{\phi(y) + \phi(0)}{|f(y)|} \quad (58)$$

for all $x \in S$. If $f(0) \neq 0$, S is a group, and g is bounded, then

$$||f(0)| - |f(x)|| \leq \frac{\phi(x) + \phi(0)}{M_g} \quad (59)$$

for all $x \in S$, where $M_g = \sup_{x \in S} |g(x)|$. If $f(0) \neq 0$ and g is unbounded, then there exists an unbounded σ -exponential function $m_\sigma : S \rightarrow \mathbb{C}$ such that

$$f(x) = f(0)m_\sigma(x), \quad |g(x) - m_\sigma(x)| \leq \frac{\phi(0)}{|f(0)|} \quad (60)$$

for all $x \in S$.

Proof. Putting $y = 0$ in (57) we have

$$|f(x) - f(0)g(x)| \leq \phi(0) \quad (61)$$

for all $x \in S$. If $f(0) = 0$, then we have

$$|f(x)| \leq \phi(0) \quad (62)$$

for all $x \in S$. Using the triangle inequality with (57) and using (62) we have

$$|g(x)f(y)| \leq \phi(y) + \phi(0) \quad (63)$$

for all $x, y \in S$. Dividing (63) by $|f(y)|$ and taking infimum of the right-hand side of the result we get

$$|g(x)| \leq \inf_{y \in S} \frac{\phi(y) + \phi(0)}{|f(y)|} \quad (64)$$

for all $x \in S$, which gives (58). If $f(0) \neq 0$, using the triangle inequality with (57), using (61), and dividing the result by $|f(0)|$, we have

$$|g(x + \sigma(y)) - g(x)f_0(y)| \leq \phi_0(y) \quad (65)$$

for all $x, y \in S$, where $f_0(y) = f(y)/f(0)$, $\phi_0(y) = (\phi(y) + \phi(0))/|f(0)|$. If g is bounded and S is a group, then by Theorem 5, f_0 satisfies

$$|1 - |f_0(x)|| \leq \frac{\phi_0(x)}{M_g} \quad (66)$$

for all $x \in S$, where $M_g = \sup_{x \in S} |g(x)|$. Multiplying (66) by $|f(0)|$ we have

$$||f(0)| - |f(x)|| \leq \frac{\phi(x) + \phi(0)}{M_g} \quad (67)$$

for all $x \in S$. If g is unbounded, then by Theorem 5, f_0 is exponential function; say $f_0 = m$. Putting $y = 0$ in (57) and dividing the result by $|f(0)|$ we have

$$|g(x) - m(x)| \leq \frac{\phi(0)}{|f(0)|} \quad (68)$$

for all $x \in S$. Multiplying (68) by $|f(y)|$,

$$|g(x)f(y) - m(x)f(y)| \leq \frac{\phi(0)|f(y)|}{|f(0)|} \quad (69)$$

for all $x, y \in S$. Using the triangle inequality with (57) and (69) we have

$$\begin{aligned} & |f(0)m(x)||m(\sigma(y)) - m(y)| \\ &= |f(x + \sigma(y)) - m(x)f(y)| \\ &\leq |f(x + \sigma(y)) - g(x)f(y)| \\ &\quad + |g(x)f(y) - m(x)f(y)| \\ &\leq \phi(y) + \frac{\phi(0)|f(y)|}{|f(0)|} \end{aligned} \quad (70)$$

for all $x, y \in S$. Since m is unbounded, it follows from (70) that $m(\sigma(y)) = m(y)$ for all $y \in S$. This completes the proof. $\square$

Theorem 8. Let $f, g : S \rightarrow \mathbb{C}$ satisfy

$$|f(x + \sigma(y)) - f(x)g(y)| \leq \phi(x) \quad (71)$$

for all $x, y \in S$. If $f(0) = 0$, then

$$|f(x)| \leq \phi(0), \quad |g(x)| \leq \inf_{y \in S} \frac{\phi(y) + \phi(0)}{|f(y)|} \quad (72)$$

for all $x \in S$. If $f(0) \neq 0$, S is a group, and g is bounded, then

$$||f(0)| - |f(x)|| \leq \frac{\phi(x) + \phi(0)}{M_g} \quad (73)$$

for all $x \in S$, where $M_g = \sup_{x \in S} |g(x)|$. If $f(0) \neq 0$ and g is unbounded, then there exists an unbounded exponential $m : S \rightarrow \mathbb{C}$ such that

$$f(x) = f(0)m(x), \quad |g(x) - m(\sigma(x))| \leq \frac{\phi(0)}{|f(0)|} \quad (74)$$

for all $x \in S$.

Proof. Putting $x = 0$ in (71) we have

$$|f(\sigma(y)) - f(0)g(y)| \leq \phi(0) \quad (75)$$

for all $y \in S$. If $f(0) = 0$, then replacing y by $\sigma(y)$ in (75) we have

$$|f(y)| \leq \phi(0) \quad (76)$$

for all $y \in S$. Using the triangle inequality with (71) and using (76) we have

$$|f(x)g(y)| \leq \phi(x) + \phi(0) \quad (77)$$

for all $x, y \in S$. Dividing (77) by $|f(x)|$ and taking infimum of the right-hand side of the result we have

$$|g(y)| \leq \inf_{x \in S} \frac{\phi(x) + \phi(0)}{|f(x)|} \quad (78)$$

for all $x \in S$, which gives (72). If $f(0) \neq 0$, using the triangle inequality with (71) and (75), replacing (x, y) by (y, x) , and dividing the result by $|f(0)|$, we have

$$|g(x + \sigma(y)) - g(x) f_0(y)| \leq \phi_0(y) \quad (79)$$

for all $x, y \in S$, where $f_0(y) = f(y)/f(0)$, $\phi_0(y) = (\phi(y) + \phi(0))/|f(0)|$. If g is bounded and S is a group, then by Theorem 5 we get (73). If g is unbounded, then by Theorem 5, f_0 is exponential; say $f_0 = m$. Putting $x = 0$ in (71), replacing y by $\sigma(x)$ in the result, and dividing the result by $|f(0)|$ we get (74). This completes the proof. $\square$

Finally, we consider the functional inequality with three unknown functions $f, g, h : S \rightarrow \mathbb{C}$:

$$|f(x + \sigma(y)) - g(x) h(y)| \leq \epsilon \quad (80)$$

for all $x, y \in S$. The inequality (80) is no more than the inequality

$$|f(x + y) - g(x) h(y)| \leq \epsilon \quad (81)$$

for all $x, y \in S$, since, if we replace y by $\sigma(y)$ in (80) and h by $h \circ \sigma$, inequality (80) is reduced to (81).

In the following we exclude the trivial cases when $g = 0$ or $h = 0$.

Theorem 9. *Let $f, g, h : S \rightarrow \mathbb{C}$ satisfy the inequality (81). If g or h is unbounded, then there exists an unbounded exponential function $m : S \rightarrow \mathbb{C}$ such that*

$$\begin{aligned} g(x) &= g(0) m(x), & h(x) &= h(0) m(x), \\ |f(x) - g(0) h(0) m(x)| &\leq \epsilon \end{aligned} \quad (82)$$

for all $x \in S$. If S is a group and g or h is bounded, then

$$\begin{aligned} ||h(x)| - |h(0)|| &\leq \frac{2\epsilon}{M_g}, & ||g(x)| - |g(0)|| &\leq \frac{2\epsilon}{M_h}, \\ ||f(x)| - |g(0) h(0)|| &\leq 3\epsilon \end{aligned} \quad (83)$$

for all $x \in S$, where $M_g = \sup_{x \in S} |g(x)|$, $M_h = \sup_{x \in S} |h(x)|$.

Proof. Replacing (x, y) by (y, x) in (81) and using the triangle inequality with the result and (81) we have

$$|g(x) h(y) - h(x) g(y)| \leq 2\epsilon \quad (84)$$

for all $x, y \in S$. Since we exclude the trivial cases when $g = 0$ or $h = 0$, it follows from (84) that g is bounded if and only if h is bounded. If $h(0) = 0$, putting $y = 0$ in (81), and if $h(0) \neq 0$, putting $y = 0$ in (81), we have $|f(x)| \leq \epsilon$ for all $x \in S$. Using the triangle inequality with (81) we have

$$|g(x) h(y)| \leq |f(x + y)| + \epsilon \leq 2\epsilon \quad (85)$$

for all $x, y \in S$. Thus, we have

$$M_g M_h \leq 2\epsilon. \quad (86)$$

Assume that g or h is unbounded. From (86) we have $g(0) \neq 0$, $h(0) \neq 0$. Putting $y = 0$ and replacing x by $x + y$ in (81) we have

$$|f(x + y) - g(x + y) h(0)| \leq \epsilon \quad (87)$$

for all $x, y \in S$. Using the triangle inequality with (81) and (87) and dividing the result by $|h(0)|$ we have

$$|g(x + y) - g(x) h_0(y)| \leq \frac{2\epsilon}{|h(0)|} \quad (88)$$

for all $x, y \in S$, where $h_0(x) = h(x)/h(0)$. Replacing y by $\sigma(y)$ in (87) and using Theorem 5 we get

$$h_0(x) = m(x), \quad g(x) = g(0) m(x), \quad \forall x \in S \quad (89)$$

for some exponential function m . Putting (89) in (81) we obtain (82). Now, we assume that g or h is bounded. If $h(0) \neq 0$, then using Theorem 5 with (88) we have

$$||h(x)| - |h(0)|| \leq \frac{2\epsilon}{M_g} \quad (90)$$

for all $x \in S$. If $g(0) = 0$, from (86) we have

$$|g(x)| \leq \frac{2\epsilon}{M_h} \quad (91)$$

for all $x \in S$. Thus, h satisfies (90) for both cases $h(0) = 0$ and $h(0) \neq 0$. Changing the role of g and h we get

$$||g(x)| - |g(0)|| \leq \frac{2\epsilon}{M_h} \quad (92)$$

for all $x \in S$. Putting $y = 0$ in (81) and using the triangle inequality we have

$$||f(x)| - |h(0) g(x)|| \leq \epsilon \quad (93)$$

for all $x \in S$. Multiplying $|h(0)|$ in (92) we have

$$||h(0) g(x)| - |g(0) h(0)|| \leq \frac{2\epsilon |h(0)|}{M_h} \leq 2\epsilon \quad (94)$$

for all $x \in S$. From (92) and (93), using the triangle inequality we have

$$||f(x)| - |g(0) h(0)|| \leq 3\epsilon \quad (95)$$

for all $x \in S$. This completes the proof. $\square$

3. Bounded Solutions

Throughout this section, we assume that G is a group and $\sigma : S \rightarrow S$ an involution. We describe bounded functions satisfying following functional inequalities:

$$|f(x + \sigma(y)) - f(x) f(y)| \leq \phi(x), \quad (96)$$

$$|f(x + \sigma(y)) - f(x) f(y)| \leq \phi(y) \quad (97)$$

for all $x, y \in G$ and for some $\phi : G \rightarrow \mathbb{R}^+$.

Theorem 10. Let $f : G \rightarrow \mathbb{C}$ be bounded functions satisfying (96) or (97). Then f satisfies

$$|f(x)| \leq \frac{1}{2} \left(1 + \sqrt{1 + 4\phi(x)} \right) \quad (98)$$

for all $x \in G$. Let $K := \{x \in G : \phi(x) < 1/4\}$. Then, either f satisfies

$$\frac{1}{2} \left(1 + \sqrt{1 - 4\phi(x)} \right) \leq |f(x)| \leq \frac{1}{2} \left(1 + \sqrt{1 + 4\phi(x)} \right) \quad (99)$$

for all $x \in K$, or else

$$|f(x)| \leq \frac{1}{2} \left(1 - \sqrt{1 - 4\phi(x)} \right) \quad (100)$$

for all $x \in K$.

Proof. By Corollaries 4 and 6, every bounded solution of (96) and (97) satisfies

$$|f(y)(1 - |f(x)|)| \leq \phi(x) \quad (101)$$

for all $x, y \in G$. Letting $y = x$ in (101) we have

$$-\phi(x) \leq |f(x)|(|f(x)| - 1) \leq \phi(x) \quad (102)$$

for all $x \in G$. Solving the inequality (102) we get (98). Also from (102), for each $x \in K$, we have

$$\frac{1}{2} \left(1 + \sqrt{1 - 4\phi(x)} \right) \leq |f(x)| \leq \frac{1}{2} \left(1 + \sqrt{1 + 4\phi(x)} \right), \quad (103)$$

or

$$|f(x)| \leq \frac{1}{2} \left(1 - \sqrt{1 - 4\phi(x)} \right). \quad (104)$$

Assume that there exist $x_1, x_2 \in K$ such that

$$\begin{aligned} |f(x_1)| &\leq \frac{1}{2} \left(1 - \sqrt{1 - 4\phi(x_1)} \right), \\ |f(x_2)| &\geq \frac{1}{2} \left(1 + \sqrt{1 - 4\phi(x_2)} \right). \end{aligned} \quad (105)$$

Putting $y = x_2, x = x_1$ in (101) we have the contradiction

$$\frac{1}{4} < |f(x_2)(1 - |f(x_1)|)| \leq \phi(x_2) < \frac{1}{4}. \quad (106)$$

Thus, f satisfies (103) for all $x \in K$ or (104) for all $x \in K$. This completes the proof. $\square$

Define $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\sigma(x, y) = (y, x)$ for all $x, y \in \mathbb{R}$. Then it is easy to see that every σ -exponential $m_\sigma : \mathbb{R}^2 \rightarrow \mathbb{C}$ has the form $m_\sigma(x, y) = m(x + y)$, $x, y \in \mathbb{R}$ for some exponential $m : \mathbb{R} \rightarrow \mathbb{C}$. Thus, by Corollary 4 and Theorem 10, we have the following.

Example 11. Let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfy

$$|f(x + v, y + u) - f(x, y)f(u, v)| \leq \frac{1}{4(u^2 + v^2)} \quad (107)$$

for all $x, y, u, v \in \mathbb{R}$. Then, if f is unbounded function, then f has the form

$$f(x, y) = m(x + y) \quad (108)$$

for all $x, y \in \mathbb{R}$, where $m : \mathbb{R} \rightarrow \mathbb{C}$ is an exponential function; if f is bounded, then f satisfies

$$|f(x, y)| \leq \frac{1}{2} \left(1 + \sqrt{1 + \frac{1}{(x^2 + y^2)}} \right) \quad (109)$$

for all $x, y \in \mathbb{R}$. Let $K = \{(x, y) : x^2 + y^2 > 1\}$. Then we have

$$\begin{aligned} \frac{1}{2} \left(1 + \sqrt{1 - \frac{1}{(x^2 + y^2)}} \right) &\leq |f(x, y)| \\ &\leq \frac{1}{2} \left(1 + \sqrt{1 + \frac{1}{(x^2 + y^2)}} \right) \end{aligned} \quad (110)$$

for all $x, y \in K$, or

$$|f(x, y)| \leq \frac{1}{2} \left(1 - \sqrt{1 - \frac{1}{(x^2 + y^2)}} \right) \quad (111)$$

for all $x, y \in K$.

Corollary 12. Let $\epsilon < 1/4$ and $f : G \rightarrow \mathbb{C}$ be a bounded function satisfying

$$|f(x + \sigma(y)) - f(x)f(y)| \leq \epsilon \quad (112)$$

for all $x, y \in G$. Then either f satisfies

$$\frac{1}{2} (1 + \sqrt{1 - 4\epsilon}) \leq |f(x)| \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}) \quad (113)$$

for all $x \in G$, or else

$$|f(x)| \leq \frac{1}{2} (1 - \sqrt{1 - 4\epsilon}) \quad (114)$$

for all $x \in G$.

In particular, if G is 2-divisible and $\sigma(x) = x$ for all $x \in G$, then we obtain the following result which is a generalized version of the result of Albert and Baker [3].

Corollary 13. Let S be 2-divisible, $\epsilon < 1/4$, and let $f : G \rightarrow \mathbb{R}$ be bounded functions satisfying

$$|f(x + y) - f(x)f(y)| \leq \epsilon \quad (115)$$

for all $x, y \in G$. Then either f satisfies

$$\frac{1}{2} (1 + \sqrt{1 - 4\epsilon}) \leq f(x) \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}) \quad (116)$$

for all $x \in G$, or else

$$-\epsilon \leq f(x) \leq \frac{1}{2} (1 - \sqrt{1 - 4\epsilon}) \quad (117)$$

for all $x \in G$.

Proof. Replacing y by x in (115) and using the triangle inequality with the result we have $f(2x) \geq f(x)^2 - \epsilon \geq -\epsilon$ for all $x \in G$, which implies that $f(x) \geq -\epsilon$ for all $x \in G$. Since $\epsilon < (1/2)(1 - \sqrt{1 - 4\epsilon})$, inequality (113) implies (116), and (114) implies (117). This completes the proof. $\square$

4. Applications

In this section we consider the stability of (8) and (9) which were dealt with in [13]. Let $\mathbb{H} = \{a+bi+cj+dk \mid a, b, c, d \in \mathbb{R}\}$ be the quaternion group. Recall that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$, and the conjugate of $q = a + bi + cj + dk \in \mathbb{H}$ is given by $q^* = a - bi - cj - dk$. We denote $\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$. We first consider unbounded functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying (10) and (11) and unbounded functions $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfying (12) and (13).

Theorem 14. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an unbounded function satisfying (10). Then f has the form*

$$f(x, y) = m\left(\sqrt{x^2 + y^2}\right) \quad (118)$$

for all $x, y \in \mathbb{R}$.

Proof. Let $F(x + yi) = f(x, y)$ and $\Phi(x + yi) = \phi(x, y)$ for all $x, y \in \mathbb{R}$. Then the functional inequality (10) is converted to

$$|F(\bar{z}w) - F(z)F(w)| \leq \Phi(w) \quad (119)$$

for all $z = x_2 + iy_2$, $w = x_1 + iy_1 \in \mathbb{C}$. Letting $\sigma(z) = \bar{z}$ and using Corollary 4 when $S = \langle \mathbb{C}, \cdot \rangle$ we have $F(z) = m_\sigma(z)$ for all $z \in \mathbb{C}$. In view of (14), $m_\sigma(z)$ is written in the form $m_\sigma(z) = m(|z|)$ for all $z \in \mathbb{C}$. Thus, we get (118). This completes the proof. $\square$

Using Corollary 6 we obtain the following.

Theorem 15. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an unbounded function satisfying (11). Then f has the form*

$$f(x, y) = m\left(\sqrt{x^2 + y^2}\right) \quad (120)$$

for all $x, y \in \mathbb{R}$.

Theorem 16. *Let $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ be an unbounded function satisfying (12). Then g has the form*

$$g(x, y, u, v) = m\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right) \quad (121)$$

for all $x, y, u, v \in \mathbb{R}$.

Proof. Let $F(x + yi + uj + vk) = g(x, y, u, v)$ and $\Psi(x + yi + uj + vk) = \psi(x, y, u, v)$ for all $x, y, u, v \in \mathbb{R}$. Then the functional inequality (12) is reduced to

$$|F(q)F(p) - F(qp^*)| \leq \Psi(q) \quad (122)$$

for all $p, q \in \mathbb{H}$. Letting $\sigma(q) = q^*$ and using Corollary 4 when $S = \langle \mathbb{H}, \cdot \rangle$ we have that $F(q) = m_\sigma(q)$ for all $q \in \mathbb{H}$. In view of (14), $m_\sigma(q)$ is written in the form $m_\sigma(q) = m(\|q\|)$ for all $q \in \mathbb{H}$. Thus, we get (121). This completes the proof. $\square$

Using Corollary 6 we obtain the following.

Theorem 17. *Let $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ be an unbounded function satisfying (13). Then g has the form*

$$g(x, y, u, v) = m\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right) \quad (123)$$

for all $x, y, u, v \in \mathbb{R}$.

Using the results in Section 3 we can also obtain bounded functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying the inequalities (10), (11) and bounded functions $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfying (12) and (13) as in [13].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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